Embedding into BD spaces and spaces with very few operators.

S.A. Argyros, D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, and D. Zisimopoulou

March 5, 2012
Definition (Lindenstrauss, Pelczynski 1968)

Let $\lambda > 1$. A Banach space $X$ is called a $L(\infty, \lambda)$-space if for every finite dimensional subspace $F$ of $X$, there is a finite dimensional subspace $E$ of $X$ such that $F \subset E$ and $d(E, \ell_\infty) < \lambda$.

$X$ is called a $L_\infty$-space if it is a $L(\infty, \lambda)$-space for some $\lambda > 1$.

Examples:
1. $c_0$ is a $L(\infty, 1+\epsilon)$-space for all $\epsilon > 0$.
2. $C(K)$ is a $L(\infty, 1+\epsilon)$-space for all $\epsilon > 0$.

Theorem (Lewis, Stegall 1973)

If a $L_\infty$ space $X$ has a separable dual $X^*$, then $X^*$ is isomorphic to $\ell_1$. 

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*If a $\mathcal{L}_{\infty}$ space $X$ has a separable dual $X^*$, then $X^*$ is isomorphic to $\ell_1$.***
Embedding into isomorphic preduals of $\ell_1$

**Theorem (F, Odell, Schlumprecht 2011)**

Let $X$ be a Banach space with separable dual. If $X$ does not contain $c_0$, then $X$ embeds into an isomorphic predual of $\ell_1$ which does not contain $c_0$. If $X$ is reflexive then $X$ embeds into an isomorphic predual of $\ell_1$ which is somewhat reflexive.
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Embedding into spaces with very few operators

Theorem (Argyros, F, Haydon, Odell, Raikoftsalis, Schlumprecht, Zisimopoulou 2012)
Let $X$ be a separable uniformly convex Banach space. $X$ embeds into a Banach space $Z$ such that $Z$ is an isomorphic predual of $\ell_1$, and $Z$ has very few operators. That is, every operator on $Z$ is equal to a scalar times the identity plus a compact operator.

Theorem (Argyros, F, Haydon, Odell, Raikoftsalis, Schlumprecht, Zisimopoulou (in preparation))
Let $X$ be a separable Banach space such that $\ell_1$ is not isomorphic to a complemented subspace of $X^*$. $X$ embeds into a Banach space $Z$ such that $Z$ is an isomorphic predual of $\ell_1$, and $Z$ has very few operators.
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Let \( \{ \Delta_i \}_{i=1}^{\infty} \) be a sequence of finite disjoint sets. We will construct a separable \( \ell_\infty \) subspace of \( \ell_\infty(\bigcup_{i=1}^{\infty} \Delta_i) \).

For each \( n \in \mathbb{N} \), let \( U_n: \ell_\infty(\bigcup_{i=1}^{n} \Delta_i) \to \ell_\infty(\Delta_n + 1) \) be some linear map. Thus if \( x \in \ell_\infty(\bigcup_{i=1}^{n} \Delta_i) \) then \((x, U_n(x)), U_{n+1}(x)) \in \ell_\infty(\bigcup_{i=1}^{n} \Delta_i \cup \Delta_n + 1 \cup \Delta_n + 2) \). Want to have:

\[(x, U_n(x), U_{n+1}(x), U_{n+2}(x), \ldots, \ldots) \in \ell_\infty(\bigcup_{i=1}^{\infty} \Delta_i) \]

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Assume there exists some constant $C \geq 1$ such that
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\|(x, U_n(x), U_{n+1}(x, U_n(x)), U_{n+2}(\ldots), \ldots)\|_\infty \leq C\|x\|_\infty \quad \forall x \in \ell_\infty(\cup_{i=1}^n \Delta_i)
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We define \( J_n : \ell_\infty (\bigcup_{i=1}^n \Delta_i) \to \ell_\infty (\bigcup_{i=1}^\infty \Delta_i) \) by:

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The space $Y_n$ is $C$-isomorphic to $\ell_\infty(\bigcup_{i=1}^\infty \Delta_i)$. Thus $Y$ is a separable $L_\infty$-subspace of $\ell_\infty(\bigcup_{i=1}^\infty \Delta_i)$. 

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Thus \( Y \) is a separable \( L_\infty \)-subspace of \( \ell_\infty(\bigcup_{i=1}^\infty \Delta_i) \).
How to ensure \( \|J_n\|_\infty \leq 2 \).

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\ell_\infty \left( \bigcup_{i=1}^n \Delta_i \right) \rightarrow \ell_\infty \left( \Delta_{n+1} \right)
\]

\[
J_n(x) = (x, U_n(x), U_{n+1}(x), U_{n+2}(x), \ldots, \ldots, U_{n+k}(x))
\]

\( \forall x \in \ell_\infty \left( \bigcup_{i=1}^n \Delta_i \right) \)

Some notation: If \( \gamma \in \Delta_{n+1} \) then

\[
u^*_\gamma(x) = U_n(x)(\gamma) \quad \text{and} \quad e^*_\gamma(x) = x(\gamma)
\]

Proposition (B-D condition)

The following condition guarantees that \( \|J_n\|_\infty \leq 2 \) for all \( n \in \mathbb{N} \).

For all \( \gamma \in \Delta_{n+1} \) there exists constants \( a_\gamma, b_\gamma \in \mathbb{R}, \) an integer \( 1 \leq k < n, \) an element \( \eta \in \Delta_k \) and a functional \( b^*_\in \mathbb{B} \ell_1 \left( \bigcup_{i=1}^{n-1} \Delta_i \right) \) such that:

\[
u^*_\gamma(x) = a_\gamma e^*_\eta(x) + b_\gamma b^*_\in(x) \quad \forall x \in \ell_\infty \left( \bigcup_{i=1}^n \Delta_i \right)
\]

\[|a_\gamma| \leq 1 \quad \text{and} \quad |b_\gamma| \leq \frac{1}{4}
\]

\( \text{or } a_\gamma = 0 \) and \( |b_\gamma| \leq \frac{1}{4} \)

\[
J_n(x) = 0 \quad \forall x \in \ell_\infty \left( \bigcup_{i=1}^k \Delta_i \right)
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*The following condition guarantees that $\|J_n\|_\infty \leq 2$ for all $n \in \mathbb{N}.$*

*For all $\gamma \in \Delta_{n+1}$ there exists constants $a_\gamma, b_\gamma \in \mathbb{R},$ an integer $1 \leq k < n,$ an element $\eta \in \Delta_k$ and a functional $b^*_\gamma \in B_{\ell_1(\bigcup_{i=1}^{n-1})}$ such that:*

\begin{enumerate}
\item $u^*_\gamma(x) = a_\gamma e^*_\eta(x) + b_\gamma b^*_\gamma(x)$ \quad $\forall x\in \ell_\infty\left(\bigcup_{i=1}^k \Delta_i\right)$
\item $|a_\gamma| \leq 1$ and $|b_\gamma| \leq 1/4$ or $a_\gamma = 0$ and $|b_\gamma| \leq 1/4$
\item $b^*_\gamma(J_n(x)) = 0$ for all $x \in \ell_\infty\left(\bigcup_{i=1}^{k-1} \Delta_i\right)$
\end{enumerate}
How to ensure $\|J_n\|_\infty \leq 2$.

$U_n : \ell_\infty(\bigcup_{i=1}^n \Delta_i) \to \ell_\infty(\Delta_{n+1})$

$J_n(x) = (x, U_n(x), U_{n+1}(x, U_n(x)), U_{n+2}(\ldots), \ldots) \quad \forall x \in \ell_\infty(\bigcup_{i=1}^n \Delta_i)$

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Let \( X^* \) be a separable dual space with a boundedly complete FDD \( (E_i^*) \).

Definition (c-decomposition)

Let \( 0 < c < 1 \) be a constant. We call a finite block sequence \((x^*_{1}, \ldots, x^*_{m})\) a \( c \)-decomposition of \( x^* \in X^* \) with respect to \( (E_i^*) \) if:

1. \[ \sum_{i=1}^{m} x^*_{i} = x^* \]
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We use the \( c \)-decomp. of a countable subset of \( B(X^*) \) to create a BD space containing \( X^* \).

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\[ u^*_\gamma(x^*_{1}, \ldots, x^*_{m}) = e^*_\gamma(x^*_{1}, \ldots, x^*_{m-1}) + \| x^*_{m} \| e^*_cd\left(\frac{x^*_{m}}{\| x^*_{m} \|}\right) \]

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Embedding into BD spaces and spaces with very few operators.
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If $X$ is a Banach space then $\psi: X \to C(B_X^*)$ defined by $\psi(x)(x^*) = x^*(x)$ is an isometry.

$X$ embeds into $Y$ in a very similar way.

We define the embedding $\phi: X \to Y \subset \ell_\infty(\bigcup_{i=1}^\infty \Delta_i)$ by:

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We have a Banach space $X$, finite sets $(\Delta_i)_{i=1}^\infty$, and a BD space $Y$ with $X \subseteq Y \subseteq \ell_\infty \left( \bigcup_{i=1}^\infty \Delta_i \right)$. We create new finite sets $(\Theta_i)_{i=1}^\infty$ and a new BD space $Z$ with $X \oplus 0 \subseteq Z \subset \ell_\infty \left( \bigcup_{i=1}^\infty \Theta_i \right) \oplus \ell_\infty \left( \bigcup_{i=1}^\infty (\Delta_i \cup \Theta_i) \right)$.

Depending on $X$, we want $Z$ to have the additional property of not containing $c_0$, being somewhat reflexive, or having very few operators. For $\gamma \in \Theta_{n+1}$, we need to define $u^*\gamma(x)$. We require that there exists constants $a_\gamma, b_\gamma \in \mathbb{R}$, an integer $1 \leq k < n$, an element $\eta \in \Theta_k$ and a functional $b^* \in B_{\ell_1} \left( \bigcup_{i=1}^{n-1} \Delta_i \cup \Theta_i \right)$ such that:

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Augmentations

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$$X \subseteq Y \subseteq \ell_\infty(\bigcup_{i=1}^\infty \Delta_i).$$

We create new finite sets $(\Theta_i)_{i=1}^\infty$ and a new BD space $Z$ with

$$X \oplus 0 \subseteq Z \subset \ell_\infty(\bigcup_{i=1}^\infty \Delta_i) \oplus \ell_\infty(\bigoplus \bigcup_{i=1}^\infty \Theta_i).$$

Depending on $X$, we want $Z$ to have the additional property of not containing $c_0$, being somewhat reflexive, or having very few operators.

For $\gamma \in \Theta_{n+1}$, we need to define $u_\gamma^*$. We require that there exists constants $a_\gamma, b_\gamma \in \mathbb{R}$, an integer $1 \leq k < n$, an element $\eta \in \Theta_k$ and a functional $b^* \in B_{\ell_1(\bigcup_{i=1}^{n-1})}$ such that:

1. $u_\gamma^*(x) = a_\gamma e_\eta^*(x) + b_\gamma b^*(x) \quad \forall x \in \ell_\infty(\bigcup_{i=1}^{n-1} \Delta_i \cup \Theta_i)$
2. $|a_\gamma| \leq 1$ and $|b_\gamma| \leq 1/4$ or $a_\gamma = 0$ and $|b_\gamma| \leq 1$
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3. $b^*(J_n(x)) = 0$ for all $x \in \ell_\infty(\bigcup_{i=1}^{k} \Delta_i \cup \Theta_i)$
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4. $b^*|_X = 0$
How to augment FOS with AH for \( X \) uniformly convex

For each \( \gamma \in AH \), there exists \( m_j \in \mathbb{N} \), called the weight of \( \gamma \), such that:

\[
u^* \gamma = m_j - 1 b^* \quad \text{or} \quad u^* \gamma = e^* \xi + m_j - 1 b^*
\]

and weight of \( \xi \) is \( m_j \).

Define:

\[
e^* \gamma = u^* \gamma + d^* \gamma.
\]

Note that \( u^* \xi \) has the same form as \( u^* \gamma \).

After repeatedly substituting, we obtain the evaluation analysis of \( \gamma \):

\[
e^* \gamma = a \sum_{i=1}^{n_j} d^* \xi + m_j - 1 \sum_{i=1}^{n_j} b^* i
\]

and \( a \leq n_j \).

In FOS, each \( \gamma \) is a \( c \)-decomposition (\( x^*_1, x^*_2, \ldots, x^*_a \)).

\[
u^* (x^*_1, x^*_2, \ldots, x^*_a) = e^* (x^*_1, x^*_2, \ldots, x^*_a - 1) + \|x^*_a\| e^* (x^*_a / \|x^*_a\|)
\]

The evaluation analysis of \( (x^*_1, x^*_2, \ldots, x^*_a) \) is:

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e^* (x^*_1, x^*_2, \ldots, x^*_a) = a \sum_{i=1}^{n_j} d^* (x^*_1, x^*_2, \ldots, x^*_i) + \|x^*_i\| a \sum_{i=1}^{n_j} e^* (x^*_i / \|x^*_i\|)
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$$u^*\gamma = m_j - 1 b^*$$

or

$$u^*\gamma = e^*\xi + m_j - 1 b^*$$

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$$e^*\gamma = u^*\gamma + d^*\gamma.$$  

Note that $u^*\xi$ has the same form as $u^*\gamma$!

After repeatedly substituting, we obtain the evaluation analysis of $\gamma$:

$$e^*\gamma = a \sum_{i=1}^{a} d^*\xi + m_j - 1 a \sum_{i=1}^{a} b^*i$$

and $a \leq n_j$

In FOS, each $\gamma$ is a c-decomposition ($x^*1$, $x^*2$, ..., $x^*a$).

$$u^*(x^*1, x^*2, ..., x^*a) = e^*(x^*1, x^*2, ..., x^*a - 1) + \|x^*a\|e^*(x^*a/\|x^*a\|)$$

The evaluation analysis of ($x^*1, x^*2, ..., x^*a$) is:

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   u^*_\gamma(x_1^*, x_2^*, \ldots, x_a^*) = e^*_\gamma(x_1^*, x_2^*, \ldots, x_a^*) + \|x_a^*\| e^*_\gamma(x_a^*/\|x_a^*\|)
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Embedding into BD spaces and spaces with very few operators.
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How to augment FOS with AH for $X$ uniformly convex

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After repeatedly substituting, we obtain the evaluation analysis of $\gamma$: 

$$e^*_\gamma = a \sum_{i=1}^{a} d^*_i$$

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How to augment FOS with AH for $X$ uniformly convex

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In FOS, each $\gamma$ is a $c$-decomposition $(x_1^*, x_2^*, \ldots, x_a^*)$.

$$u_{(x_1^*, x_2^*, \ldots, x_a^*)}^* = e_{(x_1^*, x_2^*, \ldots, x_a^*)}^* + \|x_a^*\| e_{(x_a^*/\|x_a^*\|)}^*$$

The evaluation analysis of $(x_1^*, x_2^*, \ldots, x_a^*)$ is:

$$e_{(x_1^*, x_2^*, \ldots, x_a^*)}^* = \sum_{i=1}^{a} d_{(x_1^*, x_2^*, \ldots, x_i^*)}^* + \|x_i^*\| \sum_{i=1}^{a} e_{(x_i^*/\|x_i^*\|)}^*$$
We replace each $\gamma = (x_1^*, x_2^*, ..., x_a^*)$ in FOS with $(cx_1^*, cx_2^*, ..., cx_a^*)$. 
How to augment FOS with AH for $X$ uniformly convex

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$$e^* (cx_1^*, cx_2^*, ..., cx_a^*) = a \sum_{i=1}^{a} d^* (cx_i^*, cx_2^*, ..., cx_a^*) + c a \sum_{i=1}^{a} \|x_i^*\| e^*(cx_i^*/\|x_i^*\|)$$
We replace each $\gamma = (x_1^*, x_2^*, \ldots, x_a^*)$ in FOS with $(cx_1^*, cx_2^*, \ldots, cx_a^*)$. The evaluation analysis of $(cx_1^*, cx_2^*, \ldots, cx_a^*)$ is:

$$e^{(cx_1^*, cx_2^*, \ldots, cx_a^*)} = \sum_{i=1}^{a} d^{(cx_1^*, cx_2^*, \ldots, cx_i^*)} + c \sum_{i=1}^{a} \frac{\|x_i^*\|}{c} e^{(cx_i^*/\|x_i^*\|)}$$

We may choose $m_1 = c$. 
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We may choose $m_1 = c$. If $X$ is uniformly convex then there exists $n_1 \in \mathbb{N}$ such that if $x^* \in B_{X^*}$ then $x^*$ has a $c$-decomposition $(x_1^*, x_2^*, ..., x_a^*)$ with $a \leq n_1$. 

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S.A. Argyros, D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlü.. Embedding into BD spaces and spaces with very few operators.
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u_{\gamma}^* = m_j^{-1} b^* \quad \text{or} \quad u_{\gamma}^* = e_{\xi}^* + m_j^{-1} b^* \quad \text{and weight of } \xi \text{ is } m_j
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S.A. Argyros, D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, and D. Zisimopoulou

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After augmenting FOS with AH when $X$ is uniformly convex, given any bounded operator $T$, there exists a constant $\lambda$ and a compact operator $K$ such that $T - \lambda \text{Id} - K$ factors through $X$. Thus $T - \lambda \text{Id} - K$ is weakly compact, and hence compact.
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After augmenting FOS with AH when $X$ is uniformly convex, given any bounded operator $T$, there exists a constant $\lambda$ and a compact operator $K$ such that $T - \lambda I - K$ factors through $X$. Thus $T - \lambda I - K$ is weakly compact. Thus $T^* - \lambda I^* - K^* : l_1 \to l_1$ is weakly compact, and hence compact. This gives that $T - \lambda I$ is compact.