

# Ideals of operators on $(\bigoplus E_n)_{c_0}$ and $(\bigoplus E'_n)_{\ell^1}$

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# Statement of Problem

Let  $E$  be a Banach space. The space  $L(E)$  of all bounded linear operators on  $E$  is a Banach algebra. Determine the structure of the closed ideals in  $L(E)$ .

Particularly, look at  $E$  of the form  $(\bigoplus E_n)_{c_0}$  and  $(\bigoplus E'_n)_{\ell^1}$ , where  $E_n$ 's are certain finite dimensional spaces.

- $(\bigoplus \ell^2(n))_{c_0}$  [Laustsen, Loy & Read, JFA, 2004].
- $(\bigoplus \ell^2(n))_{\ell^1}$  [Laustsen, Schlumprecht & Zsák, JOT 2006].
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# Notation

$X, Y$  separable Banach spaces.

$$\mathcal{G}_{c_0}(X, Y) = \{T \in L(X, Y) : \exists R \in L(X, c_0), S \in L(c_0, Y), T = SR\}.$$

$$\mathcal{G}^s(X, Y) = \{T \in L(X, Y) : \exists \text{ quotient } Q : \ell^1 \rightarrow X, TQ \in \overline{\mathcal{G}_{c_0}(\ell^1, Y)}\}.$$

$T_1, T_2$  operators.  $T_1 \prec T_2$  if  $T_1$  factors through  $T_2$ , i.e.,  $T_1 = AT_2B$  for some operators  $A$  and  $B$ .

## Theorems

LLR Let  $E = (\oplus \ell^2(n))_{c_0}$ . If  $T \notin \overline{\mathcal{G}_{c_0}(E)}$ , then  $1_E \prec T$ .

LOSZ Let  $E = (\oplus \ell^1(n))_{c_0}$ . If  $T \notin \mathcal{G}^s(E)$ , then  $1_E \prec T$ .

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# Steps in the proofs

Let  $E = (\oplus E_n)_{c_0}$ , where  $E_n$  is finite dimensional for each  $n$ .

If  $I \subseteq \mathbb{N}$ , let  $E_I = (\oplus_{n \in I} E_n)_{c_0}$  and  $P_I$  be the projection onto  $E_I$ .

$$P_{\{n\}} = P_n.$$

An operator  $T \in L(E)$  has a matrix representation  $(T_{mn})$ , where

$$T_{mn} = P_n T P_m.$$

$T \in L(X, Y)$ ,  $\varepsilon > 0$ . Let

$$\text{Fac}_0^\varepsilon(T) = \inf\{\|R\| \|S\| : R \in L(X, c_0), S \in L(c_0, Y), \|T - SR\| \leq \varepsilon\}.$$

- ① Let  $T \in L(E)$  and  $T_n = P_n T$ . If  $\sup_n \text{Fac}_0^\varepsilon(T_n) < \infty$ , then  $d(T, \mathcal{G}_{c_0}(E)) \leq \varepsilon$ .
- ② Let  $T_n : (\oplus_{i=1}^n \ell^2(m_i))_{\ell^\infty(n)} \rightarrow \ell^2(j_n)$  be uniformly bounded. Then either
  - ①  $\sup_n \text{Fac}_0^\varepsilon(T_n) < \infty$  for all  $\varepsilon > 0$ , or
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- ① Let  $T \in L(E)$  and  $T_n = P_n T$ . If there exists a quotient  $Q : \ell^1 \rightarrow E$  such that  $\sup_n \text{Fac}_0^\varepsilon(T_n Q) < \infty$ , then for any quotient  $\pi : \ell^1 \rightarrow E$ ,  $d(T\pi, \mathcal{G}_{c_0}(\ell^1, E)) \leq \varepsilon$ .
- ② Let  $X$  be a separable Banach spaces and let  $T_n : X \rightarrow L^1$  be uniformly bounded. Then either
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# Dual cases

Let  $E_n$  be finite dimensional spaces and let  $F = (\oplus E'_n)_{c_0}$ ,  
 $E = F' = (\oplus E_n)_{\ell^1}$ .

$$\mathcal{G}_{\ell^1}(X, Y) = \{T \in L(X, Y) : \exists R \in L(X, \ell^1), S \in L(\ell^1, Y), T = SR\}.$$

$$\mathcal{G}^i(X, Y) = \{T \in L(X, Y) : \exists \text{ embedding } J : Y \rightarrow \ell^\infty, JT \in \overline{\mathcal{G}_{\ell^1}(X, \ell^\infty)}\}.$$

## Theorems

LSZ Let  $E = (\oplus \ell^2(n))_\ell$ . If  $T \notin \overline{\mathcal{G}_\ell(E)}$ , then  $1_E \prec T$ .

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# Preliminaries

$$T : X \rightarrow Y, \varepsilon > 0.$$

$$\text{Fac}_1^\varepsilon(T) = \inf\{\|R\| \|S\| : R \in L(X, \ell^1), S \in L(\ell^1, Y), \|T - SR\| \leq \varepsilon\}.$$

- $T = A_1 + A_2 + A_3, A_i \in L(X, Y)$ .  $\varepsilon > 0$ . Then

$$\text{Fac}_1^{3\varepsilon}(T) \leq 3 \max_{1 \leq i \leq 3} \text{Fac}_1^\varepsilon(A_i).$$

If  $\|A_i - S_i R_i\| \leq \varepsilon$ , let  $R : X \rightarrow \ell^1 \oplus_1 \ell^1 \oplus_1 \ell^1$  be  
 $Rx = R_1 x \oplus R_2 x \oplus R_3 x$ , and  $S : \ell^1 \oplus_1 \ell^1 \oplus_1 \ell^1 \rightarrow Y$  be  
 $S(z_1 \oplus z_2 \oplus z_3) = S_1 z_1 + S_2 z_2 + S_3 z_3$ .

$$E = (\oplus E_n)_{\ell^1}. T \in L(E). T_n = TP_n.$$

- ① If  $\sup_n \text{Fac}_1^\varepsilon(T_n) < \infty$ , then  $d(T, \mathcal{G}_{\ell^1}(E)) \leq \varepsilon$ .
- ② If there exists an embedding  $J : Y \rightarrow \ell^\infty$  such that  
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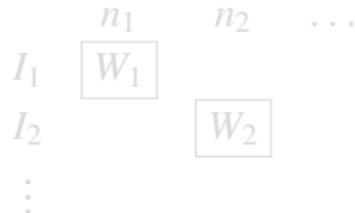
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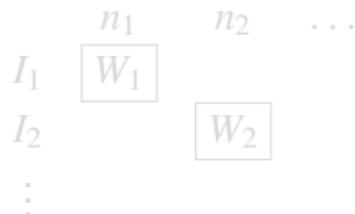
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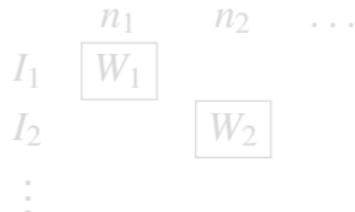
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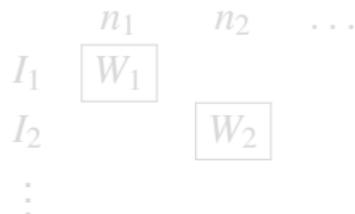
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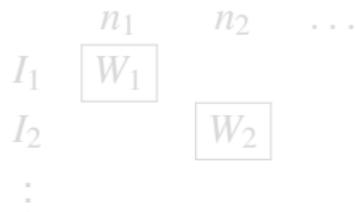
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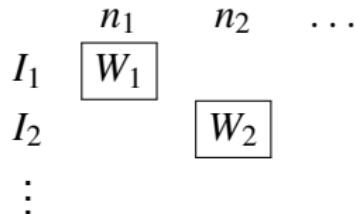
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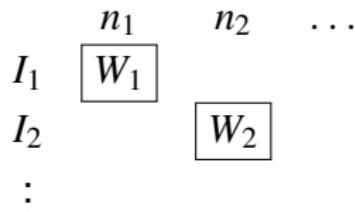
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$\|i_{mn}\| = \|i_{n-1} \cdots i_m : E_m \rightarrow E_n\| \leq C$ .

$\|q_{mn}\| = \|q_m \cdots q_{n-1} : E_n \rightarrow E_m\| \leq C$ .

$$T = \begin{pmatrix} W_1 + V_1 \\ W_2 + V_2 \\ \vdots \end{pmatrix}, W_n : E_{I_n} \rightarrow E_{I_n}, V_n : (\oplus_{m > I_n} E_m)_{\ell^1} \rightarrow E_{I_n}.$$

$$1_{E_n} = R_n W_n S_n, \|R_n : E_{I_n} \rightarrow E_n\| \leq 1, \|S_n : E_n \rightarrow E_{I_n}\| \leq C.$$

Let  $V_{m,n,j} = V_j S_n i_{mn} : E_m \rightarrow E_{I_j}, j, m < n.$

Up to small perturbation, we may assume  $V_{m,n,j} = V_{m,n',j}$  if  $j, m < n, n'.$

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Consider the matrix equation

$$(0 \quad R_2 \quad 0 \quad R_4 \quad \dots) \begin{pmatrix} 0 \\ W_2 + V_2 \\ 0 \\ W_4 + V_4 \\ 0 \\ W_6 + V_6 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ S_2 i_{12} x_1 \\ -S_3 i_{23} x_2 \\ S_4 i_{24} x_2 \\ -S_5 i_{35} x_3 \\ S_6 i_{36} x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ R_2 W_2 S_2 i_{12} x_1 \\ 0 \\ R_4 W_4 S_4 i_{24} x_2 \\ \vdots \end{pmatrix}$$

$$= 0 \oplus i_{12} x_1 \oplus 0 \oplus i_{24} x_2 \oplus \dots$$

$i_{12} E_1 \xrightarrow{c} E_2, i_{24} E_2 \xrightarrow{c} E_4, \dots$  with uniform constants.

It follows that  $1_{(\bigoplus E_n)_{\ell^1}} \prec$  submatrix of  $T \prec T$ .

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*Thank you*