Strictly singular non-compact operators in asymptotic ℓ_p spaces

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1. D.Kutzarova, A.Manoussakis, A.Pelczar-Barwacz, *Isomorphisms and strictly singular operators in mixed Tsirelson spaces*, J. Math. Anal. Appl. 388 (2012) 1040-1060.
2. A.Pelczar-Barwacz, *Strictly singular operators in asymptotic ℓ_p spaces*, submitted.

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◦ Th.Schlumprecht: Class 1 and Class 2.

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- Y is minimal (ℓ_p, c_0, Schlumprecht space, dual to Tsirelson space).

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In general: let X be a mixed Tsirelson space $T[(\mathcal{A}_n, \frac{c_n}{\sqrt[q]{n}})_n]$, with some $q > 1$ and $(c_n) \subset (0,1)$.

- If $c_n \to 0$, then X is of Class 2 (Schlumprecht space type).
- If $\inf c_n > 0$, then X is saturated with asymptotic ℓ_p spaces of Class 1, with $\frac{1}{p} + \frac{1}{q} = 1$ (Tzafriri space type).

Schreier families $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ (D.Alspach, S.Argyros)

$$\mathcal{S}_0 = \{\{n\} : \ n \in \mathbb{N}\} \cup \{\emptyset\}$$

$$\mathcal{S}_{\alpha+1} = \{\cup_{i=1}^{k} F_i : \ k \leq F_1 < \cdots < F_k, \ F_i \in \mathcal{S}_\alpha, \ k \in \mathbb{N}\}, \quad \alpha < \omega_1$$

Given a limit $\alpha < \omega_1$ pick $\alpha_n \nearrow \alpha$ and let

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A sequence $(x_1, \ldots, x_k)$ in X is called $\mathcal{S}_\alpha$-*admissible*, if it is a block sequence with $(\min \operatorname{supp} x_1, \ldots, \min \operatorname{supp} x_k) \in \mathcal{S}_\alpha$.

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An $(\mathcal{S}_\alpha, \theta)$-*operation*, $0 < \theta < 1$, is an operation which associates with any vectors $f_1, \ldots, f_k \in c_{00}$ satisfying $(\min \operatorname{supp} f_1, \ldots, \min \operatorname{supp} f_k) \in \mathcal{S}_\alpha$ the vector
$$f = \theta(f_1 + \cdots + f_k) \in c_{00}.$$

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Mixed Tsirelson space and its modified version

Let $D \subset c_{00}$ be the smallest set containing $\{\pm e_i^*\}_i$ and closed under $(\mathcal{S}_{k_n}, \theta_n)$-operations on block sequences (resp. sequences of vectors with pairwise disjoint supports), for all $n \in \mathbb{N}$.

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Taking $k_n = k \in \mathbb{N}$ and $\theta_n = \theta \in (0,1)$ for any $n \in \mathbb{N}$ we obtain a classical Tsirelson type space $T[\mathcal{S}_k, \theta]$ or its modified version.

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Each (modified) mixed Tsirelson space is isometric to a "regular" space $T[(\mathcal{S}_n, \theta_n)_n]$ (resp. $T_M[(\mathcal{S}_n, \theta_n)_n]$) with $\theta_n^{1/n} \to \theta \in (0,1]$.

Let X be an asymptotic ℓ_p space with a basis, $p \in [1, \infty)$, i.e. for any $\mathcal{S}_1$-admissible sequence $(x_1, \dots, x_k)$ and universal $C \geq 1$ we have

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For any $n \in \mathbb{N}$ define a *lower asymptotic constant* θ_n as the biggest constant such that any $\mathcal{S}_n$-admissible sequence $(x_1, \ldots, x_k)$ satisfies

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Let $\theta = \lim_n \theta_n^{1/n}$. $T^{(p)}[\mathcal{S}_1, \theta]$ denotes the p-convexification of $T[\mathcal{S}_1, \theta]$. Note, that $T^{(p)}[\mathcal{S}_1, \theta] = \ell_p$ for $\theta = 1$.

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Theorem [KMP, P]

Any block subspace of X contains for any $M \in \mathbb{N}$ an infinite normalized block sequence (x_i) satisfying

$$3\|\sum_{i\in G} a_i x_i\| \geq \|\sum_{i\in G} a_i e_{\min \operatorname{supp} x_i}\|_{T^{(p)}[\mathcal{S}_1,\theta]}$$

for any $G \in \mathcal{S}_M$ and scalars $(a_i)_{i\in G}$.

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Let Z be a modified mixed Tsirelson space $T_M[(\mathcal{S}_n, \theta_n)_n]$ with $(\frac{\theta_n}{\theta^n}) \searrow$.

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- Z is sequentially minimal,
- if moreover $\frac{\theta_n}{\theta^n} \longrightarrow 0$, then Z is arbitrarily distortable.

Construction of non-compact strictly singular operators on subspaces based on different types of asymptotic behaviour of basic sequences with respect to an auxiliary basic sequence:
◦ G.Androulakis, E.Odell, Th.Schlumprecht, N.Tomczak-Jaegermann, G.Androulakis, F.Sanacory: construction based on Krivine theorem and spreading models,
◦ Th.Schlumprecht: construction based on asymptotic behaviour of higher order of basic sequences with respect to the u.v.b of ℓ_1.

Theorem [P]

Let X be a Banach space with an $\mathcal{S}_\alpha$-unconditional basis, for limit $\alpha < \omega_1$.
Let E be a Banach space with an unconditional basis (e_i) dominated by all its subsequences, not containing uniformly c_0^n's (on vectors with disjoint supports).

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Assume that for some $\alpha_n \nearrow \alpha$ the following is satisfied

1. X contains a normalized block sequence (x_i) satisfying

$$\|\sum_i a_i x_i\| \leq \max_{n\in\mathbb{N}} \delta_n \max_{n\leq F\in\mathcal{S}_{\alpha_n}} \|\sum_{i\in F} a_i e_{\min \operatorname{supp} x_i}\|_E$$

for any scalars (a_i) and universal $\delta_n \to 0$.

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   for any scalars (a_i) and universal $\delta_n \to 0$.

2. for any $n \in \mathbb{N}$ there is a normalized block sequence $(x_i^n)_i$ such that $(x_i^n)_{i\in F}$ C-dominates $(e_i)_{i\in F}$ for any $F \in \mathcal{S}_{\alpha_n}$ and universal $C \geq 1$

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Then X admits a bounded strictly singular non-compact operator on a subspace.

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Assume that X contains a normalized block sequence (x_i) satisfying

$$\|\sum_i a_i x_i\| \leq \max_{n \in \mathbb{N}} \delta_n \max_{n \leq F \in \mathcal{S}_n} \|\sum_{i \in F} a_i e_{\min \operatorname{supp} x_i}\|_{T^{(p)}[\mathcal{S}_1, \theta]}$$

for any scalars (a_i) and some universal sequence $\delta_n \to 0$.

Corollary [P]

Let X be an asymptotic ℓ_p space, $p \in [1, \infty)$, with lower asymptotic constants (θ_n) and an $\mathcal{S}_\omega$-unconditional basis.
Assume that X contains a normalized block sequence (x_i) satisfying

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for any scalars (a_i) and some universal sequence $\delta_n \to 0$.
Then X admits a bounded strictly singular non-compact operator on a subspace.

Corollary [P]

Let $Z = T^{(p)}[(\mathcal{S}_n, \theta_n)_n]$ be a p-convexified mixed Tsirelson space with $\frac{\theta_n}{\theta^n} \to 0$, $p \in [1, \infty)$.

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Let X be a Banach space with a basis (g_i) $\mathcal{S}_\omega$-equivalent to the u.v.b. of Z, i.e. $(g_i)_{i\in F}$ and $(e_i)_{i\in F}$ are C-equivalent for any $F \in \mathcal{S}_\omega$ and universal $C \geq 1$.

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Applications include spaces built on the basis of spaces $T[(\mathcal{S}_n, \theta_n)_n]$ and their convexifications including HI spaces (G.Androulakis-K.Beanland, I.Deliyanni-A.Manoussakis). The construction relies on the unconditional components building the considered space.

Thank you