Maximal ideals in the algebra of bounded linear operators on Orlicz sequence spaces

Bentuo Zheng
University of Memphis, TN

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This is a joint work with Peikee Lin and Bunyamin Sari.
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A classical result of Calkin says that the only nontrivial proper closed ideal in the algebra $L(\ell_2)$ of bounded linear operators on a separable Hilbert space is the ideal of compact operators. The same was shown to be true for $\ell_p$ $(1 \leq p < \infty)$ and $c_0$ by Gohberg, Markus and Feldman. Apart from these, there are only few Banach spaces for which the closed ideals in the algebra of bounded linear operators are completely determined.
Theorem (Argyros and Haydon, 2011)

There is a H.I. space $X$ on which every bounded linear operator is a scalar multiple of the identity plus a compact operator. As a consequence, the only nontrivial norm closed ideal in $L(X)$ is the space of compact operators.
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Theorem (Laustsen, Loy and Read, 2004)
Let $X = (\oplus \ell_2^n)c_0$. Then there are exactly two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through $c_0$. 
Theorem (Laustsen, Schlumprecht and Zsak, 2006)
Let $X = (\oplus \ell^r_2)_{\ell_1}$. Then there are precisely two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through $\ell_1$. 
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Theorem (Gramsch, Luft and Daws)
Let $I$ be an infinite set, and let $X = \ell_p(I)$ for $1 \leq p < \infty$, or $X = c_0(I)$. If $J$ is a closed ideal in $L(X)$, then $J = K_\alpha(X)$ for some cardinal $\alpha$. 
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An operator $T \in L(X)$ is in $K_\alpha(X)$ if for every $\epsilon > 0$, there is a subset $E$ of $B_X$ with $|E| < \alpha$ so that for all $x \in B_X$

$$\inf\{\|Tx - Ty\| : y \in E\} < \epsilon.$$
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Theorem (Tarbard)
For each natural number $n$, there is a Banach space $X$ so that $L(X)$ contains exactly $n$ nontrivial closed ideals generated by the powers of a single nilpotent, strictly singular, non-compact operator.
Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Let $X$ be a Banach space. We denote by $M_X$ the set of all bounded linear operators $T$ on $X$ so that the identity operator on $X$ does not factor through $T$. There are quite a number of spaces for which $M_X$ is the unique maximal ideal. The following is a list of spaces recently found to be in the family.

1. $L^p(1 \leq p < \infty)$ (Dosev, Johnson and Schechtman, 2011)
2. $C([0,\omega_1])$ (Kania and Laustsen, 2011)
3. $(\sum \ell_q \ell_p)$ ($1 \leq q < p < \infty$) (Chen, Johnson and Zheng, 2011)
4. $d_{w,p}, w_1 = 1, w_n \to 0, \sum_{i=1}^{\infty} w_i = \infty$ (Kaminska, Popov, Spinu, Tcaciuc and Troisky, 2011)
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Recall that an *Orlicz function* $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing and convex function such that $M(0) = 0$ and $\lim_{t \to \infty} M(t) = \infty$. 

To any Orlicz function $M$ we associate a sequence space $\ell^M$ of all sequences of scalars $x = (a_1, a_2, ...)$ such that

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\sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) < \infty
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for some $\rho > 0$. The space $\ell^M$ equipped with the norm

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\|x\| = \inf\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) \leq 1\}
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is a Banach space usually called an Orlicz sequence space.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition if

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3. $\ell_M$ contains no copy of $\ell_\infty$.

Recall that for $\Lambda > 0$, $C_{M,\Lambda}$ is the norm-closed convex hull in $[0,1]$ of the set $E_{M,\Lambda} = \{M(\lambda t) ; 0 < \lambda < \Lambda\}$. We let $E_M = \bigcap_{\Lambda > 0} E_{M,\Lambda}$ and $C_M = \bigcap_{\Lambda > 0} C_{M,\Lambda}$.

If $M$ satisfies the $\Delta_2$-condition then an Orlicz sequence space $\ell_N$ is isomorphic to a subspace of $\ell_M$ if and only if $N$ is equivalent to some function in $C_M$.1
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If $M$ satisfies the $\Delta_2$-condition then an Orlicz sequence space $\ell_N$ is isomorphic to a subspace of $\ell_M$ if and only if $N$ is equivalent to some function in $C_{M,1}$. 
Let $1 < p < \infty$ and let $M$ be an Orlicz function. We say that the Orlicz sequence space $\ell_M$ is close to $\ell_p$ if the following conditions hold:

1. The unit vector basis of $\ell_M$ and the unit vector basis of $\ell_p$ are the only, up to equivalence, symmetric basic sequences in $\ell_M$;
2. Normalized block basic sequences of $\ell_M$ uniformly dominate the unit vector basis of $\ell_p$; i.e. there is a $C > 0$ so that every normalized block bases $(x_i)$ of $\ell_M$ satisfies for all $(a_i) \subset \mathbb{R}$

$$\| \sum a_i x_i \| \geq C \left( \sum |a_i|^p \right)^{1/p}.$$
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$$\| \sum a_i x_i \| \geq C \left( \sum |a_i|^p \right)^{1/p}.$$ 

$M$ is said to be $p$-regular if $\lim_{\lambda \to 0} \frac{M(\lambda t)}{M(\lambda)} = t^p$, $0 < t \leq 1$. 

Remark. It immediate from the definition that if $\ell_M$ is close to $\ell_p$, then $M$ satisfies the $\Delta_2$-condition and $\ell_M$ is reflexive. Actually since $\ell_M$ does not contain $c_0$, $M$ satisfies the $\Delta_2$-condition. Since $\ell_M$ does not contain $\ell_1$, $\ell_M$ is reflexive. Moreover, if $M$ is $p$-regular, then $E_M = C_M = \{t^p\}$.
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**Theorem (Lin, Sari and Zheng, 2012)**

Let $1 < p < \infty$. Let $\ell_M$ be an Orlicz space close to $\ell_p$ and $M$ be $p$-regular. Then $M_{\ell_M}$ is the unique maximal ideal in $L(\ell_M)$. 
Lemma (1)

Let $1 < p < \infty$ and $\ell_M$ be an Orlicz sequence space close to $\ell_p$. Let $(u_j)$ be a normalized block basis in $\ell_M$. Then $(u_i)$ is $K$-dominated by the unit vector basis of $\ell_M$ for some constant $K$ independent of $(u_i)$ and there exists a subsequence of $(u_j)$ which is either equivalent to the unit vector basis of $\ell_M$ or to the unit vector basis of $\ell_p$. If, in addition, $M$ is regular, and
\[ \lim_{j} \|u_j\|_\infty = 0, \]
then a subsequence of $(u_j)$ is equivalent to the unit vector basis of $\ell_p$. 
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Lemma (2)

Let $1 < p < \infty$. Suppose that $\ell_M$ is an Orlicz sequence space close to $\ell_p$, and $M$ is $p$-regular. Let $(u_i)$ be a normalized block basis of $\ell_M$ which is either equivalent to the unit vector basis of $\ell_p$ or the unit vector basis of $\ell_M$. Then $[(u_i)]$ is complemented in $\ell_M$. 
Lemma (3)

Let $1 < p < \infty$. Let $\ell_M$ be an Orlicz space close to $\ell_p$ and $M$ be $p$-regular. Then $M_{\ell_M}$ is an ideal in $L(\ell_M)$.
Lemma (3)

Let $1 < p < \infty$. Let $\ell_M$ be an Orlicz space close to $\ell_p$ and $M$ be $p$-regular. Then $M_{\ell_M}$ is an ideal in $L(\ell_M)$.

Proof. An operator $T$ is in $M_{\ell_M}$ if and only if $T$ does not preserve a copy of $\ell_M$ (i.e. $T$ is $\ell_M$-strictly singular).

Let $S$ and $T$ be two $\ell_M$-strictly singular operators. Suppose that $S + T$ is not $\ell_M$-strictly singular. Then we can find a normalized sequence $(x_i)$ in $\ell_M$ so that both $(x_i)$ and $(Sx_i + Tx_i)$ are equivalent to the unit vector basis of $\ell_M$. By passing to a subsequence of $(x_i)$ and perturbing, without loss of generality we assume that both $(x_i)$ and $(Sx_i + Tx_i)$ are block bases in $\ell_M$.

By Lemma (1), there exists a $\delta > 0$ so that $\|Sx_i + Tx_i\|_{\infty} > \delta$.

By passing to a further subsequence $(y_i)$ of $(x_i)$, we get either $\|Sy_i\|_{\infty} > \delta/2$ for all $i \in \mathbb{N}$ or $\|Ty_i\|_{\infty} > \delta/2$ for all $i \in \mathbb{N}$. But this implies that either $(Sy_i)$ or $(Ty_i)$ is equivalent to the unit vector basis of $\ell_M$ since the unit vector basis of $\ell_M$ dominates every block basis of $\ell_M$ (by Lemma (1) again). Hence either $S$ or $T$ is preserve a copy of $\ell_M$ which contradicts our hypothesis.
Theorem (Dosev and Johnson, 2010)

Let $X$ be a Banach space. If $M_X$ is an ideal, then it is automatically maximal.

Theorem (Lin, Sari and Zheng, 2012)

Let $1 < p < \infty$. Let $\ell M$ be an Orlicz space close to $\ell p$ and $M$ be $p$-regular. Then $M \ell M$ is the unique maximal ideal in $L(\ell M)$. 
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Let $1 < p < \infty$. Let $\ell_M$ be an Orlicz space close to $\ell_p$ and $M$ be $p$-regular. Then $M_{\ell_M}$ is the unique maximal ideal in $L(\ell_M)$. 
Let $\ell_M$ be an Orlicz space. We use $\Gamma^{\ell_p}$ to denote the ideal of all operators in $L(\ell_M)$ which factor through $\ell_p$. Let $\bar{\Gamma}^{\ell_p}$ be the closure of $\Gamma^{\ell_p}$.
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**Theorem**

Let $1 < p \leq 2$ and $M$ be $p$-regular. Let $\ell_M$ be an Orlicz sequence space close to $\ell_p$ but not isomorphic to $\ell_p$. Then $\bar{\Gamma}^{\ell_p}$ is a proper subset of $M_{\ell_M}$. 
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**Lemma (4)**

Let $1 \leq p < \infty$ and let $X$ be a complemented subspace of an Orlicz space $\ell_M$ and let $P$ be a projection from $\ell_M$ onto $X$. If $P$ is in $\bar{\Gamma}^{\ell_p}$, then $X$ is isomorphic to $\ell_p$. 
Lemma (5)

Let $1 < p \leq 2$ and $M$ be $p$-regular. Let $\ell_M$ be an Orlicz sequence space close to $\ell_p$ but not isomorphic to $\ell_p$. Then there exists an infinite dimensional complemented subspace $X$ of $\ell_M$ which is not isomorphic to $\ell_p$ so that $\ell_M$ does not embed into $X$. 

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Let $1 < p \leq 2$ and $M$ be $p$-regular. Let $\ell_M$ be an Orlicz sequence space close to $\ell_p$ but not isomorphic to $\ell_p$. Then $\bar{\Gamma}_{\ell_p}$ is a proper subset of the $\ell_M$-strictly singular operators.

Proof. Let $X$ be the complemented subspace of $\ell_M$ as in Lemma (5). Let $P$ be a projection from $\ell_M$ onto $X$. Since $X$ is not isomorphic to $\ell_p$, by Lemma (4), $P$ is not in $\bar{\Gamma}_{\ell_p}$. Since $\ell_M$ does not embed into $X$, $P$ is $\ell_M$ strictly singular.
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Proof. Let $X$ be the complemented subspace of $\ell_M$ as in Lemma (5). Let $P$ be a projection from $\ell_M$ onto $X$. Since $X$ is not isomorphic to $\ell_p$, by Lemma (4), $P$ is not in $\bar{\Gamma}^{\ell_p}$. Since $\ell_M$ does not embed into $X$, $P$ is $\ell_M$ strictly singular.
Let $I_{\ell_M \to \ell_p}$ be the formal identity from $\ell_M$ into $\ell_p$. By composing with an isomorphic embedding of $\ell_p$ into $\ell_M$, it is considered as an operator on $\ell_M$. It is easy to prove that if $\ell_M$ is close to $\ell_p$, then the closed ideal $\Gamma_{I_{\ell_M \to \ell_p}}$ generated by $I_{\ell_M \to \ell_p}$ is an immediate successor of the compacts.
Let \( I_{\ell M \to \ell p} \) be the formal identity from \( \ell M \) into \( \ell p \). By composing with an isomorphic embedding of \( \ell p \) into \( \ell M \), it is considered as an operator on \( \ell M \). It is easy to prove that if \( \ell M \) is close to \( \ell p \), then the closed ideal \( \bar{\Gamma}^{I_{\ell M \to \ell p}} \) generated by \( I_{\ell M \to \ell p} \) is an immediate successor of the compacts.

**Theorem**

*Let \( \ell M \) be an Orlicz space close to \( \ell p \). If \( T \in L(\ell_M) \) is not compact, then \( \bar{\Gamma}^{I_{\ell M \to \ell p}} \) is a subspace of \( \bar{\Gamma}^T \).*
Let \( I_{\ell_M \to \ell_p} \) be the formal identity from \( \ell_M \) into \( \ell_p \). By composing with an isomorphic embedding of \( \ell_p \) into \( \ell_M \), it is considered as an operator on \( \ell_M \). It is easy to prove that if \( \ell_M \) is close to \( \ell_p \), then the closed ideal \( \tilde{\Gamma}^{I_{\ell_M \to \ell_p}} \) generated by \( I_{\ell_M \to \ell_p} \) is an immediate successor of the compacts.

**Theorem**

Let \( \ell_M \) be an Orlicz space close to \( \ell_p \). If \( T \in L(\ell_M) \) is not compact, then \( \tilde{\Gamma}^{I_{\ell_M \to \ell_p}} \) is a subspace of \( \tilde{\Gamma}^T \).

**Question:** Is \( \tilde{\Gamma}^{I_{\ell_M \to \ell_p}} \) a proper subspace of \( \tilde{\Gamma}^{\ell_p} \)?