Spin(9), complex structures and vector fields on spheres

Maurizio Parton[1]   Paolo Piccinni[2]   Victor Vuletescu[3]

[1]Università di Chieti-Pescara, Italy

[2]Università di Roma "La Sapienza", Italy

[3]University of Bucharest, Romania

Manifolds with special holonomy and their calibrated submanifolds and connections

Banff, Tuesday, 2012, May 1st

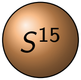
MP, Paolo Piccinni.
$\mathrm{Spin}(9)$ and almost complex structures on 16-dimensional manifolds.
Ann. Global An. Geom., 41 (2012), 321-345.

MP, Paolo Piccinni.
Spheres with more than 7 vector fields: all the fault of $\mathrm{Spin}(9)$.
arXiv: 1107.0462v2.

MP, Paolo Piccinni, Victor Vuletescu.
16-dimensional manifolds with a locally conformal parallel $\mathrm{Spin}(9)$ metric.
Work in progress.

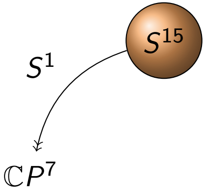
First characterization: Hopf fibrations

S^{15} is the only sphere involved in three different Hopf fibrations.



First characterization: Hopf fibrations

S^{15} is the only sphere involved in three different Hopf fibrations.



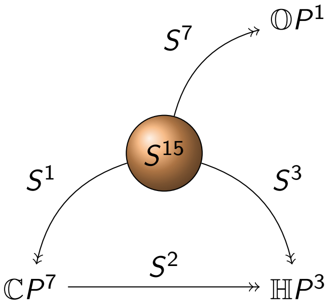
First characterization: Hopf fibrations

S^{15} is the only sphere involved in three different Hopf fibrations.

$$
\begin{tikzcd}
 & S^{15} \arrow[dl, bend right, two heads, "S^1"'] \arrow[dr, bend left, two heads, "S^3"] & \\
\mathbb{C}P^7 \arrow[rr, two heads, "S^2"] & & \mathbb{H}P^3
\end{tikzcd}
$$

First characterization: Hopf fibrations

S^{15} is the only sphere involved in three different Hopf fibrations.



Remark

The complex and quaternionic Hopf fibrations are not subfibrations of the octonionic one [Loo-Verjovsky, Topology 1992].

Second characterization: Einstein metrics

S^{15} is the only sphere with three homogeneous Einstein metrics

[Ziller, Math. Ann. 1982].

- Round metric.
- Einstein metric on $\mathrm{Sp}(4)/\mathrm{Sp}(3)$ [Jensen, J. Diff. Geom. 1973].
- Einstein metric on $\mathrm{Spin}(9)/\mathrm{Spin}(7)$

  [Bourguignon-Karcher, Ann. Sci. Ec. Norm. Sup. 1978].

Third characterization: vector fields on spheres

S^{15} is the lowest dimensional sphere admitting more than 7 vector fields

[Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].

Third characterization: vector fields on spheres

S^{15} is the lowest dimensional sphere admitting more than 7 vector fields

[Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].

- Number $\sigma(m)$ of linearly independent vector fields on S^{m-1}?
- If $m = (2k+1)2^p 16^q$, with $0 \leq p \leq 3$, then

$$\sigma(m) = 8q + 2^p - 1$$

Third characterization: vector fields on spheres

S^{15} is the lowest dimensional sphere admitting more than 7 vector fields

[Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].

- Number $\sigma(m)$ of linearly independent vector fields on S^{m-1}?
- If $m = (2k+1)2^p 16^q$, with $0 \le p \le 3$, then

$$\sigma(m) = \quad 8q \quad + \boxed{2^p - 1}$$

$\mathbb{C}, \mathbb{H}, \mathbb{O}$ contribution

Third characterization: vector fields on spheres

S^{15} is the lowest dimensional sphere admitting more than 7 vector fields

[Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].

- Number $\sigma(m)$ of linearly independent vector fields on S^{m-1}?
- If $m = (2k+1)2^p 16^q$, with $0 \le p \le 3$, then

$$\sigma(m) = \underbrace{8q}_{\text{Spin}(9)\text{ contribution}} + \underbrace{2^p - 1}_{\mathbb{C},\mathbb{H},\mathbb{O}\text{ contribution}}$$

Berger's list and $\mathrm{Spin}(9)$ refutation

Holonomy of simply connected, irreducible, nonsymmetric?

Berger's list and $\mathrm{Spin}(9)$ refutation

Holonomy of simply connected, irreducible, nonsymmetric?

$\boxed{\mathrm{U}(n)}$ $\boxed{\mathrm{SO}(n)}$

$\boxed{\mathrm{SU}(n)}$

$\boxed{\mathrm{Spin}(9)}$ $\boxed{\mathrm{Spin}(7)}$

$\boxed{\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)}$

$\boxed{\mathrm{G}_2}$

$\boxed{\mathrm{Sp}(n)}$

Berger's list and $\mathrm{Spin}(9)$ refutation

Holonomy of simply connected, irreducible, nonsymmetric?

$\mathrm{U}(n)$

$\mathrm{SO}(n)$

$\mathrm{SU}(n)$

$\mathrm{Spin}(9)$

$\mathrm{Spin}(7)$

$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$

G_2

$\mathrm{Sp}(n)$

Simply connected, complete, holonomy $\mathrm{Spin}(9)$

$$\Leftrightarrow$$

$$\mathbb{O}P^2 = \frac{\mathrm{F}_4}{\mathrm{Spin}(9)}(s>0), \quad \mathbb{R}^{16}(\text{flat}), \quad \mathbb{O}H^2 = \frac{\mathrm{F}_{4(-20)}}{\mathrm{Spin}(9)}(s<0)$$

[Alekseevsky, Funct. Anal. Prilozhen 1968].

Berger's list and $\mathrm{Spin}(9)$ refutation

Holonomy of simply connected, irreducible, nonsymmetric?

$\mathrm{U}(n)$

$\mathrm{SO}(n)$

$\mathrm{SU}(n)$

$\cancel{\mathrm{Spin}(9)}$

$\mathrm{Spin}(7)$

$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$

G_2

$\mathrm{Sp}(n)$

Simply connected, complete, holonomy $\mathrm{Spin}(9)$

$$\Leftrightarrow$$

$$\mathbb{O}P^2 = \frac{\mathrm{F}_4}{\mathrm{Spin}(9)} (s > 0), \quad \mathbb{R}^{16} (\text{flat}), \quad \mathbb{O}H^2 = \frac{\mathrm{F}_{4(-20)}}{\mathrm{Spin}(9)} (s < 0)$$

[Alekseevsky, Funct. Anal. Prilozhen 1968].

What is $\mathrm{Spin}(9)$?

Definition

$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

What is $\mathrm{Spin}(9)$?

Definition

$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

- $\Lambda^8(\mathbb{R}^{16}) \overset{\mathrm{Spin}(9)}{=} \Lambda^8_1 + \dots$ [Friedrich, Asian Journ. Math 2001].
- $\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of any element of Λ^8_1 [Brown-Gray, Diff. Geom. in honor of K. Yano 1972].

What is Spin(9)?

Definition

$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

- $\Lambda^8(\mathbb{R}^{16}) \overset{\mathrm{Spin}(9)}{=} \Lambda^8_1 + \dots$ [Friedrich, Asian Journ. Math 2001].
- $\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of any element of Λ^8_1

  [Brown-Gray, Diff. Geom. in honor of K. Yano 1972].

Definition

$\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of the 8-form

$$\Phi_{\mathrm{Spin}(9)} \overset{\mathrm{utc}}{=} \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

▸ Details

[Berger, Ann. Éc. Norm. Sup. 1972].

What is Spin(9)?

Definition

$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

- $\Lambda^8(\mathbb{R}^{16}) \overset{\mathrm{Spin}(9)}{=} \Lambda^8_1 + \dots$ [Friedrich, Asian Journ. Math 2001].
- $\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of any element of Λ^8_1

[Brown-Gray, Diff. Geom. in honor of K. Yano 1972].

Definition

$\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of the 8-form

$$\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

Details

[Berger, Ann. Éc. Norm. Sup. 1972].

Up to constants

Time check

Are we left with 32 or more minutes?

Yes, go ahead as planned   No, skip quaternionic analogy

A close relative: the quaternionic case

- $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(8)$ is the group of symmetries of the Hopf fibration $\mathbb{H}^2 \supset S^7 \stackrel{S^3}{\to} S^4 \cong \mathbb{H}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].
- $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ is the stabilizer in $\mathrm{SO}(8)$ of the 4-form $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$ defined by

$$\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)} = \int_{\mathbb{H}P^1} p_l^* \nu_l \, dl$$

[Berger, Ann. Éc. Norm. Sup. 1972].

Five involutions for $\mathrm{Spin}(5)$

- Consider in $\mathrm{Sp}(2)$ the matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r, u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$.

Five involutions for $\mathrm{Spin}(5)$

- Consider in $\mathrm{Sp}(2)$ the matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r, u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$.

- The choice of $(r, u) = (1, 0)$, $(0, 1)$, $(0, i)$, $(0, j)$, $(0, k)$ gives

$$\mathcal{I}_1, \dots, \mathcal{I}_5 \in \mathrm{SO}(8)$$

Details

Five involutions for $\mathrm{Spin}(5)$

- Consider in $\mathrm{Sp}(2)$ the matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r,u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$.

- The choice of $(r,u) = (1,0)$, $(0,1)$, $(0,i)$, $(0,j)$, $(0,k)$ gives

$$\mathcal{I}_1, \dots, \mathcal{I}_5 \in \mathrm{SO}(8)$$

Details

- $\mathcal{I}_1, \dots, \mathcal{I}_5$ satisfy

$$\mathcal{I}_\alpha^2 = \mathrm{Id}, \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

- The Kähler forms $\theta_{\alpha\beta}$ give rise to a 5×5 skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

- The Kähler forms $\theta_{\alpha\beta}$ give rise to a 5×5 skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$

Remark

Denote by $\tau_2(\theta)$ the second coefficient of the characteristic polynomial of $\theta = (\theta_{\alpha\beta})$. Then

$$\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)} \overset{\text{utc}}{=} \tau_2(\theta)$$

Nine involutions for $\mathrm{Spin}(9)$

- $\mathrm{Spin}(9)$ is the subgroup of $\mathrm{SO}(16)$ generated by matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r, u) \in S^8 \subset \mathbb{R} \times \mathbb{O}$ and $\mathbb{O}^2 \cong \mathbb{R}^{16}$

[Harvey, Spinors and Calibrations 1990].

Nine involutions for $\mathrm{Spin}(9)$

- $\mathrm{Spin}(9)$ is the subgroup of $\mathrm{SO}(16)$ generated by matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r, u) \in S^8 \subset \mathbb{R} \times \mathbb{O}$ and $\mathbb{O}^2 \cong \mathbb{R}^{16}$

[Harvey, Spinors and Calibrations 1990].

- The choice of $(r, u) = (1, 0)$, $(0, 1)$, $(0, i)$, $(0, j)$, $(0, k)$, $(0, e)$, $(0, f)$, $(0, g)$, $(0, h)$ gives

$$\mathcal{I}_1, \dots, \mathcal{I}_9 \in \mathrm{SO}(16)$$

Details

Nine involutions for Spin(9)

- $\mathrm{Spin}(9)$ is the subgroup of $\mathrm{SO}(16)$ generated by matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r, u) \in S^8 \subset \mathbb{R} \times \mathbb{O}$ and $\mathbb{O}^2 \cong \mathbb{R}^{16}$

[Harvey, Spinors and Calibrations 1990].

- The choice of $(r, u) = (1, 0)$, $(0, 1)$, $(0, i)$, $(0, j)$, $(0, k)$, $(0, e)$, $(0, f)$, $(0, g)$, $(0, h)$ gives

$$\mathcal{I}_1, \dots, \mathcal{I}_9 \in \mathrm{SO}(16)$$

Details

- $\mathcal{I}_1, \dots, \mathcal{I}_9$ satisfy

$$\mathcal{I}_\alpha^2 = \mathrm{Id}, \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

- Their Kähler forms $\theta_{\alpha\beta}$ give rise to a 9×9 skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

- Their Kähler forms $\theta_{\alpha\beta}$ give rise to a 9×9 skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$

Remark

$$\Lambda^2(\mathbb{R}^{16}) = \quad \Lambda^2_{36} \quad \oplus \quad \Lambda^2_{84} \quad = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

- Their Kähler forms $\theta_{\alpha\beta}$ give rise to a 9×9 skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$

Remark

$$\Lambda^2(\mathbb{R}^{16}) = \Lambda^2_{36} \oplus \Lambda^2_{84} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

generated by $\theta_{\alpha\beta}$

From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \qquad \text{for } \alpha < \beta$$

- Their Kähler forms $\theta_{\alpha\beta}$ give rise to a 9×9 skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$

Remark

$$\Lambda^2(\mathbb{R}^{16}) = \underbrace{\Lambda^2_{36}}_{\text{generated by } \theta_{\alpha\beta}} \oplus \underbrace{\Lambda^2_{84}}_{\text{generated by } \theta_{\alpha\beta\gamma}} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

From the Kähler forms to the $\mathrm{Spin}(9)$ form

Theorem [P-Piccinni, Ann. Gl. An. Geom. 2012]

Denote the characteristic polynomial of θ by

$$t^9 + \tau_2(\theta)t^7 + \tau_4(\theta)t^5 + \tau_6(\theta)t^3 + \tau_8(\theta)t$$

From the Kähler forms to the $\mathrm{Spin}(9)$ form

Theorem [P-Piccinni, Ann. Gl. An. Geom. 2012]

Denote the characteristic polynomial of θ by

$$t^9 + \tau_2(\theta)t^7 + \tau_4(\theta)t^5 + \tau_6(\theta)t^3 + \tau_8(\theta)t$$

Then

$$\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \tau_4(\theta)$$

An explicit formula for $\Phi_{\mathrm{Spin}(9)}$

- From $\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \tau_4(\theta)$, we obtain an ▸ explicit formula

An explicit formula for $\Phi_{\mathrm{Spin}(9)}$

- From $\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \tau_4(\theta)$, we obtain an explicit formula
- The $\binom{16}{8} = 12870$ integrals of

$$\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

can be computed with the help of *Mathematica*.

Show all 702 terms | Show only 70 terms | Computational challenge

An explicit formula for $\Phi_{\mathrm{Spin}(9)}$

- From $\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \tau_4(\theta)$, we obtain an explicit formula
- The $\binom{16}{8} = 12870$ integrals of

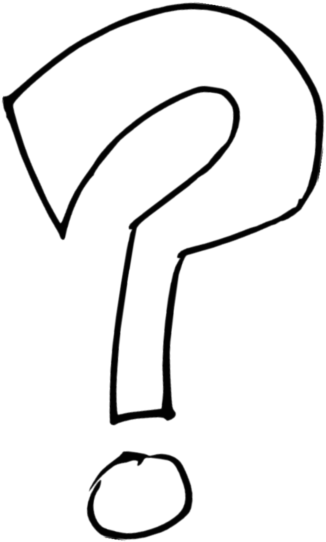
$$\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

can be computed with the help of *Mathematica*.

Show all 702 terms | Show only 70 terms | Computational challenge

Previous work for $\Phi_{\mathrm{Spin}(9)}$ in [Abe-Matsubara, Korea Japan Conf. Transf. Groups 1997], [Friedrich, Asian J. Math. 2001], [C. Lopez-Gadea-Mykytyuk, int. J. Geom. Methods 2010].

Questions to the audience



Questions to the audience

$\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$ and $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)} = \int_{\mathbb{H}P^1} p_l^* \nu_l \, dl$ share the following general pattern:

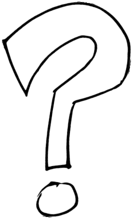
$$\Phi = \int_{\text{Gr(calibrated subspaces)}} p^* \nu_{\text{calibrated subspaces}}$$

Questions to the audience

$\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$ and $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)} = \int_{\mathbb{H}P^1} p_l^* \nu_l \, dl$ share the following general pattern:

$$\Phi = \int_{\mathsf{Gr(calibrated\ subspaces)}} p^* \nu_{\mathsf{calibrated\ subspaces}}$$

- $\Phi_{\mathrm{G}_2} \in \Lambda^3(\mathbb{R}^7)$ is a calibration, with associative subspaces as calibrated submanifolds. The Grassmannian in this case is $\mathrm{G}_2/\mathrm{SO}(4)$: is it true that
$$\Phi_{\mathrm{G}_2} = \int_{\frac{\mathrm{G}_2}{\mathrm{SO}(4)}} p_l^* \nu_l \, dl$$
- Same question for $\Phi_{\mathrm{Spin}(7)} \in \Lambda^4(\mathbb{R}^8)$: is it true that
$$\Phi_{\mathrm{Spin}(7)} = \int_{\mathsf{CAY}} p_l^* \nu_l \, dl$$



QttA/2

The forms $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$, Φ_{G_2}, $\Phi_{\mathrm{Spin}(7)}$ and $\Phi_{\mathrm{Spin}(9)}$ are finite sums of 14, 7, 14 and 702 terms respectively.

QttA/2

The forms $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$, Φ_{G_2}, $\Phi_{\mathrm{Spin}(7)}$ and $\Phi_{\mathrm{Spin}(9)}$ are finite sums of 14, 7, 14 and 702 terms respectively.

- Why these numbers?
- Are these numbers related to finite subgroups of $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$ and $\mathrm{Spin}(9)$ respectively?
- Why do Φ_{G_2} and $\Phi_{\mathrm{Spin}(7)}$ have coefficients ± 1, whereas $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$ and $\Phi_{\mathrm{Spin}(9)}$ do not?

QttA/2

The forms $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$, Φ_{G_2}, $\Phi_{\mathrm{Spin}(7)}$ and $\Phi_{\mathrm{Spin}(9)}$ are finite sums of 14, 7, 14 and 702 terms respectively.

- Why these numbers?
- Are these numbers related to finite subgroups of $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$ and $\mathrm{Spin}(9)$ respectively?
- Why do Φ_{G_2} and $\Phi_{\mathrm{Spin}(7)}$ have coefficients ± 1, whereas $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$ and $\Phi_{\mathrm{Spin}(9)}$ do not?

In the framework of Clifford structures [Moroianu-Semmelmann, Adv. Math. 2011], one can associate to any rank r even Clifford structure a skew-symmetric $r \times r$ matrix of Kähler forms.

QttA/2

The forms $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$, Φ_{G_2}, $\Phi_{\mathrm{Spin}(7)}$ and $\Phi_{\mathrm{Spin}(9)}$ are finite sums of 14, 7, 14 and 702 terms respectively.

- Why these numbers?
- Are these numbers related to finite subgroups of $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$ and $\mathrm{Spin}(9)$ respectively?
- Why do Φ_{G_2} and $\Phi_{\mathrm{Spin}(7)}$ have coefficients ± 1, whereas $\Phi_{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}$ and $\Phi_{\mathrm{Spin}(9)}$ do not?

In the framework of Clifford structures [Moroianu-Semmelmann, Adv. Math. 2011], one can associate to any rank r even Clifford structure a skew-symmetric $r \times r$ matrix of Kähler forms.

- Do the coefficients of the characteristic polynomial have any particular geometrical meaning?

How many vector fields on spheres?

- Spheres $S^{m-1} \subset \mathbb{R}^m$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1$, 2 or 3 in

$$m = (2k+1)2^p$$

How many vector fields on spheres?

- Spheres $S^{m-1} \subset \mathbb{R}^m$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1$, 2 or 3 in

$$m = (2k+1)2^p$$

- In the general case

$$m = (2k+1)2^p 16^q \qquad \text{with } q \geq 0 \quad \text{and} \quad p = 0, 1, 2, 3$$

the maximum number of vector fields is

$$\sigma(m) = \ 8q \ + \ 2^p - 1$$

How many vector fields on spheres?

- Spheres $S^{m-1} \subset \mathbb{R}^m$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1$, 2 or 3 in
$$m = (2k+1)2^p$$
- In the general case
$$m = (2k+1)2^p 16^q \qquad \text{with } q \geq 0 \quad \text{and} \quad p = 0, 1, 2, 3$$
the maximum number of vector fields is
$$\sigma(m) = 8q + \boxed{2^p - 1}$$

Spin(9) contribution

$\mathbb{C}, \mathbb{H}, \mathbb{O}$ contribution

How many vector fields on spheres?

- Spheres $S^{m-1} \subset \mathbb{R}^m$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1$, 2 or 3 in

$$m = (2k+1)2^p$$

- In the general case

$$m = (2k+1)2^p 16^q \qquad \text{with } q \geq 0 \quad \text{and} \quad p = 0, 1, 2, 3$$

the maximum number of vector fields is

$$\sigma(m) = \underbrace{8q}_{\mathrm{Spin}(9)\text{ contribution}} + \underbrace{2^p - 1}_{\mathbb{C}, \mathbb{H}, \mathbb{O}\text{ contribution}}$$

The lowest dimensional sphere with more than 7 vector field is S^{15}

[Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].

The lowest dimension: S^{15}

- Coordinates on S^{15}:

$$N = (x, y) = (x_1, \dots, x_8, y_1, \dots, y_8) \qquad \text{unit normal vector field}$$

The lowest dimension: S^{15}

- Coordinates on S^{15}:

$$N = (x, y) = (x_1, \dots, x_8, y_1, \dots, y_8) \qquad \text{unit normal vector field}$$

- Among the 36 complex structures $\mathcal{I}_\alpha \circ \mathcal{I}_\beta$ on $\mathbb{R}^{16}$ associated to the $\mathrm{Spin}(9)$ structure, choose $J_\alpha = \mathcal{I}_\alpha \circ \mathcal{I}_9$, for $\alpha = 1, \dots, 8$.

The lowest dimension: S^{15}

- Coordinates on S^{15}:

$$N = (x, y) = (x_1, \dots, x_8, y_1, \dots, y_8) \qquad \text{unit normal vector field}$$

- Among the 36 complex structures $\mathcal{I}_\alpha \circ \mathcal{I}_\beta$ on $\mathbb{R}^{16}$ associated to the $\mathrm{Spin}(9)$ structure, choose $J_\alpha = \mathcal{I}_\alpha \circ \mathcal{I}_9$, for $\alpha = 1, \dots, 8$.

Proposition

A maximal system of 8 orthonormal vector fields on S^{15} is given by

$$J_1 N, \dots, J_8 N$$

The lowest dimension: S^{15}

- Coordinates on S^{15}:

$$N = (x, y) = (x_1, \dots, x_8, y_1, \dots, y_8) \qquad \text{unit normal vector field}$$

- Among the 36 complex structures $\mathcal{I}_\alpha \circ \mathcal{I}_\beta$ on $\mathbb{R}^{16}$ associated to the $\mathrm{Spin}(9)$ structure, choose $J_\alpha = \mathcal{I}_\alpha \circ \mathcal{I}_9$, for $\alpha = 1, \dots, 8$.

Proposition

A maximal system of 8 orthonormal vector fields on S^{15} is given by

$$J_1 N, \dots, J_8 N$$

Remark

The eight complex structures $\{J_1, \dots, J_8\}$ play a role analogous to that of the units in $\mathbb{C}, \mathbb{H}, \mathbb{O}$.

Next spheres with $\sigma(m) > 7$: $S^{2^p 16-1}$, $p = 1, 2, 3$

Group coordinates in 16-ples s^α, and split each s^α as a pair (x^α, y^α) of 8-ples. Define a conjugation D by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.

Next spheres with $\sigma(m) > 7$: $S^{2^p 16-1}$, $p = 1, 2, 3$

Group coordinates in 16-ples s^α, and split each s^α as a pair (x^α, y^α) of 8-ples. Define a conjugation D by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.

Proposition

The following table gives a maximal system of $\sigma(m)$ orthonormal vector fields on $S^{2^p 16-1}$, for $p = 1, 2, 3$:

Sphere	$\sigma(m)$	Vector fields	Notations	Involved structures
$p = 1$: S^{31}	$8+1$	$J_1 N, \dots, J_8 N$ $D(L_i N)$	$N = s^1 + is^2, L_i N = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9)+\mathbb{C}$
$p = 2$: S^{63}	$8+3$	$J_1 N, \dots, J_8 N$ $D(L_i N), D(L_j N), D(L_k N)$	$N = s^1 + is^2 + js^3 + ks^4$ L_i, L_j, L_k and D as above	$\mathrm{Spin}(9)+\mathbb{H}$
$p = 3$: S^{127}	$8+7$	$J_1 N, \dots, J_8 N$ $D(L_i N), \dots, D(L_h N)$	$N = s^1 + is^2 + js^3 + ks^4 + es^5 + fs^6 + gs^7 + hs^8$ $L_i, \dots, L_h$ and D as above	$\mathrm{Spin}(9)+\mathbb{O}$

S^{255}: $\sigma(m) = 8 + 8$

- Again, group coordinates in 16-ples s^α, and split each s^α as a pair (x^α, y^α) of 8-ples. Define D by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.

S^{255}: $\sigma(m) = 8 + 8$

- Again, group coordinates in 16-ples s^α, and split each s^α as a pair (x^α, y^α) of 8-ples. Define D by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.
- Act on the (column) 16-ples of 16-ples $(s^1, \ldots, s^{16})^T$ by $J_1, \ldots, J_8$, and call $\mathrm{block}(J_1), \ldots, \mathrm{block}(J_8)$ the resulting automorphisms.

S^{255}: $\sigma(m) = 8 + 8$

- Again, group coordinates in 16-ples s^α, and split each s^α as a pair (x^α, y^α) of 8-ples. Define D by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.
- Act on the (column) 16-ples of 16-ples $(s^1, \dots, s^{16})^T$ by $J_1, \dots, J_8$, and call $\mathrm{block}(J_1), \dots, \mathrm{block}(J_8)$ the resulting automorphisms.

Proposition

A maximal system of orthonormal vector fields on S^{255} is given by:

Sphere	$\sigma(m)$	Vector fields	Notations	Involved structures
$p = 1$: S^{31}	$8+1$	$J_1N, \dots, J_8N$ $D(L_iN)$	$N = s^1 + is^2$, $L_iN = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9)$+$\mathbb{C}$
$p = 2$: S^{63}	$8+3$	$J_1N, \dots, J_8N$ $D(L_iN), D(L_jN), D(L_kN)$	$N = s^1 + is^2 + js^3 + ks^4$ L_i, L_j, L_k and D as above	$\mathrm{Spin}(9)$+$\mathbb{H}$
$p = 3$: S^{127}	$8+7$	$J_1N, \dots, J_8N$ $D(L_iN), \dots, D(L_hN)$	$N = s^1+is^2+js^3+ks^4+es^5+fs^6+gs^7+hs^8$ $L_i, \dots, L_h$ and D as above	$\mathrm{Spin}(9)$+$\mathbb{O}$
S^{255}	$8+8$	$J_1N, \dots, J_8N$ $D(\mathrm{block}(J_1)N), \dots, D(\mathrm{block}(J_8)N)$	$N = (s^1, \dots, s^{16})$ $\mathrm{block}(J_1), \dots, \mathrm{block}(J_8)$ and D as above	$\mathrm{Spin}(9)$+$\mathrm{Spin}(9)$

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.
- Imitating the $\mathbb{R}^{32}$ case, group coordinates in 256-ples (s^1, s^2), and define $L_i(s^1, s^2) = (-s^2, s^1)$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.
- Imitating the $\mathbb{R}^{32}$ case, group coordinates in 256-ples (s^1, s^2), and define $L_i(s^1, s^2) = (-s^2, s^1)$.

Proposition

The vector field $D(L_i N)$ is orthogonal to $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.
- Imitating the $\mathbb{R}^{32}$ case, group coordinates in 256-ples (s^1, s^2), and define $L_i(s^1, s^2) = (-s^2, s^1)$.

Proposition

[FALSE]

The vector field $D(L_iN)$ is orthogonal to $\{L_iN,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.
- Imitating the $\mathbb{R}^{32}$ case, group coordinates in 256-ples (s^1, s^2), and define $L_i(s^1, s^2) = (-s^2, s^1)$.

Proposition

[FALSE]

The vector field $D(L_i N)$ is orthogonal to $\{J_\alpha N, D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$.

- Next try: split each s^α as a pair (x^α, y^α) of 128-ples, and define a conjugation D_2 by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\ldots,8}$.
- Imitating the $\mathbb{R}^{32}$ case, group coordinates in 256-ples (s^1, s^2), and define $L_i(s^1, s^2) = (-s^2, s^1)$.

Proposition

[FALSE]

The vector field $D(L_i N)$ is orthogonal to $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\ldots,8}$.

- Next try: split each s^α as a pair (x^α, y^α) of 128-ples, and define a conjugation D_2 by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.

Proposition

The vector field $D_2(L_i N)$ is orthogonal to $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\ldots,8}$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

- 16 vector fields are given by $\{J_\alpha N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\ldots,8}$.
- Imitating the $\mathbb{R}^{32}$ case, group coordinates in 256-ples (s^1, s^2), and define $L_i(s^1, s^2) = (-s^2, s^1)$.

Proposition

[FALSE]

The vector field $D(L_i N)$ is orthogonal to $\{L_i N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\ldots,8}$.

- Next try: split each s^α as a pair (x^α, y^α) of 128-ples, and define a conjugation D_2 by $(x^\alpha, y^\alpha) \mapsto (x^\alpha, -y^\alpha)$.

Proposition

[FALSE]

The vector field $D_2(L_i N)$ is orthogonal to $\{L_i N,\ D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\ldots,8}$.

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

Proposition

The vector field $D(D_2(L_i N))$ is orthogonal to

$$\{J_\alpha N, D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$$

S^{511}: $\sigma(m) = 2 \cdot 8 + 1$

Proposition

The vector field $D(D_2(L_i N))$ is orthogonal to

$$\{J_\alpha N, D(\mathrm{block}(J_\alpha)N)\}_{\alpha=1,\dots,8}$$

1. S^{15} and $\mathrm{Spin}(9)$
   - S^{15} is "more equal" than other spheres
   - $\mathrm{Spin}(9)$ and Hopf fibrations
2. The $\mathrm{Spin}(9)$ fundamental form
   - Quaternionic analogy
   - $\mathrm{Spin}(9)$ and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\mathrm{Spin}(9)}$
3. Vector fields on spheres
   - Maximum number and examples
   - The general case
4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem

Remark

Abuse of notation in previous slides: $J_\alpha \in \mathrm{Mat}_{16}$, but for instance in this row $J_\alpha \in \mathrm{Mat}_{32}$:

$p = 1$: S^{31}	$8+1$	$J_1 N, \dots, J_8 N$ $D(L_i N)$	$N = s^1 + is^2, L_i N = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9){+}\mathbb{C}$

Remark

Abuse of notation in previous slides: $J_\alpha \in \mathrm{Mat}_{16}$, but for instance in this row $J_\alpha \in \mathrm{Mat}_{32}$:

$p = 1$: S^{31}	$8+1$	$J_1 N, \ldots, J_8 N$ $D(L_i N)$	$N = s^1 + is^2, L_i N = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9)+\mathbb{C}$

To state and prove the general case, we need to formalize the above notation.

Remark

Abuse of notation in previous slides: $J_\alpha \in \mathrm{Mat}_{16}$, but for instance in this row $J_\alpha \in \mathrm{Mat}_{32}$:

$p=1$: S^{31}	$8+1$	$J_1N,\dots,J_8N$ $D(L_iN)$	$N = s^1 + is^2, L_iN = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9)+\mathbb{C}$

To state and prove the general case, we need to formalize the above notation.

- Get rid of N: identify vector fields on S^{m-1} with $\mathfrak{so}(m)$.

Remark

Abuse of notation in previous slides: $J_\alpha \in \mathrm{Mat}_{16}$, but for instance in this row $J_\alpha \in \mathrm{Mat}_{32}$:

$p = 1$: S^{31}	$8 + 1$	$J_1 N, \dots, J_8 N$ $D(L_i N)$	$N = s^1 + is^2, L_i N = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	Spin(9)+$\mathbb{C}$

To state and prove the general case, we need to formalize the above notation.

- Get rid of N: identify vector fields on S^{m-1} with $\mathfrak{so}(m)$.
- $A \in \mathfrak{so}(m)$ has lenght 1 if and only if $A^2 = -\mathrm{Id}_m$.
- A is orthogonal to $B \in \mathfrak{so}(m)$ if and only if $AB + BA = 0$.

Remark

Abuse of notation in previous slides: $J_\alpha \in \mathrm{Mat}_{16}$, but for instance in this row $J_\alpha \in \mathrm{Mat}_{32}$:

$p = 1$: S^{31}	$8+1$	$J_1N, \dots, J_8N$ $D(L_iN)$	$N = s^1 + is^2, L_iN = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9){+}\mathbb{C}$

To state and prove the general case, we need to formalize the above notation.

- Get rid of N: identify vector fields on S^{m-1} with $\mathfrak{so}(m)$.
- $A \in \mathfrak{so}(m)$ has lenght 1 if and only if $A^2 = -\mathrm{Id}_m$.
- A is orthogonal to $B \in \mathfrak{so}(m)$ if and only if $AB + BA = 0$.
- Orthonormality is reduced to matrices computation.

Definition

Define $\mathrm{diag}_{m,n} : \mathrm{Mat}_m \to \mathrm{Mat}_{mn}$ by

$$\mathrm{diag}_{m,n}(A) = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$$

Definition

Define $\mathrm{diag}_{m,n} : \mathrm{Mat}_m \to \mathrm{Mat}_{mn}$ by

$$\mathrm{diag}_{m,n}(A) = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$$

repeat the $m \times m$ matrix A diagonally n times

Definition

Define $\mathrm{diag}_{m,n} : \mathrm{Mat}_m \to \mathrm{Mat}_{mn}$ by

$$\mathrm{diag}_{m,n}(A) = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$$

repeat the $m \times m$ matrix A diagonally n times

Example

$$\mathrm{diag}_{16,2}(J_\alpha) = \begin{pmatrix} J_\alpha & 0 \\ 0 & J_\alpha \end{pmatrix}$$

formalizes $J_1 N, \ldots, J_8 N$ in

$p = 1$: S^{31}	$8+1$	$J_1 N, \ldots, J_8 N$ $D(L_i N)$	$N = s^1 + is^2, L_i N = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	$\mathrm{Spin}(9)+\mathbb{C}$

Definition

If $A = (a_{\alpha\beta})_{\alpha,\beta=1,\dots,m}$, define $\mathrm{block}_{m,n} : \mathrm{Mat}_m \to \mathrm{Mat}_{mn}$ by

$$\mathrm{block}_{m,n}(A) \; = (a_{\alpha\beta}\,\mathrm{Id}_n)_{\alpha,\beta=1,\dots,m}$$

Definition

If $A = (a_{\alpha\beta})_{\alpha,\beta=1,\ldots,m}$, define $\mathrm{block}_{m,n} : \mathrm{Mat}_m \to \mathrm{Mat}_{mn}$ by

$$\mathrm{block}_{m,n}(A) \;= (a_{\alpha\beta}\,\mathrm{Id}_n)_{\alpha,\beta=1,\ldots,m}$$

each element of the $m \times m$ matrix A is a $n \times n$ matrix

Definition

If $A = (a_{\alpha\beta})_{\alpha,\beta=1,\ldots,m}$, define $\mathrm{block}_{m,n} : \mathrm{Mat}_m \to \mathrm{Mat}_{mn}$ by

$$\mathrm{block}_{m,n}(A) \; = (a_{\alpha\beta}\,\mathrm{Id}_n)_{\alpha,\beta=1,\ldots,m}$$

each element of the $m \times m$ matrix A is a $n \times n$ matrix

Example

$$\mathrm{block}_{2,16}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{Id}_{16} \\ \mathrm{Id}_{16} & 0 \end{pmatrix}$$

formalizes $L_i N$ in

$p=1$: S^{31}	$8+1$	$J_1N, \ldots, J_8N$ $D(L_iN)$	$N = s^1 + is^2, L_iN = -s^2 + is^1$ $D : (x^\alpha, y^\alpha) \to (x^\alpha, -y^\alpha)$	Spin(9)+$\mathbb{C}$

Definition

The *basic conjugation* in $\mathbb{R}^{16^s}$ is

$$\mathrm{D}_s \ = \mathrm{block}_{2,\frac{16^s}{2}}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in \mathrm{Mat}_{16^s}$$

Definition

The *basic conjugation* in $\mathbb{R}^{16^s}$ is

$$\mathrm{D}_s = \mathrm{block}_{2,\frac{16^s}{2}}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) \in \mathrm{Mat}_{16^s}$$

D_s swaps the signs of the last $\frac{16^s}{2}$ coordinates of a vector in $\mathbb{R}^{16^s}$

Definition

The *basic conjugation* in $\mathbb{R}^{16^s}$ is

$$\mathrm{D}_s = \mathrm{block}_{2,\frac{16^s}{2}}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in \mathrm{Mat}_{16^s}$$

D_s swaps the signs of the last $\frac{16^s}{2}$ coordinates of a vector in $\mathbb{R}^{16^s}$

Definition

Let $t \geq 2$ and $s = 1, \ldots, t-1$. Then

$$\mathrm{D}_{t,s} = \mathrm{diag}_{16^s,16^{t-s}}(\mathrm{D}_s) \in \mathrm{Mat}_{16^t}$$

Definition

The *basic conjugation* in $\mathbb{R}^{16^s}$ is

$$\mathrm{D}_s = \mathrm{block}_{2,\frac{16^s}{2}}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in \mathrm{Mat}_{16^s}$$

D_s swaps the signs of the last $\frac{16^s}{2}$ coordinates of a vector in $\mathbb{R}^{16^s}$

Definition

Let $t \geq 2$ and $s = 1, \ldots, t-1$. Then

$$\mathrm{D}_{t,s} = \mathrm{diag}_{16^s,16^{t-s}}(\mathrm{D}_s) \in \mathrm{Mat}_{16^t}$$

$\mathrm{D}_{2,1}$ is the conjugation D in $\mathbb{R}^{256}$ in the following row:

S^{255}	$8+8$	$J_1N, \ldots, J_8N$ $D(\mathrm{block}(J_1)N), \ldots, D(\mathrm{block}(J_8)N)$	$N = (s^1, \ldots, s^{16})$ $\mathrm{block}(J_1), \ldots, \mathrm{block}(J_8)$ and D as above	$\mathrm{Spin}(9)+\mathrm{Spin}(9)$

Main theorem for $m = 16^q$

For any $q \geq 1$, the $8q$ vector fields on $S^{16^q - 1}$ given by

$$\{B^q(t, J_\alpha) = \mathrm{diag}_{16^t, 16^{q-t}}(\prod_{s=1}^{t-1} \mathrm{D}_{t,s} \mathrm{block}_{16, 16^{t-1}}(J_\alpha))\}_{\substack{t=1,\dots,q \\ \alpha=1,\dots,8}}$$

are a maximal orthonormal set.

Definition

- $\mathrm{C}_t = \prod_{s=1}^{t-1} \mathrm{D}_{t,s}$.
- $\mathcal{G}^0 = \emptyset$.
- $\mathcal{G}^1 = \{L_i^{\mathbb{C}}\} \subset \mathrm{Mat}_2$.
- $\mathcal{G}^2 = \{L_i^{\mathbb{H}}, L_j^{\mathbb{H}}, L_k^{\mathbb{H}}\} \subset \mathrm{Mat}_4$.
- $\mathcal{G}^3 = \{L_i, L_j, L_k, L_e, L_f, L_g, L_h\} \subset \mathrm{Mat}_8$.

Definition

- $\mathrm{C}_t = \prod_{s=1}^{t-1} \mathrm{D}_{t,s}$.
- $\mathcal{G}^0 = \emptyset$.
- $\mathcal{G}^1 = \{L_i^{\mathbb{C}}\} \subset \mathrm{Mat}_2$.
- $\mathcal{G}^2 = \{L_i^{\mathbb{H}}, L_j^{\mathbb{H}}, L_k^{\mathbb{H}}\} \subset \mathrm{Mat}_4$.
- $\mathcal{G}^3 = \{L_i, L_j, L_k, L_e, L_f, L_g, L_h\} \subset \mathrm{Mat}_8$.

Theorem: $\sigma(m) > 7$? All the fault of $\mathrm{Spin}(9)$!

Let $k \geq 0$, $q \geq 1$ and $p = 0, 1, 2$ or 3. The $8q + 2^p - 1$ vector fields on $S^{(2k+1)2^p16^q-1}$ given by

$$\{B^{k,p,q}(t, J_\alpha) = \mathrm{diag}_{16^t,(2k+1)2^p16^{q-t}}(\mathrm{C}_t \mathrm{block}_{16,16^{t-1}}(J_\alpha))\}_{\substack{t=1,\dots,q \\ \alpha=1,\dots,8}}$$

$$\{L^{k,p,q}(G) = \mathrm{diag}_{2^p16^q,2k+1}(\mathrm{diag}_{16^q,2^p}(\mathrm{C}_q)\mathrm{block}_{2^p,16^q}(G))\}_{G\in\mathcal{G}^p}$$

are a maximal orthonormal set.

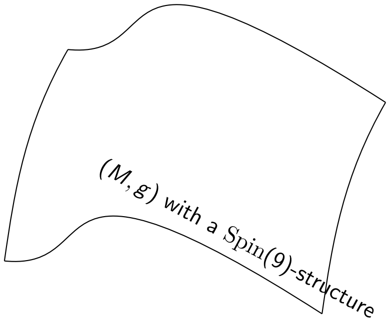
1. S^{15} and $\mathrm{Spin}(9)$
   - S^{15} is "more equal" than other spheres
   - $\mathrm{Spin}(9)$ and Hopf fibrations
2. The $\mathrm{Spin}(9)$ fundamental form
   - Quaternionic analogy
   - $\mathrm{Spin}(9)$ and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\mathrm{Spin}(9)}$
3. Vector fields on spheres
   - Maximum number and examples
   - The general case
4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem

Definition

A locally conformal parallel $\mathrm{Spin}(9)$ manifold is a 16-dimensional $\mathrm{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\mathrm{Spin}(9)$.

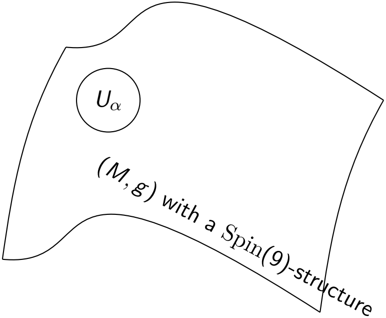
Definition

A locally conformal parallel $\mathrm{Spin}(9)$ manifold is a 16-dimensional $\mathrm{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\mathrm{Spin}(9)$.



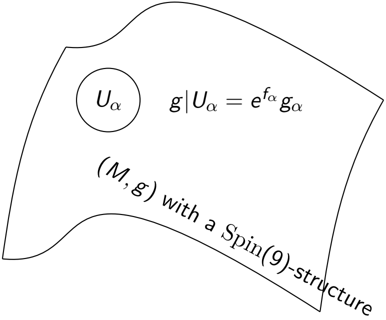
Definition

A locally conformal parallel $\mathrm{Spin}(9)$ manifold is a 16-dimensional $\mathrm{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\mathrm{Spin}(9)$.



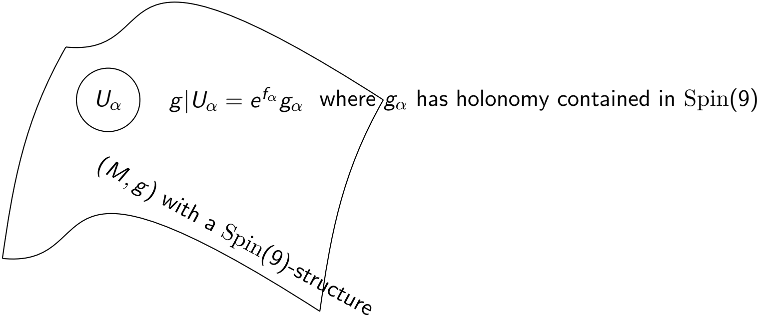
Definition

A locally conformal parallel $\mathrm{Spin}(9)$ manifold is a 16-dimensional $\mathrm{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\mathrm{Spin}(9)$.



Definition

A locally conformal parallel $\mathrm{Spin}(9)$ manifold is a 16-dimensional $\mathrm{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\mathrm{Spin}(9)$.



Examples

The product $S^{15} \times S^1 = \frac{\mathbb{O}^2 - 0}{\mathbb{Z}} =$ cone over S^{15} with the (conformal class) of the flat metric.

Examples

The product $S^{15} \times S^1 = \frac{\mathbb{O}^2 - 0}{\mathbb{Z}} =$ cone over S^{15} with the (conformal class) of the flat metric.

The trivial S^1-bundle $\mathbb{R}P^{15} \times S^1$, with the metric induced by the flat cone $C(S^{15})$.

Examples

The product $S^{15} \times S^1 = \frac{\mathbb{O}^2 - 0}{\mathbb{Z}}$ = cone over S^{15} with the (conformal class) of the flat metric.

The trivial S^1-bundle $\mathbb{R}P^{15} \times S^1$, with the metric induced by the flat cone $C(S^{15})$.

The non-trivial S^1-bundle over $\mathbb{R}P^{15}$, with the metric induced by the flat cone $C(S^{15})$.

Structure of compact locally conformal parallel $\mathrm{Spin}(9)$ manifolds

Theorem [P-Piccinni-Vuletescu]

Let (M, g) be a compact, locally conformal but not globally conformal parallel $\mathrm{Spin}(9)$ manifold. Then

$$M = C(N)/\mathbb{Z}$$

where $C(N)$ is a flat cone over a compact 15-dimensional manifold N with finite fundamental group.

Proof

1. On each U_α it is defined a ∇^α-parallel 8-form Φ_α.

Proof

1. On each U_α it is defined a ∇^α-parallel 8-form Φ_α.
2. There is a 8-form Φ on M locally given by $e^{4f_\alpha}\Phi_\alpha$.

Proof

1. On each U_α it is defined a ∇^α-parallel 8-form Φ_α.
2. There is a 8-form Φ on M locally given by $e^{4f_\alpha}\Phi_\alpha$.
3. There is a closed 1-form ω (the Lee form) on M, locally given by $4df_\alpha$, such that $d\Phi = \omega \wedge \Phi$.

Proof

1. On each U_α it is defined a ∇^α-parallel 8-form Φ_α.
2. There is a 8-form Φ on M locally given by $e^{4f_\alpha}\Phi_\alpha$.
3. There is a closed 1-form ω (the Lee form) on M, locally given by $4df_\alpha$, such that $d\Phi = \omega \wedge \Phi$.
4. The 1-form ω defines a closed Weyl connection D on M by $Dg = \omega \otimes g$.

Proof

1. On each U_α it is defined a ∇^α-parallel 8-form Φ_α.
2. There is a 8-form Φ on M locally given by $e^{4f_\alpha}\Phi_\alpha$.
3. There is a closed 1-form ω (the Lee form) on M, locally given by $4df_\alpha$, such that $d\Phi = \omega \wedge \Phi$.
4. The 1-form ω defines a closed Weyl connection D on M by $Dg = \omega \otimes g$.
5. Since the local metrics g_α are Einstein, D is Einstein-Weyl.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.
7. On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f}\tilde{g})$ is the metric cone $C(\tilde{N})$.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.
7. On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f}\tilde{g})$ is the metric cone $C(\tilde{N})$.
8. The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.
7. On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f}\tilde{g})$ is the metric cone $C(\tilde{N})$.
8. The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.
9. Ricci-flat + holonomy $\mathrm{Spin}(9) \Rightarrow$ flat.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.
7. On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f}\tilde{g})$ is the metric cone $C(\tilde{N})$.
8. The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.
9. Ricci-flat + holonomy $\mathrm{Spin}(9) \Rightarrow$ flat.
10. Since $\pi_1(M)$ acts by homotheties on $C(\tilde{N})$, and $\tilde{N}$ is compact, $\pi_1(M)$ contains a finite normal subgroup I of isometries.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.
7. On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f}\tilde{g})$ is the metric cone $C(\tilde{N})$.
8. The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.
9. Ricci-flat + holonomy $\mathrm{Spin}(9) \Rightarrow$ flat.
10. Since $\pi_1(M)$ acts by homotheties on $C(\tilde{N})$, and $\tilde{N}$ is compact, $\pi_1(M)$ contains a finite normal subgroup I of isometries.
11. We obtain $\pi_1(M) = I \rtimes \mathbb{Z}$, and $M = C(\tilde{N}/I)/\mathbb{Z}$.

End of talk. Thank you for your attention!

Details for $\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$

- ν_l = volume form on the octonionic lines $l = \{(x, mx)\}$ or $l = \{(0, y)\}$ in $\mathbb{O}^2$.
- $p_l : \mathbb{O}^2 \to l$ = projection on l.
- $p_l^* \nu_l$ = 8-form in $\mathbb{O}^2 = \mathbb{R}^{16}$.
- The integral over $\mathbb{O}P^1$ can be computed over $\mathbb{O}$ with polar coordinates.
- The formula arise from distinguished 8-planes in the $\mathrm{Spin}(9)$-geometry $\to$ (forthcoming) calibrations.

Go back

The five involutions of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ as 8×8 matrices

$$\mathcal{I}_2 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right)$$

$$\mathcal{I}_3 = \left(\begin{array}{c|c} 0 & -R_i^{\mathbb{H}} \\ \hline R_i^{\mathbb{H}} & 0 \end{array}\right)$$

Go back

$$\mathcal{I}_1 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right)$$

$$\mathcal{I}_4 = \left(\begin{array}{c|c} 0 & -R_j^{\mathbb{H}} \\ \hline R_j^{\mathbb{H}} & 0 \end{array}\right)$$

$$\mathcal{I}_5 = \left(\begin{array}{c|c} 0 & -R_k^{\mathbb{H}} \\ \hline R_k^{\mathbb{H}} & 0 \end{array}\right)$$

The nine involutions of $\mathrm{Spin}(9)$ as 16×16 matrices

$$\mathcal{I}_4 = \left(\begin{array}{c|c} 0 & -R_j \\ \hline R_j & 0 \end{array}\right)$$

$$\mathcal{I}_3 = \left(\begin{array}{c|c} 0 & -R_i \\ \hline R_i & 0 \end{array}\right)$$

$$\mathcal{I}_2 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right)$$

$$\mathcal{I}_5 = \left(\begin{array}{c|c} 0 & -R_k \\ \hline R_k & 0 \end{array}\right)$$

Go back

$$\mathcal{I}_1 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right)$$

$$\mathcal{I}_6 = \left(\begin{array}{c|c} 0 & -R_e \\ \hline R_e & 0 \end{array}\right)$$

$$\mathcal{I}_9 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right)$$

$$\mathcal{I}_7 = \left(\begin{array}{c|c} 0 & -R_f \\ \hline R_f & 0 \end{array}\right)$$

$$\mathcal{I}_8 = \left(\begin{array}{c|c} 0 & -R_g \\ \hline R_g & 0 \end{array}\right)$$

Explicit formula for Φ_{G_2}

Denote by $x_1, \ldots, x_7$ the coordinates in $\mathbb{R}^7$. Then $\mathrm{G}_2 =$ stabilizer in $\mathrm{SO}(7)$ of

$$\begin{aligned}\Phi_{\mathrm{G}_2} = {} & dx_1 \wedge dx_2 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5 + dx_3 \wedge dx_4 \wedge dx_6 \\ & + dx_4 \wedge dx_5 \wedge dx_7 + dx_5 \wedge dx_6 \wedge dx_1 + dx_6 \wedge dx_7 \wedge dx_2 \\ & + dx_7 \wedge dx_1 \wedge dx_3\end{aligned}$$

As a shortcut, we could write

$$\Phi_{\mathrm{G}_2} = \mathbf{124} + \mathbf{235} + \mathbf{346} + \mathbf{457} + \mathbf{561} + \mathbf{672} + \mathbf{713}$$

Go back

351 terms of $\Phi_{\mathrm{Spin}(9)}$

Go back

12345678		-14	123456	1′2′	2	123456	3′4′	-2	123456	5′6′	-2	123456	7′8′	-2
123457	1′3′	2	123457	2′4′	2	123457	5′7′	-2	123457	6′8′	2	123458	1′4′	2
123458	2′3′	-2	123458	5′8′	-2	123458	6′7′	-2	123467	1′4′	-2	123467	2′3′	2
123467	5′8′	-2	123467	6′7′	-2	123468	1′3′	2	123468	2′4′	2	123468	5′7′	2
123468	6′8′	-2	123478	1′2′	-2	123478	3′4′	2	123478	5′6′	-2	123478	7′8′	-2
1234	1′2′3′4′	-2	1234	5′6′7′8′	-2	123567	1′5′	-2	123567	2′6′	-2	123567	3′7′	-2
123567	4′8′	2	123568	1′6′	-2	123568	2′5′	2	123568	3′8′	-2	123568	4′7′	-2
123578	1′7′	-2	123578	2′8′	2	123578	3′5′	2	123578	4′6′	2	1235	1′2′3′5′	-1
1235	1′2′4′6′	-1	1235	1′3′4′7′	-1	1235	1′5′6′7′	-1	1235	2′3′4′8′	1	1235	2′5′6′8′	1
1235	3′5′7′8′	1	1235	4′6′7′8′	1	123678	1′8′	-2	123678	2′7′	-2	123678	3′6′	2
123678	4′5′	-2	1236	1′2′3′6′	-1	1236	1′2′4′5′	1	1236	1′3′4′8′	-1	1236	1′5′6′8′	-1
1236	2′3′4′7′	-1	1236	2′5′6′7′	-1	1236	3′6′7′8′	1	1236	4′5′7′8′	-1	1237	1′2′3′7′	-1
1237	1′2′4′8′	1	1237	1′3′4′5′	1	1237	1′5′7′8′	-1	1237	2′3′4′6′	1	1237	2′6′7′8′	-1
1237	3′5′6′7′	-1	1237	4′5′6′8′	1	1238	1′2′3′8′	-1	1238	1′2′4′7′	-1	1238	1′3′4′6′	1
1238	1′6′7′8′	-1	1238	2′3′4′5′	-1	1238	2′5′7′8′	1	1238	3′5′6′8′	-1	1238	4′5′6′7′	-1
124567	1′6′	2	124567	2′5′	-2	124567	3′8′	-2	124567	4′7′	-2	124568	1′5′	-2
124568	2′6′	-2	124568	3′7′	2	124568	4′8′	-2	124578	1′8′	-2	124578	2′7′	-2
124578	3′6′	-2	124578	4′5′	2	1245	1′2′3′6′	1	1245	1′2′4′5′	-1	1245	1′3′4′8′	-1
1245	1′5′6′8′	-1	1245	2′3′4′7′	-1	1245	2′5′6′7′	-1	1245	3′6′7′8′	-1	1245	4′5′7′8′	1
124678	1′7′	2	124678	2′8′	-2	124678	3′5′	2	124678	4′6′	2	1246	1′2′3′5′	-1
1246	1′2′4′6′	-1	1246	1′3′4′7′	1	1246	1′5′6′7′	1	1246	2′3′4′8′	-1	1246	2′5′6′8′	-1
1246	3′5′7′8′	1	1246	4′6′7′8′	1	1247	1′2′3′8′	-1	1247	1′2′4′7′	-1	1247	1′3′4′6′	-1
1247	1′6′7′8′	1	1247	2′3′4′5′	1	1247	2′5′7′8′	-1	1247	3′5′6′8′	-1	1247	4′5′6′7′	-1
1248	1′2′3′7′	1	1248	1′2′4′8′	-1	1248	1′3′4′5′	1	1248	1′5′7′8′	-1	1248	2′3′4′6′	1
1248	2′6′7′8′	-1	1248	3′5′6′7′	1	1248	4′5′6′8′	-1	125678	1′2′	-2	125678	3′4′	-2
125678	5′6′	2	125678	7′8′	-2	1256	1′2′5′6′	-2	1256	3′4′7′8′	-2	1257	1′2′5′7′	-1
1257	1′2′6′8′	1	1257	1′3′5′6′	-1	1257	1′3′7′8′	1	1257	2′4′5′6′	-1	1257	2′4′7′8′	1
1257	3′4′5′7′	-1	1257	3′4′6′8′	1	1258	1′2′5′8′	-1	1258	1′2′6′7′	-1	1258	1′4′5′6′	-1
1258	1′4′7′8′	1	1258	2′3′5′6′	1	1258	2′3′7′8′	-1	1258	3′4′5′8′	-1	1258	3′4′6′7′	-1
1267	1′2′5′8′	-1	1267	1′2′6′7′	-1	1267	1′4′5′6′	1	1267	1′4′7′8′	-1	1267	2′3′5′6′	-1
1267	2′3′7′8′	1	1267	3′4′5′8′	-1	1267	3′4′6′7′	-1	1268	1′2′5′7′	1	1268	1′2′6′8′	-1
1268	1′3′5′6′	-1	1268	1′3′7′8′	1	1268	2′4′5′6′	-1	1268	2′4′7′8′	1	1268	3′4′5′7′	1
1268	3′4′6′8′	-1	1278	1′2′7′8′	-2	1278	3′4′5′6′	-2	12	1′2′3′4′5′6′	2	12	1′2′3′4′7′8′	-2
12	1′2′5′6′7′8′	-2	12	3′4′5′6′7′8′	-2	134567	1′7′	2	134567	2′8′	2	134567	3′5′	-2
134567	4′6′	2	134568	1′8′	2	134568	2′7′	-2	134568	3′6′	-2	134568	4′5′	-2
134578	1′5′	-2	134578	2′6′	2	134578	3′7′	-2	134578	4′8′	-2	1345	1′2′3′7′	1
1345	1′2′4′8′	1	1345	1′3′4′5′	-1	1345	1′5′7′8′	-1	1345	2′3′4′6′	1	1345	2′6′7′8′	1
1345	3′5′6′7′	-1	1345	4′5′6′8′	-1	134678	1′6′	-2	134678	2′5′	-2	134678	3′8′	-2
134678	4′7′	2	1346	1′2′3′8′	1	1346	1′2′4′7′	-1	1346	1′3′4′6′	-1	1346	1′6′7′8′	-1
1346	2′3′4′5′	-1	1346	2′5′7′8′	-1	1346	3′5′6′8′	1	1346	4′5′6′7′	1	1347	1′2′3′5′	-1
1347	1′2′4′6′	1	1347	1′3′4′7′	-1	1347	1′5′6′7′	1	1347	2′3′4′8′	-1	1347	2′5′6′8′	1
1347	3′5′7′8′	-1	1347	4′6′7′8′	1	1348	1′2′3′6′	-1	1348	1′2′4′5′	-1	1348	1′3′4′8′	-1
1348	1′5′6′8′	1	1348	2′3′4′7′	1	1348	2′5′6′7′	-1	1348	3′6′7′8′	-1	1348	4′5′7′8′	-1
135678	1′3′	-2	135678	2′4′	2	135678	5′7′	2	135678	6′8′	2	1356	1′2′5′7′	-1
1356	1′2′6′8′	-1	1356	1′3′5′6′	-1	1356	1′3′7′8′	-1	1356	2′4′5′6′	1	1356	2′4′7′8′	1
1356	3′4′5′7′	1	1356	3′4′6′8′	1	1357	1′3′5′7′	-2	1357	2′4′6′8′	-2	1358	1′3′5′8′	-1
1358	1′3′6′7′	-1	1358	1′4′5′7′	-1	1358	1′4′6′8′	-1	1358	2′3′5′7′	1	1358	2′3′6′8′	1
1358	2′4′5′8′	1	1358	2′4′6′7′	1	1367	1′3′5′8′	-1	1367	1′3′6′7′	-1	1367	1′4′5′7′	1
1367	1′4′6′8′	1	1367	2′3′5′7′	-1	1367	2′3′6′8′	-1	1367	2′4′5′8′	1	1367	2′4′6′7′	1
1368	1′3′6′8′	-2	1368	2′4′5′7′	-2	1378	1′2′5′7′	1	1378	1′2′6′8′	1	1378	1′3′5′6′	-1
1378	1′3′7′8′	-1	1378	2′4′5′6′	1	1378	2′4′7′8′	1	1378	3′4′5′7′	-1	1378	3′4′6′8′	-1
13	1′2′3′4′5′7′	2	13	1′2′3′4′6′8′	2	13	1′3′5′6′7′8′	-2	13	2′4′5′6′7′8′	2	145678	1′4′	-2
145678	2′3′	-2	145678	5′8′	2	145678	6′7′	-2	1456	1′2′5′8′	-1	1456	1′2′6′7′	1
1456	1′4′5′6′	-1	1456	1′4′7′8′	-1	1456	2′3′5′6′	-1	1456	2′3′7′8′	-1	1456	3′4′5′8′	1
1456	3′4′6′7′	-1	1457	1′3′5′8′	-1	1457	1′3′6′7′	1	1457	1′4′5′7′	-1	1457	1′4′6′8′	1
1457	2′3′5′7′	-1	1457	2′3′6′8′	1	1457	2′4′5′8′	-1	1457	2′4′6′7′	1	1458	1′4′5′8′	-2
1458	2′3′6′7′	-2	1467	1′4′6′7′	-2	1467	2′3′5′8′	-2	1468	1′3′5′8′	-1	1468	1′3′6′7′	1
1468	1′4′5′7′	1	1468	1′4′6′8′	-1	1468	2′3′5′7′	1	1468	2′3′6′8′	-1	1468	2′4′5′8′	-1
1468	2′4′6′7′	1	1478	1′2′5′8′	1	1478	1′2′6′7′	-1	1478	1′4′5′6′	-1	1478	1′4′7′8′	-1
1478	2′3′5′6′	-1	1478	2′3′7′8′	-1	1478	3′4′5′8′	-1	1478	3′4′6′7′	1	14	1′2′3′4′5′8′	2
14	1′2′3′4′6′7′	-2	14	1′4′5′6′7′8′	-2	14	2′3′5′6′7′8′	-2	1567	1′2′3′5′	-1	1567	1′2′4′6′	1
1567	1′3′4′7′	1	1567	1′5′6′7′	-1	1567	2′3′4′8′	1	1567	2′5′6′8′	-1	1567	3′5′7′8′	-1
1567	4′6′7′8′	1	1568	1′2′3′6′	-1	1568	1′2′4′5′	-1	1568	1′3′4′8′	1	1568	1′5′6′8′	-1
1568	2′3′4′7′	-1	1568	2′5′6′7′	1	1568	3′6′7′8′	-1	1568	4′5′7′8′	-1	1578	1′2′3′7′	-1
1578	1′2′4′8′	-1	1578	1′3′4′5′	-1	1578	1′5′7′8′	-1	1578	2′3′4′6′	1	1578	2′6′7′8′	1
1578	3′5′6′7′	1	1578	4′5′6′8′	1	15	1′2′3′5′6′7′	-2	15	1′2′4′5′6′8′	-2	15	1′3′4′5′7′8′	-2
15	2′3′4′6′7′8′	2	1678	1′2′3′8′	-1	1678	1′2′4′7′	1	1678	1′3′4′6′	-1	1678	1′6′7′8′	-1
1678	2′3′4′5′	-1	1678	2′5′7′8′	-1	1678	3′5′6′8′	1	1678	4′5′6′7′	-1	16	1′2′3′5′6′8′	-2
16	1′2′4′5′6′7′	2	16	1′3′4′6′7′8′	-2	16	2′3′4′5′7′8′	-2	17	1′2′3′5′7′8′	-2	17	1′2′4′6′7′8′	2
17	1′3′4′5′6′7′	2	17	2′3′4′5′6′8′	2	18	1′2′3′6′7′8′	-2	18	1′2′4′5′7′8′	-2	18	1′3′4′5′6′8′	2
18	2′3′4′5′6′7′	-2												

70 terms of $\Phi_{\mathrm{Spin}(9)}$

12345678		-14	123456	1′2′	2	123456	3′4′	-2	123456	5′6′	-2	123456	7′8′	-2
123457	1′3′	2	123457	2′4′	2	123457	5′7′	-2	123457	6′8′	2	123458	1′4′	2
123458	2′3′	-2	123458	5′8′	-2	123458	6′7′	-2	123467	1′4′	-2	123467	2′3′	2
123467	5′8′	-2	123467	6′7′	-2	123468	1′3′	2	123468	2′4′	2	123468	5′7′	2
123468	6′8′	-2	123478	1′2′	-2	123478	3′4′	2	123478	5′6′	-2	123478	7′8′	-2
1234	1′2′3′4′	-2	1234	5′6′7′8′	-2	123567	1′5′	-2	123567	2′6′	-2	123567	3′7′	-2
123567	4′8′	2	123568	1′6′	-2	123568	2′5′	2	123568	3′8′	-2	123568	4′7′	-2
123578	1′7′	-2	123578	2′8′	2	123578	3′5′	2	123578	4′6′	2	1235	1′2′3′5′	-1
1235	1′2′4′6′	-1	1235	1′3′4′7′	-1	1235	1′5′6′7′	-1	1235	2′3′4′8′	1	1235	2′5′6′8′	1
1235	3′5′7′8′	1	1235	4′6′7′8′	1	123678	1′8′	-2	123678	2′7′	-2	123678	3′6′	2
123678	4′5′	-2	1236	1′2′3′6′	-1	1236	1′2′4′5′	1	1236	1′3′4′8′	-1	1236	1′5′6′8′	-1
1236	2′3′4′7′	-1	1236	2′5′6′7′	-1	1236	3′6′7′8′	1	1236	4′5′7′8′	-1	1237	1′2′3′7′	-1
1237	1′2′4′8′	1	1237	1′3′4′5′	1	1237	1′5′7′8′	-1	1237	2′3′4′6′	1	1237	2′6′7′8′	-1
1237	3′5′6′7′	-1	1237	4′5′6′8′	1	1238	1′2′3′8′	-1	1238	1′2′4′7′	-1	1238	1′3′4′6′	1

- $\{\mathbf{1,2,3,4,5,6,7,8,1',2',3',4',5',6',7',8'}\}$ are (indexes of) coordinates in $\mathbb{R}^{16}$.
- A table entry $||\mathbf{123578} \quad \mathbf{1'7'} \quad -2||$ means that
  $\Phi_{\mathrm{Spin}(9)} = \cdots - 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_7 \wedge dx_8 \wedge dx'_1 \wedge dx'_7 + \dots$
- Table obtained from Berger's definition of $\Phi_{\mathrm{Spin}(9)}$ with the help of *Mathematica*.
- The coefficients are normalized in such a way that they are all integers with gcd $= 1$.

Go back

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) Ricci; (2) EDC; (3) DIY;

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) EDC; (3) DIY;

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;
- The implementation of the wedge product can be reduced to a sorting problem:

$$\begin{aligned} \texttt{Wedge}(dx_1 \wedge dx_4, dx_2 \wedge dx_3) &\overset{\text{concatenation}}{=} dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3 \\ &\overset{\text{sorting}}{=} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;
- The implementation of the wedge product can be reduced to a sorting problem:

$$\begin{aligned} \texttt{Wedge}(dx_1 \wedge dx_4, dx_2 \wedge dx_3) &\overset{\text{concatenation}}{=} dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3 \\ &\overset{\text{sorting}}{=} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

- *Divide and conquer* paradigm can be used: break the problem into subproblems, recursively solve these subproblems, combine the solutions into a solution to the original problem.

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;
- The implementation of the wedge product can be reduced to a sorting problem:

$$\begin{aligned} \texttt{Wedge}(dx_1 \wedge dx_4, dx_2 \wedge dx_3) &\overset{\text{concatenation}}{=} dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3 \\ &\overset{\text{sorting}}{=} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

- *Divide and conquer* paradigm can be used: break the problem into subproblems, recursively solve these subproblems, combine the solutions into a solution to the original problem.

sorting $dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3$

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;
- The implementation of the wedge product can be reduced to a sorting problem:

$$\begin{aligned} \texttt{Wedge}(dx_1 \wedge dx_4, dx_2 \wedge dx_3) &\overset{\text{concatenation}}{=} dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3 \\ &\overset{\text{sorting}}{=} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

- *Divide and conquer* paradigm can be used: break the problem into subproblems, recursively solve these subproblems, combine the solutions into a solution to the original problem.

sorting $dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3$

sorting $dx_1 \wedge dx_4$ and $dx_2 \wedge dx_3$

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;
- The implementation of the wedge product can be reduced to a sorting problem:

$$\begin{aligned} \texttt{Wedge}(dx_1 \wedge dx_4, dx_2 \wedge dx_3) &\overset{\text{concatenation}}{=} dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3 \\ &\overset{\text{sorting}}{=} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

- *Divide and conquer* paradigm can be used: break $\boxed{\text{the problem}}$ into $\boxed{\text{subproblems,}}$ recursively $\boxed{\text{solve}}$ these subproblems, combine the solutions into a solution to the original problem.

$\boxed{\text{sorting } dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3}$

$\boxed{\text{sorting } dx_1 \wedge dx_4 \text{ and } dx_2 \wedge dx_3}$

$\boxed{\text{already solved}}$

Computational challenge for $\Phi_{\mathrm{Spin}(9)}$

- Differential geometry in Mathematica? (1) ~~Ricci~~; (2) ~~EDC~~; (3) DIY;
- The implementation of the wedge product can be reduced to a sorting problem:

$$\texttt{Wedge}(dx_1 \wedge dx_4, dx_2 \wedge dx_3) \overset{\text{concatenation}}{=} dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3$$
$$\overset{\text{sorting}}{=} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

- *Divide and conquer* paradigm can be used: break the problem into subproblems, recursively solve these subproblems, combine the solutions into a solution to the original problem.

sorting $dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3$

sorting $dx_1 \wedge dx_4$ and $dx_2 \wedge dx_3$

already solved

next slide

Code to merge 2 sorted lists

[Adapted from the classical *mergesort* algorithm, thanks to Gianluca Amato and Francesca Scozzari]

```
(*Take care of sign when swapping*)
sign = 1;

(*Induction base: what to do when one or both the arguments are empty*)
formWedge[{}, {}] = {};
formWedge[{}, right_] := right;
formWedge[left_, {}] := left;

(*Compare first terms, and recursively build the ordered list*)
formWedge[left_, right_] :=
    Switch[Order[left[[1]], right[[1]]],
        1,
            Return[Prepend[formWedge[Drop[left, 1], right], left[[1]]]],
        -1,
            sign = sign*(-1)^Length[left];
            Return[Prepend[formWedge[left, Drop[right, 1]], right[[1]]]],
        0,
            Abort[]
    ]
```

Go back

From Pfaffians to $\Phi_{\mathrm{Spin}(9)}$

$$\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \sum_{1\le\alpha_1<\alpha_2<\alpha_3<\alpha_4\le 9} (\psi_{\alpha_1\alpha_2}\wedge\psi_{\alpha_3\alpha_4}-\psi_{\alpha_1\alpha_3}\wedge\psi_{\alpha_2\alpha_4}+\psi_{\alpha_1\alpha_4}\wedge\psi_{\alpha_2\alpha_3})^2$$

$\psi_{12} = (-12+34+56-78)-(\ )'$ $\psi_{13} = (-13-24+57+68)-(\ )'$ $\psi_{14} = (-14+23+58-67)-(\ )'$

$\psi_{15} = (-15-26-37-48)-(\ )'$ $\psi_{16} = (-16+25-38+47)-(\ )'$ $\psi_{17} = (-17+28+35-46)-(\ )'$

$\psi_{18} = (-18-27+36+45)-(\ )'$ $\psi_{23} = (-14+23-58+67)+(\ )'$ $\psi_{24} = (13+24+57+68)+(\ )'$

$\psi_{25} = (-16+25+38-47)+(\ )'$ $\psi_{26} = (15+26-37-48)+(\ )'$ $\psi_{27} = (18+27+36+45)+(\ )'$

$\psi_{28} = (-17+28-35+46)+(\ )'$ $\psi_{34} = (-12+34-56+78)+(\ )'$ $\psi_{35} = (-17-28+35+46)+(\ )'$

$\psi_{36} = (-18+27+36-45)+(\ )'$ $\psi_{37} = (+15-26+37-48)+(\ )'$ $\psi_{38} = (16+25+38+47)+(\ )'$

$\psi_{45} = (-18+27-36+45)+(\ )'$ $\psi_{46} = (17+28+35+46)+(\ )'$ $\psi_{47} = (-16-25+38+47)+(\ )'$

$\psi_{48} = (15-26-37+48)+(\ )'$ $\psi_{56} = (-12-34+56+78)+(\ )'$ $\psi_{57} = (-13+24+57-68)+(\ )'$

$\psi_{58} = (-14-23+58+67)+(\ )'$ $\psi_{67} = (14+23+58+67)+(\ )'$ $\psi_{68} = (-13+24-57+68)+(\ )'$

$\psi_{78} = (12+34+56+78)+(\ )'$

$\psi_{19} = -11'-22'-33'-44'-55'-66'-77'-88'$ $\psi_{29} = -12'+21'+34'-43'+56'-65'-78'+87'$

$\psi_{39} = -13'-24'+31'+42'+57'+68'-75'-86'$ $\psi_{49} = -14'+23'-32'+41'+58'-67'+76'-85'$

$\psi_{59} = -15'-26'-37'-48'+51'+62'+73'+84'$ $\psi_{69} = -16'+25'-38'+47'-52'+61'-74'+83'$

$\psi_{79} = -17'+28'+35'-46'-53'+64'+71'-82'$ $\psi_{89} = -18'-27'+36'+45'-54'-63'+72'+81'$

Go back

Berger and calibrations

Curiosity

Berger appears to know about the fact that $\Phi_{\mathrm{Spin}(9)}$ is a calibration on $\mathbb{O}P^2$ already in 1970 [Berger, L'Enseignement Math. 1970] and more explicitly in 1972 [Berger, Ann. Éc. Norm. Sup. 1972, Theorem 6.3], very early with respect to the forthcoming calibration theory.