

# Swarm dynamics and equilibria for a nonlocal aggregation model

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## Self-organizing animal aggregations

- Animal groups with a high structural order
- The behaviour of individuals is so coordinated, that the group moves as a single coherent entity
- Examples of self-organizing biological groups
  - schooling fish
  - herds of ungulates
  - swarming insects
  - zigzagging flocks of birds

## Mathematical models

- The existing models fall into 2 categories: **Lagrangian** and **Eulerian**
- **Lagrangian** models: trajectories of all individuals of a species are tracked according to a set of interaction and decision rules
  - a large set of coupled ODE's
  - a large set of coupled difference equations (discrete time)
- **Eulerian** models: the problem is cast as an evolution equation for the population density field
  - parabolic
  - hyperbolic

## A nonlocal Eulerian PDE swarming model

- We study the PDE aggregation model in  $\mathbb{R}^n$ :

- continuity equation for the density  $\rho$ :

$$\rho_t + \nabla \cdot (\rho v) = 0$$

- the velocity  $v$  is assumed to have a functional dependence on the density

$$v = -\nabla K * \rho$$

- the potential  $K$  incorporates social interactions: attraction and repulsion

- The model was first suggested by Mogilner and Keshet, *J. Math. Biol.* [1999]
- Literature on this model has been very rich in recent years

## Lagrangian description

$N$  individuals

$X_i(t)$  = spatial location of the  $i$ -th individual at time  $t$

$$\frac{dX_i}{dt} = -\frac{1}{N} \sum_{\substack{j=1\dots N \\ j \neq i}} \nabla_i K(X_i - X_j), \quad i = 1 \dots N$$

PDE: continuum approximation, as  $N \rightarrow \infty$

Assumption: social interactions depend only on the relative distance between the individuals

- radially symmetric potentials

$$K(x) = K(|x|)$$

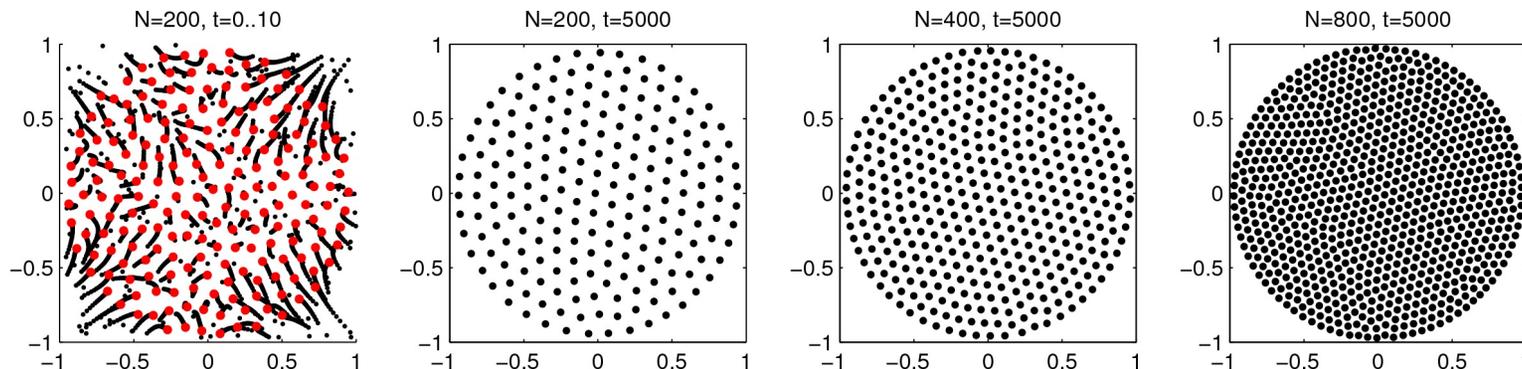
Notation:  $F(r) = -K'(r)$

$$\frac{dX_i}{dt} = \frac{1}{N} \sum_{\substack{j=1 \dots N \\ j \neq i}} F(|X_i - X_j|) \frac{X_i - X_j}{|X_i - X_j|}, \quad i = 1 \dots N$$

$F(|X_i - X_j|)$  = magnitude of the force that the individual  $X_j$  exerts on the individual  $X_i$ , along  $X_i - X_j$

**Repulsion** ( $F(r) > 0$ ) acts at **short** ranges, **attraction** ( $F(r) < 0$ ) at **long** ranges.

Example:  $n = 2$ ,  $F(r) = 1/r - r$ ; random initial conditions inside the unit square. The solution approaches a constant density in the unit disk.



## Motivation for this work

- **Equilibria** of the model should have biologically relevant features:
  - finite densities
  - sharp boundaries
  - relatively constant internal population
- The main **motivation** for this work is to
  - **design** interaction potentials  $K$  which lead to such equilibria
  - investigate analytically and numerically the well-posedness and long time behaviour of solutions

## Interaction potential $K$

$$\begin{aligned} K(x) &= K_r + K_a \\ &= \phi(x) + \frac{1}{q}|x|^q, \quad q \geq 2 \end{aligned}$$

$\phi(x)$  = the free-space Green's function for  $-\Delta$ :

$$\phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2 \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

Continuity equation:  $\rho_t + v \cdot \nabla \rho = -\rho \operatorname{div} v$

Calculate  $\operatorname{div} v$ :

$$\begin{aligned} \operatorname{div} v &= \operatorname{div}(-\nabla K * \rho) \\ &= -\Delta K * \rho \\ &= \rho - \Delta\left(\frac{1}{q}|x|^q\right) * \rho \end{aligned}$$

The repulsion term has become **local**!

## Lagrangian approach

Characteristic curves:  $\frac{d}{dt}X(\alpha, t) = v(X(\alpha, t), t)$ ,  $X(\alpha, 0) = \alpha$

Evolution equation for  $\rho(X(\alpha, t), t)$ :

$$\frac{D\rho}{Dt} = -\rho^2 + \rho \Delta\left(\frac{1}{q}|x|^q\right) * \rho$$

Special case  $q = 2$ : explicit calculations

$$\Delta\left(\frac{1}{2}|x|^2\right) = n, \quad \Delta\left(\frac{1}{2}|x|^2\right) * \rho = n \underbrace{\int_{\mathbb{R}^n} \rho(y) dy}_{=M}$$

ODE along characteristics:  $\frac{D\rho}{Dt} = -\rho(\rho - nM)$

Exact solution:  $\rho(X(\alpha, t), t) = \frac{nM}{1 + \left(\frac{nM}{\rho_0(\alpha)} - 1\right) e^{-nMt}}$

Asymptotic behaviour as  $t \rightarrow \infty$ ?

## Asymptotic behaviour

Density:  $\rho(X, t) \rightarrow nM$ , as  $t \rightarrow \infty$ , along particle paths with  $\rho_0(\alpha) \neq 0$

Asymptotic behaviour of trajectories:  $R_\alpha = \lim_{t \rightarrow \infty} |X(\alpha, t)|$

For radial solutions, it can be proved that trajectories are mapped into the ball of  $\mathbb{R}^n$  of radius  $R_\alpha = \frac{1}{(n\omega_n)^{\frac{1}{n}}}$ .

Numerics suggest that *all* solutions have this asymptotic behaviour.

**Global attractor:** constant, compactly supported density:

$$\bar{\rho}(x) = \begin{cases} nM & \text{if } |x| < \frac{1}{(n\omega_n)^{\frac{1}{n}}} \\ 0 & \text{otherwise} \end{cases}$$

## Global existence of particle paths

$$v(x) = \int_{\mathbb{R}^n} k(x - y) \rho(y) dy - Mx, \quad (1)$$

where

$$k(x) = \frac{1}{n\omega_n} \frac{x}{|x|^n}$$

The convolution kernel  $k$  is singular, homogeneous of degree  $1 - n$ .

Equation (1) is analogous to Biot-Savart law, where vorticity  $\omega$  is now replaced by density  $\rho$ .

Existence and uniqueness of particle paths follow similarly to that for incompressible Euler equations.

Extension to global existence: Beale-Kato-Majda criterion

$$\int_0^t \|\rho(\cdot, s)\|_{L^\infty} ds < \infty, \text{ for all finite times } t$$

## Case $q > 2$ : Non-constant steady states

Numerics suggests that attractors are radially symmetric.

Assume the model admits a radial steady state supported on a ball  $B(0, R)$ .

Recall formula for  $\operatorname{div} v$ :  $\operatorname{div} v = \rho - \Delta\left(\frac{1}{q}|x|^q\right) * \rho$

Equilibria supported on  $B(0, R)$ :

$$v = 0, \text{ hence } \operatorname{div} v = 0 \text{ in } B(0, R)$$

A steady state  $\bar{\rho}$  satisfies:

$$\bar{\rho} - (n + q - 2) \int_{\mathbb{R}^n} |x - y|^{q-2} \bar{\rho}(y) dy = 0 \quad \text{in } B(0, R)$$

Use radial symmetry  $\bar{\rho}(x) = \bar{\rho}(r)$ .

## Radial steady states

The density  $\bar{\rho}$  satisfies the homogeneous Fredholm integral equation

$$\bar{\rho}(r) = c(q, n) \int_0^R (r')^{n-1} \bar{\rho}(r') I(r, r') dr', \quad 0 \leq r < R,$$

$$I(r, r') = \int_0^\pi (r^2 + (r')^2 - 2rr' \cos \theta)^{q/2-1} \sin^{n-2} \theta d\theta.$$

In other words,  $\bar{\rho}$  is an eigenfunction of the linear operator  $T_R$ :

$$T_R \bar{\rho}(r) = c(q, n) \int_0^R (r')^{n-1} \bar{\rho}(r') I(r, r') dr',$$

that corresponds to eigenvalue one:  $T_R \bar{\rho}(r) = \bar{\rho}(r), \quad r < R$

The **eigenvalue problem**: find  $\bar{\rho}$  **and** the radius  $R$  of the support

## Krein-Rutman theorem

Consider case  $R = 1$  first.

The kernel  $c(q, n)(r')^{n-1}I(r, r')$  is nonnegative, continuous and bounded.

$T_1$  is a linear, strongly positive, compact operator that maps the space of continuous functions  $C([0, 1], \mathbb{R})$  into itself.

Krein-Rutman theorem: there exists a *positive* eigenfunction  $\bar{\rho}_1$  such that

$$T_1 \bar{\rho}_1 = \lambda \bar{\rho}_1 \quad (2)$$

$\lambda(q, n)$  is the spectral radius of  $T_1$ ; it is a simple eigenvalue and there is no other eigenvalue with a positive eigenvector.

Define, by rescaling:  $\bar{\rho}(r) = \bar{\rho}_1(r/R)$ .

## Existence and uniqueness of equilibria

Introduce  $\bar{\rho}(r) = \bar{\rho}_1(r/R)$  in (2):

$$T_R \bar{\rho}(r) = R^{n+q-2} \lambda \bar{\rho}(r)$$

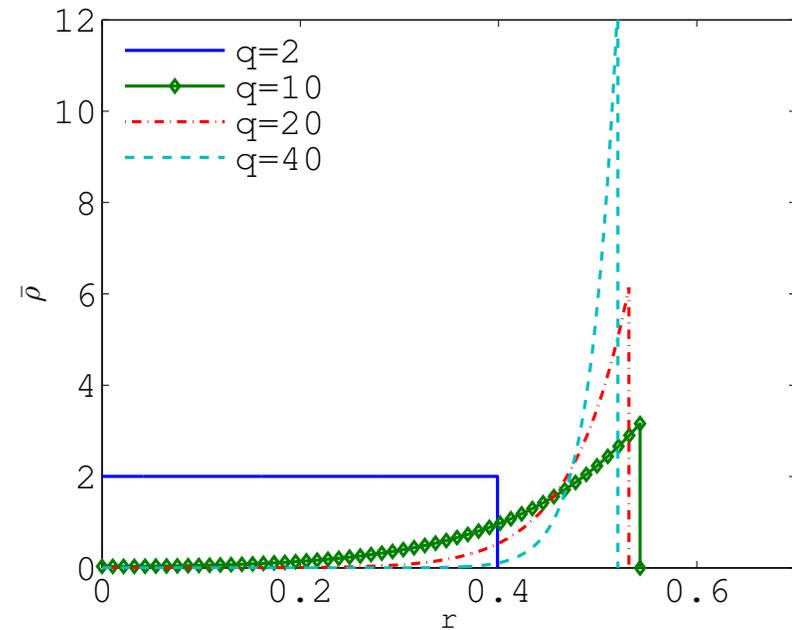
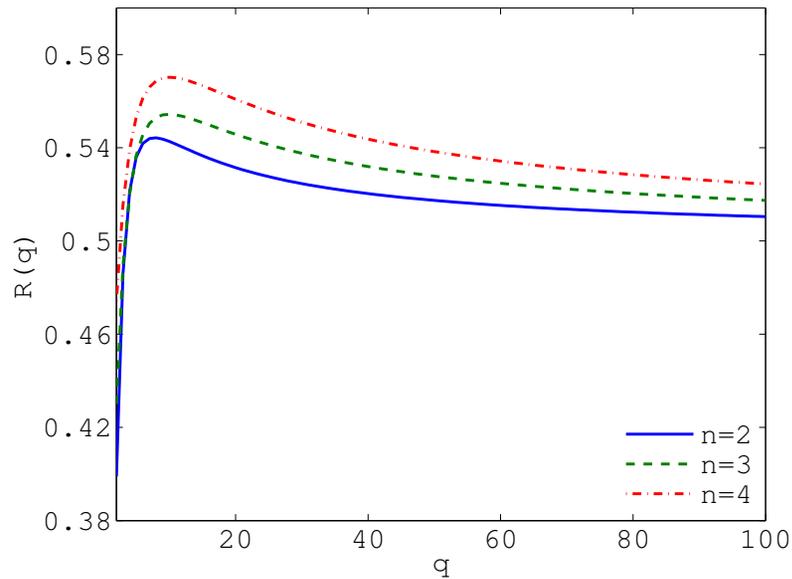
Ask that  $\bar{\rho}$  is an eigenfunction of  $T_R$  corresponding to e-value 1:

$$R = \lambda^{-\frac{1}{n+q-2}}$$

This gives the radius of the support as a function of  $q$  and  $n$ .

Once a mass  $M$  for  $\bar{\rho}$  is set, uniqueness can be inferred from the uniqueness properties of the spectral radius of  $T_1$  and its associated eigenfunction  $\bar{\rho}_1$ .

## Equilibria: numerical results



**Left:** Plot of the radius of the support  $R$  of the steady states as a function of the exponent  $q$ , for various space dimensions  $n$ .

The plot suggests that the radius  $R$  approaches a constant, as  $q \rightarrow \infty$ .

**Right:** Normalized radially symmetric steady states  $\bar{\rho}(r)$  in two dimensions for various values of the exponent  $q$ .

For  $q = 2$  the steady state is the constant solution in a disk. As  $q$  increases, mass aggregates toward the edge of the swarm, creating an increasingly void region in the centre.

## Even $q$ : polynomial steady states

Kernel  $I(r, r') = \int_0^\pi (r^2 + (r')^2 - 2rr' \cos \theta)^{q/2-1} \sin^{n-2} \theta d\theta$  is separable when  $q$  is even.

Define the  $i$ -th order moments of the density ( $m_0 = M$ ):

$$m_i = n\omega_n \int_0^R r^{n+i-1} \bar{\rho}(r) dr. \quad (3)$$

**Example:**  $q = 4$

$$I(r, r') = (r^2 + (r')^2) \int_0^\pi \sin^{n-2} \theta d\theta$$

and

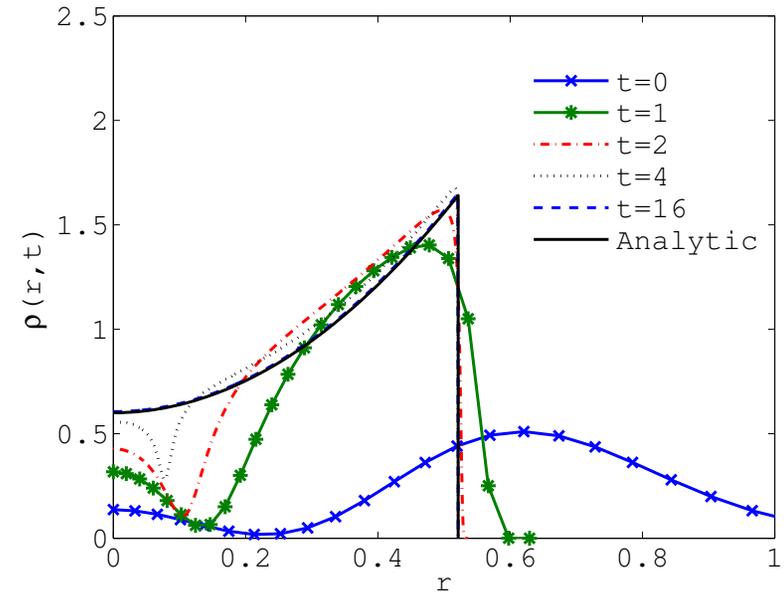
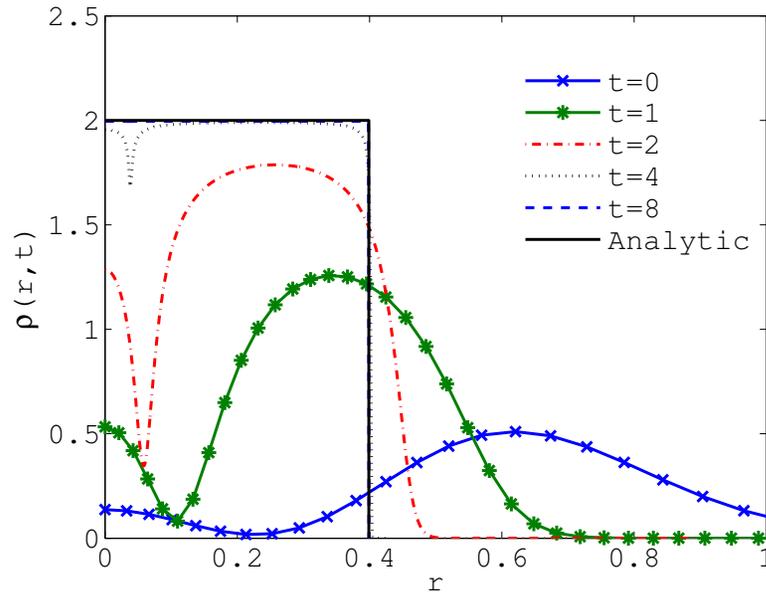
$$\begin{aligned} \bar{\rho}(r) &= n(n+2)\omega_n \int_0^R (r')^{n-1} (r^2 + (r')^2) \bar{\rho}(r') dr' \\ &= (n+2)m_0 r^2 + (n+2)m_2 \end{aligned} \quad (4)$$

Plug (4) into (3): linear system to find  $R$  and  $m_2$

$$\begin{pmatrix} m_0 \\ m_2 \end{pmatrix} = \begin{pmatrix} n\omega_n R^{n+2} & (n+2)\omega_n R^n \\ \frac{n(n+2)}{n+4}\omega_n R^{n+4} & n\omega_n R^{n+2} \end{pmatrix} \begin{pmatrix} m_0 \\ m_2 \end{pmatrix} \quad (5)$$

**General**  $q$  even:  $\bar{\rho}(r)$  is a polynomial of even powers of  $r$ , of degree  $q - 2$ .

## Dynamic evolution: numerical results



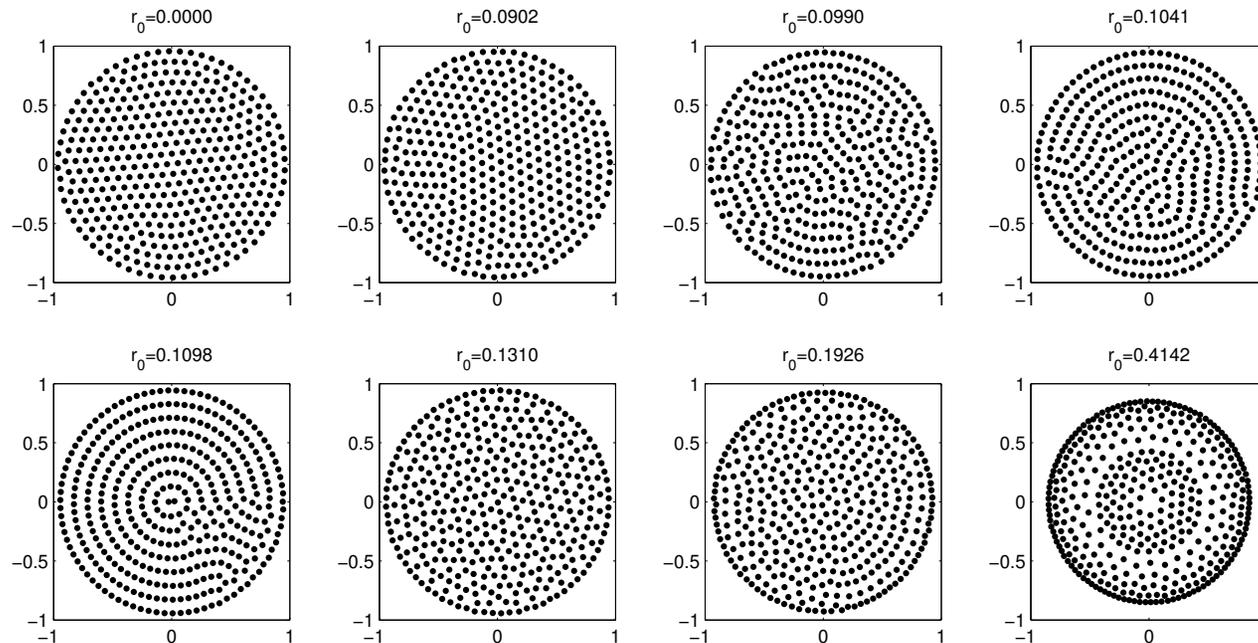
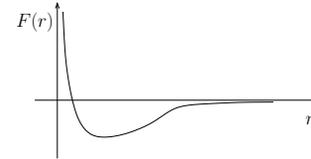
Time evolution of a radially symmetric solution to the aggregation model with  $q = 2$  (left) and  $q = 4$  (right) in two dimensions

**Left:** As predicted by the analytical results, the solution approaches asymptotically a constant, compactly supported steady state.

**Right:** The solution approaches asymptotically the steady state computed analytically

## Regularized potentials

$$F(r) = \frac{1}{r} - r \quad \xrightarrow{\text{regularize}} \quad F(r) = \begin{cases} C_1, & 0 \leq r < r_0 \\ \frac{1}{r} - r, & r_0 \leq r \leq 2 \\ -C_2 \exp(-r), & 2 < r \end{cases} \quad (6)$$



Equilibrium states for the regularized interaction force (6). Initial conditions were chosen at random in the unit square. For  $r_0 < 0.09$ , the steady state is the same as for  $r_0 = 0$  (uniform density in the unit circle).

## Inverse problem: custom designed potentials

*Inverse problem:* given a density  $\bar{\rho}(x)$ , can we find a potential  $K$  for which  $\bar{\rho}(x)$  is a steady state of the model?

*Answer:* Yes, provided  $\bar{\rho}(x)$  is radial and is a polynomial in  $|x|$ .

**Theorem:** In one dimension, consider an even density  $\bar{\rho}$  of the form

$$\bar{\rho}(x) = \begin{cases} b_0 + b_2x^2 + b_4x^4 + \dots + b_{2d}x^{2d} & |x| < R \\ 0 & \text{otherwise.} \end{cases}$$

Define the moments  $m_i$  as in (3). Then  $\bar{\rho}(x)$  is the steady state corresponding to the force  $F$ :

$$F(x) = \frac{1}{2} - \sum_{i=0}^d \frac{a_{2i}}{2i+1} x^{2i+1}$$

where the constants  $a_0, a_2, \dots, a_{2d}$ , are computed from  $b_0, b_2, \dots, b_{2d}$  by solving the following linear system:

$$b_{2k} = \sum_{j=k}^d a_{2j} \binom{2j}{2k} m_{2(j-k)}, \quad k = 0 \dots d. \quad (7)$$

Moreover, system (7) has a unique solution.

## Inverse problem: numerical results

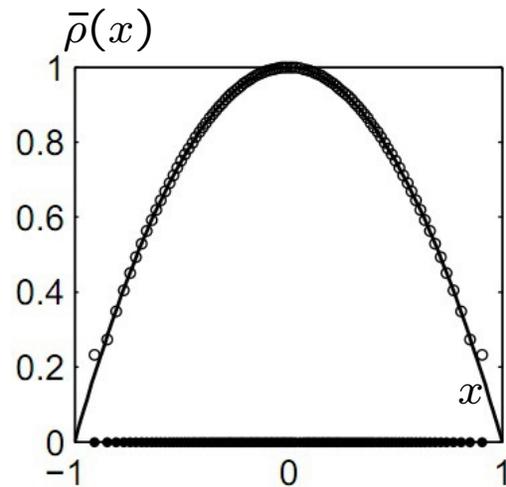
Examples:  $R = 1$  and:

$$(a) \quad \bar{\rho}(x) = 1 - x^2; \quad (b) \quad \bar{\rho}(x) = x^2; \quad (c) \quad \bar{\rho}(x) = \frac{1}{2} + x^2 - x^4$$

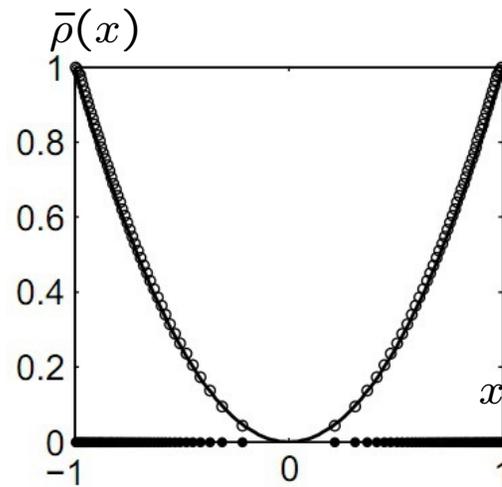
The corresponding forces given by the Theorem are:

$$(a) \quad F(x) = \frac{1}{2} - \frac{9}{10}x + \frac{1}{4}x^3; \quad (b) \quad F(x) = \frac{1}{2} + \frac{9}{10}x - \frac{1}{2}x^3;$$

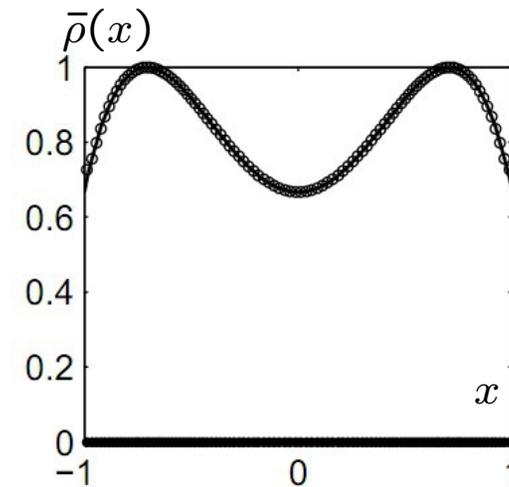
$$(c) \quad F(x) = \frac{1}{2} + \frac{209425}{672182}x - \frac{2075}{2527}x^3 + \frac{3}{19}x^5.$$



(a)



(b)



(c)

Filled circles along the  $x$ -axis: the steady states reached by numerical time evolution. Empty circles: density function as computed from the filled circles. Solid line: analytical expression for  $\bar{\rho}$ .

## Bibliography

1. R.C. Fetecau, Y. Huang and T. Kolokolnikov [2011]. Swarm dynamics and equilibria for a nonlocal aggregation model, *Nonlinearity*, Vol. 24, No. 10, pp. 2681-2716 (featured article)

## Other recent work / Future Directions

- Studied  $q < 2$ , in particular the case  $q \rightarrow 2 - n$ , when attraction becomes as singular as repulsion (Newtonian potential)
- Investigated properties of the steady states: monotonicity, asymptotic behaviour ( $q \rightarrow \infty$ ,  $q \rightarrow 2 - n$ )
- Energy considerations: local/ global minima

$$E[\rho] = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - y) \rho(x) \rho(y) dy dx$$

The model is a gradient flow with respect to this energy:

$$\frac{d}{dt} E[\rho] = - \int_{\mathbb{R}^n} \rho(x) |\nabla K * \rho(x)|^2 dx \leq 0$$