# On model-theoretic connected components in some group extensions 2

#### Jakub Gismatullin (joint work with Krzysztof Krupiński)

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- $(G, \cdot, \ldots)$  a group with some first order structure
- $G^*$  saturated extension of  $(G, \cdot, ...)$  (model monstrum,  $\overline{\kappa}$ -saturated,  $\overline{\kappa}$ -strongly homogeneus)
- $B \subset G^*$  some small set of parameters  $(|B| < \overline{\kappa})$

#### Definition

• 
$$G^{*0}_{B} = \bigcap \{ H < G^* : H \text{ is } B \text{-def. and } [G^* : H] < \omega \}$$

• 
$$G^{*00}_{B} = \bigcap \{ H < G^* : H \text{ is } B \text{-type def. and } [G^* : H] < \overline{\kappa} \}$$

•  $G^{*000}_{B} = \bigcap \{H < G^* : H \text{ is } \operatorname{Aut}(G^*/B) \text{-inv. and } [G^* : H] < \overline{\kappa} \}$ 

We say, that  $G^{*000}$  exists, if for every small  $B \subset G^*$ ,

$$G^*{}^{000}_B = G^*{}^{000}_\emptyset.$$

E.g. when G has NIP,  $G^{*000}$ ,  $G^{*00}$  and  $G^{*0}$  exist.

Let G be a group acting by automorphisms on an abelian group A, where G, A and the action of G on A are  $\emptyset$ -definable in a structure  $\mathcal{G}$ . Suppose

 $h: G \times G \to A$ 

is a 2-cocycle which is *B*-definable in  $\mathcal{G}$  and with finite image  $Im(h) \subset B$  (for some finite parameter set  $B \subset \mathcal{G}$ ). Denote  $A_0 = \langle Im(h) \rangle$ .

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#### Theorem

Assume that:

(i) the induced 2-cocycle  $\overline{h}$ :  $G_B^{*00} \times G_B^{*00} \to A_0 / (A^{*0} \cap A_0)$  is non-splitting, (ii)  $A_0 / (A^{*0} \cap A_0)$  is torsion free (and so  $\cong \mathbb{Z}^n$  for some natural n).

Then  $\widetilde{G^*}_B^{000} \neq \widetilde{G^*}_B^{00}$ .

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Assume that:

(i) the induced 2-cocycle h: G<sup>\*00</sup><sub>B</sub> × G<sup>\*00</sup><sub>B</sub> → A<sub>0</sub>/ (A<sup>\*0</sup> ∩ A<sub>0</sub>) is non-splitting,
(ii) A<sub>0</sub>/ (A<sup>\*0</sup> ∩ A<sub>0</sub>) is torsion free (and so ≅ Z<sup>n</sup> for some natural n). Then G<sup>\*000</sup><sub>B</sub> ≠ G<sup>\*00</sup><sub>B</sub>.
Suppose furthermore that G<sup>\*000</sup><sub>B</sub> = G<sup>\*</sup>, and for every proper, type-definable over B in G<sup>\*</sup> and invariant under the action of G<sup>\*</sup> subgroup H of A<sup>\*</sup> with bounded index, the induced 2-cocycle h: G<sup>\*</sup> × G<sup>\*</sup> → A<sub>0</sub>/ (H ∩ A<sub>0</sub>) is non-splitting. Then G<sup>\*00</sup><sub>B</sub> = G<sup>\*</sup>.

#### Corollary

#### Assume that:

- (1) The 2-cocycle h:  $G \times G \rightarrow A_0$  is non-splitting (via a function taking values in  $A_0$ ).
- (2)  $A^{*0} \cap A_0$  is trivial and  $A_0$  is torsion free (and so  $A_0 \cong \mathbb{Z}^n$  for some n).

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Then  $\widetilde{G}_{B}^{*00} \neq \widetilde{G}_{B}^{*00}$ .  
Under some additional assumptions, we also get  $\widetilde{G}_{B}^{*00} = \widetilde{G}^{*}$ .

Notation: for  $c, d \in \mathbb{R}$  define  $c(d) = \left\{ egin{array}{cc} c & : c 
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## Fact (Asai, '70)

The topological universal cover  $SL_2(\mathbb{R})$  of  $SL_2(\mathbb{R})$  is defined by means of the following 2-cocycle h:  $SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \rightarrow \{-1,0,1\} \subset \mathbb{Z}$ .

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Clearly, h is definable in  $(\mathbb{R}, +, \cdot, 0, 1, <)$ .

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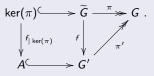
Clearly, *h* is definable in  $(\mathbb{R}, +, \cdot, 0, 1, <)$ .

### Fact (Hrushovski-Peterzil-Pillay, '11)

Let G be a definable Lie group, i.e. a definably connected group definable in an o-minimal expansion of RCF. The 2-cocycle  $h: G \times G \rightarrow \pi_1(G)$  corresponding to the topological universal cover  $\widehat{G}$  of G has finite image.

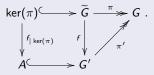
### Definition (Classical)

A central extension  $\ker(\pi)^{\smile} \longrightarrow \widetilde{G} \xrightarrow{\pi} G$  is called *universal* if for any central extension  $\pi' \colon G' \to G$  of G by A, there exists a unique homomorphism  $f \colon \widetilde{G} \to G$  such that  $\pi' \circ f = \pi$ , that is the following diagram commutes



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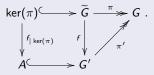
#### Fact (Classical)

{ central extensions of G by A}  $\approx$  { 2-cocycles h: G  $\times$  G  $\rightarrow$  A}  $\approx$  Hom(ker( $\pi$ ), A)

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Each perfect group G possesses a universal central extension, which is unique up to isomorphism over G.

Let k be an arbitrary infinite field. SL<sub>2</sub>(k) has a universal central extension

$$\ker(\pi) = \mathsf{K}_2^{\mathrm{sym}}(k) \xrightarrow{\pi} \mathsf{SL}_2(k) \xrightarrow{\pi} \mathsf{SL}_2(k).$$

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Let k be an arbitrary infinite field. SL<sub>2</sub>(k) has a universal central extension

$$\ker(\pi) = \mathsf{K}_2^{\mathrm{sym}}(k)^{\subset} \longrightarrow \mathsf{St}_2(k) \xrightarrow{\pi} \mathsf{SL}_2(k).$$

#### Theorem (Moore '68, Matsumoto '69)

The group  $K_2^{sym}(k)$  can be presented abstractly as

 $\langle c(x,y) \mid (S1), (S2), (S3) \rangle_{x,y \in k^{\times}},$ 

where c(x, y) for  $x, y \in k^{\times}$  are generators satisfying the following relations (S1) c(x, y) c(xy, z) = c(x, yz) c(y, z), (S2) c(1, 1) = 1,  $c(x, y) = c(x^{-1}, y^{-1})$ , (S3) c(x, y) = c(x, (1 - x)y) for  $x \neq 1$ . Let k be an arbitrary infinite field. SL<sub>2</sub>(k) has a universal central extension

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Let A be an abelian group. Then every homomorphism  $K_2^{\text{sym}}(k) \to A$  corresponds to a *symplectic Steinberg symbol*, that is a mapping  $c': k^{\times} \times k^{\times} \to A$  satisfying (S1), (S2) and (S3).

#### Corollary

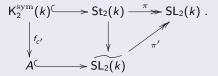
Let  $H: SL_2(k) \times SL_2(k) \to K_2^{sym}(k)$  be a 2-cocycle defining the universal central extension  $K_2^{sym}(k)^{\subset} \longrightarrow St_2(k) \xrightarrow{\pi} SL_2(k)$ , for a suitable section  $s: SL_2(k) \to St_2(k)$ .

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Let  $H: SL_2(k) \times SL_2(k) \to K_2^{sym}(k)$  be a 2-cocycle defining the universal central extension  $K_2^{sym}(k)^{\subset} \longrightarrow St_2(k) \xrightarrow{\pi} SL_2(k)$ , for a suitable section  $s: SL_2(k) \to St_2(k)$ . Let  $c': k^{\times} \times k^{\times} \to A$  be a symplectic Steinberg symbol. Then there exists a unique homomorphism  $f_{c'}: K_2^{sym}(k) \to A$  satisfying  $f_{c'}(c(x,y)) = c'(x,y)$  for all  $x, y \in k^{\times}$ .

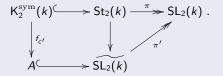
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#### Fact (Important for finite image)

Every value of the 2-cocycle H from the corollary is a linear combination of two Steinberg symbols. For example, if  $d_1c_2^2 + c_1a_2c_2 \neq 2$ ,  $H\left(\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix}, \begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix}\right) = c\left(-\frac{c_2}{c_1}, \frac{c_1}{d_1c_2^2+c_1a_2c_2}\right) - c\left(-\frac{c_2}{c_1}, \frac{1}{c_2}\right)$ .

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#### Theorem

Suppose that  $c': k^{\times} \times k^{\times} \to \mathbb{Z}$  is a symplectic Steinberg symbol such that c'(-1,-1) = 1, char(k) = 0 and

 $SL_2(\mathbb{Q}) < G < SL_2(k).$ 

Then  $H_{c'}$  restricted to G is a non-splitting 2-cocycle (actually a stronger result about  $H_{c'}$  is true).

# Examples

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Suppose  $(k, +, \cdot, <)$  is a an ordered field. The following mapping  $c' \colon k^{\times} \times k^{\times} \to \mathbb{Z}$  is a symplectic Steinberg symbol

$$c'(x,y) = \begin{cases} 1 & \text{if } x < 0 \text{ and } y < 0 \\ 0 & \text{otherwise} \end{cases}$$

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Let  $\mathcal{G} = ((\mathbb{Z}, +), (k, +, \cdot, <))$ ,  $\mathcal{G} = SL_2(k)$ ,  $A = (\mathbb{Z}, +)$  and  $B = \{-1, 0, 1\} \subseteq \mathbb{Z}$ (the action of  $\mathcal{G}$  on A is trivial). Suppose  $\widetilde{\mathcal{G}}$  is defined by means of the 2-cocycle  $H_{c'}$ :  $SL_2(k) \times SL_2(k) \rightarrow \mathbb{Z}$ .

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$$\widetilde{G^*}_B^{000} \neq \widetilde{G^*}_B^{00} = \widetilde{G^*}.$$

Moreover, the quotient  $\widetilde{G^*}_B^{00}/\widetilde{G^*}_B^{00}$  is abelian. In fact,  $\widetilde{G^*}_B^{000} = (\mathbb{Z}^{*0} + \mathbb{Z}) \times G^*$ , and  $\widetilde{G^*}_B^{00}/\widetilde{G^*}_B^{000}$  is isomorphic to  $\widehat{\mathbb{Z}}/\mathbb{Z}$ , where  $\widehat{\mathbb{Z}}$  is the profinite completion of  $\mathbb{Z}$ .

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Example (1) generalizes to any group G such that  $SL_2(\mathbb{Q}) < G < SL_2(k)$ .

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There exists an extension  $\widetilde{G}$  of  $SL_2(k)$  by  $SO_2(k)$  which is definable in  $\mathcal{G} := (k, +, \cdot, <)$  and such that  $\widetilde{G^*}_B^{00} \neq \widetilde{G^*}_B^{000}$ .

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#### Example

Let  $g \in SO_2(k)$  be of infinite order and  $B := \{-g, 0, g\}$ . Consider the following 2-cocycle  $H' : SL_2(k) \times SL_2(k) \rightarrow SO_2(k)$ ,

$$H'(x,y) = H_{c'}(x,y) \cdot g.$$

Then  $\widetilde{G}$  is definable in  $(k, +, \cdot, <)$  and  $\widetilde{G^*}_B^{00} \neq \widetilde{G^*}_B^{000}$ .

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Take G, c' and B from Example (1), and suppose

$$\operatorname{ker}(f) \xrightarrow{f} H \xrightarrow{f} G$$

is an extension of G by ker(f).

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Let  $\mathcal{H}$  be any expansion of  $\mathcal{G}$  in which H and f are  $\emptyset$ -definable (e.g.  $\mathcal{H}$  is the expansion of  $\mathcal{G}$  by the new sort H together with the function f), and let

 $\mathcal{H}^* \succ \mathcal{H}$ 

be a monster model. Assume additionally that  $Hom(ker(f^*), \mathbb{Z})$  is trivial.

#### Example

Take G, c' and B from Example (1), and suppose

$$\operatorname{ker}(f) \xrightarrow{f} G$$

is an extension of G by ker(f).

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$$\mathcal{H}^* \succ \mathcal{H}$$

be a monster model. Assume additionally that  $\operatorname{Hom}(\ker(f^*),\mathbb{Z})$  is trivial. Put

$$h' := H_{c'} \circ (f, f) \colon H \times H \to \mathbb{Z}$$

a 2-cocycle definable in  $\mathcal{H}$  over B. Let  $\widetilde{\mathcal{H}}$  be the extension of H by  $\mathbb{Z}$  corresponding to h'. Then

$$h'_{|H^*{}^{00}_B \times H^*{}^{00}_B} \colon H^*{}^{00}_B \times H^*{}^{00}_B \to \mathbb{Z}$$

is non-splitting, and so  $\widetilde{H^*}_B^{000} \neq \widetilde{H^*}_B^{00}.$ 

 If k is not formally real and char(k) ≠ 2, then one can prove that every symplectic Steinberg symbol c': k<sup>×</sup> × k<sup>×</sup> → Z with finite image is trivial.

- If k is not formally real and char(k) ≠ 2, then one can prove that every symplectic Steinberg symbol c': k<sup>×</sup> × k<sup>×</sup> → Z with finite image is trivial.
- We have almost generalized our result from SL<sub>2</sub>(k) to all symplectic groups Sp<sub>n</sub>(k), where k is an arbitrary ordered field.

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