Kolakoski (1944–1997)

American Mathematical Monthly (AMM, 1965)

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

$$1\ 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1 \cdots.$$

What is the nth term? Is the sequence periodic?

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122112122122112112212112122112112122122112

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Solved by Walter Bluger, H. Brandt Corstius* (Netherlands), Paul Cull*, Jack Dix, R. F. Jackson, Norman Miller, Julius Nadas, C. E. Olson, C. B. A. Peck, J. R. Purdy, Donald Quiring, and Judith Richman. Nadas generalized the given problem by considering sequences as above but using r integers, Miller observed that the first 42 digits in the given sequence of I's and 2's is obtained when each vowel is replaced by 1, each consonant by 2 in the following: Indian and Ethiopian, Syrian, Israeli, Arab, Persian.

Solution by Necdet Üçoluk, Clarkson College of Technology. Let $\{x_n\}$ be a sequence not eventually constant. One can consider a sequence of blocks over $\{x_n\}$ by grouping all consecutive equal numbers in the same blocks. The lengths of these blocks will give a sequence $\{\hat{x}_n\}$ associated with $\{x_n\}$. The given sequence $\{a_n\}$ can be defined as follows: $a_1=1$, $a_n=1$ or 2, and $\{a_n\}=\{\hat{a}_n\}$, for all $n=1, 2, \cdots$. So $\{a_n\}$ is constructed uniquely. Indeed, $a_1=1$, so $\hat{a}_1=1$, hence the first block of $\{a_n\}$ is of length only 1, therefore $a_2\neq 1$, so $a_2=2$. Since $\hat{a}_2=a_2=2$, the second block in $\{a_n\}$ will consist of two 2's, so a_3 and then $\hat{a}_3$ equals 2. Since the third block in $\{a_n\}$ must start with 1, it will consist of two 1's, because $\hat{a}_3=2$. Therefore $a_4=a_5=1$. Continuing in this fashion, $\{a_n\}$ is constructed recursively. If $\{s_n\}$ is the sequence of partial sums of $\sum \hat{a}_n$, then

$a_n=\frac{1}{2}(3+(-1)^m)$ with $s_{m-1}<n\leqq s_m$, since consecutive blocks contain different numbers, odd-numbered blocks contain 1's while the others consist of 2's.

$\{a_n\}$ cannot be periodic. If $\{a_n\}$ were a periodic sequence, say after $n=n_0$, with the minimum period N, then $\{\hat{a}_n\}$ would be periodic after n_0 with the period N. The periodicity of $\{\hat{a}_n\}$ would induce another period $N_1>N$ for $\{a_n\}$ after a certain index n_1 (actually $n_0<n_1<2n_0$). But $N<N_1<2N$, since $\{a_n\}$ can have only blocks of length one or two, therefore a segment of $\{\hat{a}_n\}$ having N elements produces a segment in $\{a_n\}$ of length more than N but less than $2N$. This contradicts the fact that N is the minimum period for $\{a_n\}$. But N_1 must also be a multiple of N, and the non-periodicity of $\{a_n\}$ now follows from this contradiction.

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But .....

among the many solutions

1221121221221121122121121221121121221221*

which one did he really expect?

What did William Kolakoski have in mind?

An old question of Allouche

An old question of Allouche

From jpallouc@graceland.uwaterloo.ca Thu Jul 23 17:19 EDT 1998
To: brlek@math.uqam.ca
Subject: hello

Bonjour Srecko, comment vas-tu ?
je suis a Waterloo avec Jeff et nous bossons comme des betes sur notre "fameux" bouquin. [A part 3 jours a SIAM a Toronto et 3 jours de ... camping pour la 1ere fois de ma vie].

J'ai deux questions pour toi :

-- la suite des "runs" dans Thue-Morse qui t'a permis de calculer la complexite d'icelle est-elle a ta connaissance apparue quelque part avant ton papier ?

………

meilleures amities et bonjour de Waterloo
jp

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Run-Length Encoding (RLE)

RLE history

RLE history

A.E. Laemmel 1951: Coding processes for Bandwith Reduction in Picture transmission

The basic idea in this scheme is to transmit only the lengths of the black and white runs in a picture as they occur in successive scanning lines.

Implemented
M. A. Treuhaft, 1953 : "Description of a System for Transmission of Line Drawings with Bandwidth-Time Compression,"

J. Capon 1959 : Probabilistic model for run-length coding

The process of scanning reduces a picture from a twodimensional array of cells (resolution elements) to a onedimensional sequence of cells. In the case of a black and white picture such a sequence would consist of a succession of black and white cells. A section of this sequence might appear as below.

. . . BBBWWBBBBBBWWWWBWBBWBWW . . .

*Thus, in our subsequent discussion, when we use the word "picture" we shall, in reality, be referring to a onedimensional
sequence of cells which results from scanning a picture.*

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Huffman 1972 : introduced minimum redundancy codes

CCITT T4 1980 (adopted in 1988)

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The engineering world

...... in Symbolic Dynamics

Morse and Hedlund (1938) p824

A trajectory can have more than one distinct limit trajectory. Letting a^n stand for a block $a \cdots a$ of length n we see that a trajectory of the form

$$(6.2) \qquad \cdots \beta^4\alpha^3\beta^2\alpha\beta^2\alpha^3\beta^4 \cdots$$

has the trajectories $\cdots \alpha\alpha\alpha \cdots$ and $\cdots \beta\beta\beta \cdots$ as limit trajectories. More generally let the set of admissible blocks be enumerated in the form

...... previously?

Morse and Hedlund (1938) p843

9. The derivation of recurrent trajectories.

Projection. The trajectory obtained from (c) by omitting a_k whenever it occurs in (c) will be said to be the k-th projection of (c).

Reduction. If the generating symbols consist of a finite set of integers, the trajectory obtained by reducing the elements c_j mod p will be said to be derived by reduction mod p.

Association. Let m be a positive integer at least 2, and let B_i be the m-block of (c) whose first index is i. There is at most a finite set of different blocks B_i. We introduce the I-trajectory

$$\text{(B)} \qquad \cdots B_{-1}B_0B_1 \cdots,$$

regarding the blocks B_i as the generating symbols.

Substitution. Let $A_1, \cdots, A_\mu$ be s-blocks of the generating symbols, $s > 0$. The trajectory (c^*) of a_i's obtained by replacing a_i in (c) by A_i and expanding will be said to be derived from (c) by block substitution.

Generalized substitution. Let $r_1, \cdots, r_m$ be a fixed set of integers, positive, negative, or zero. Let $\phi(x_1, \cdots, x_m)$ be a single-valued function (either

but they forgot RLE

Morse and Hedlund (1938) p824

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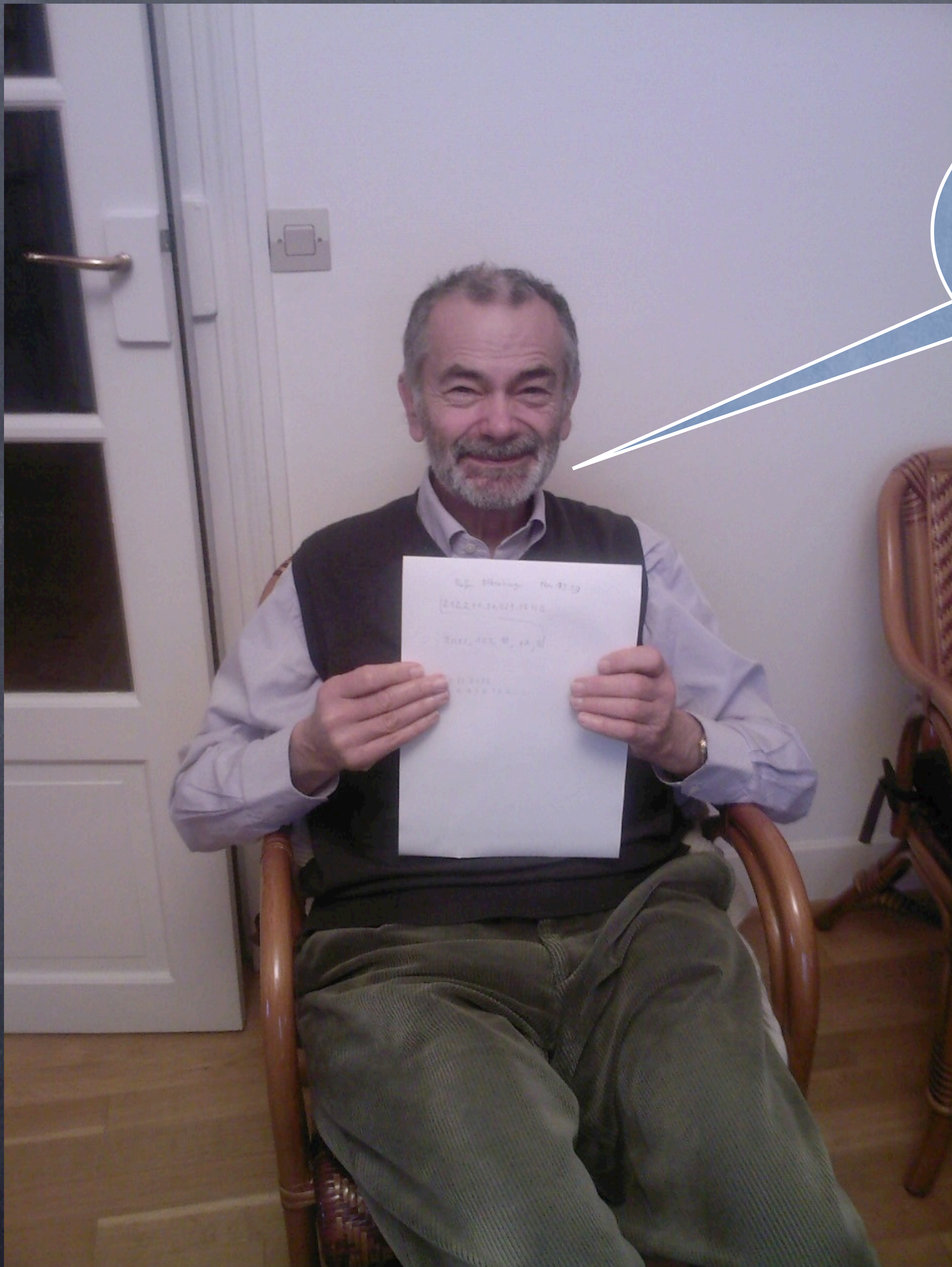
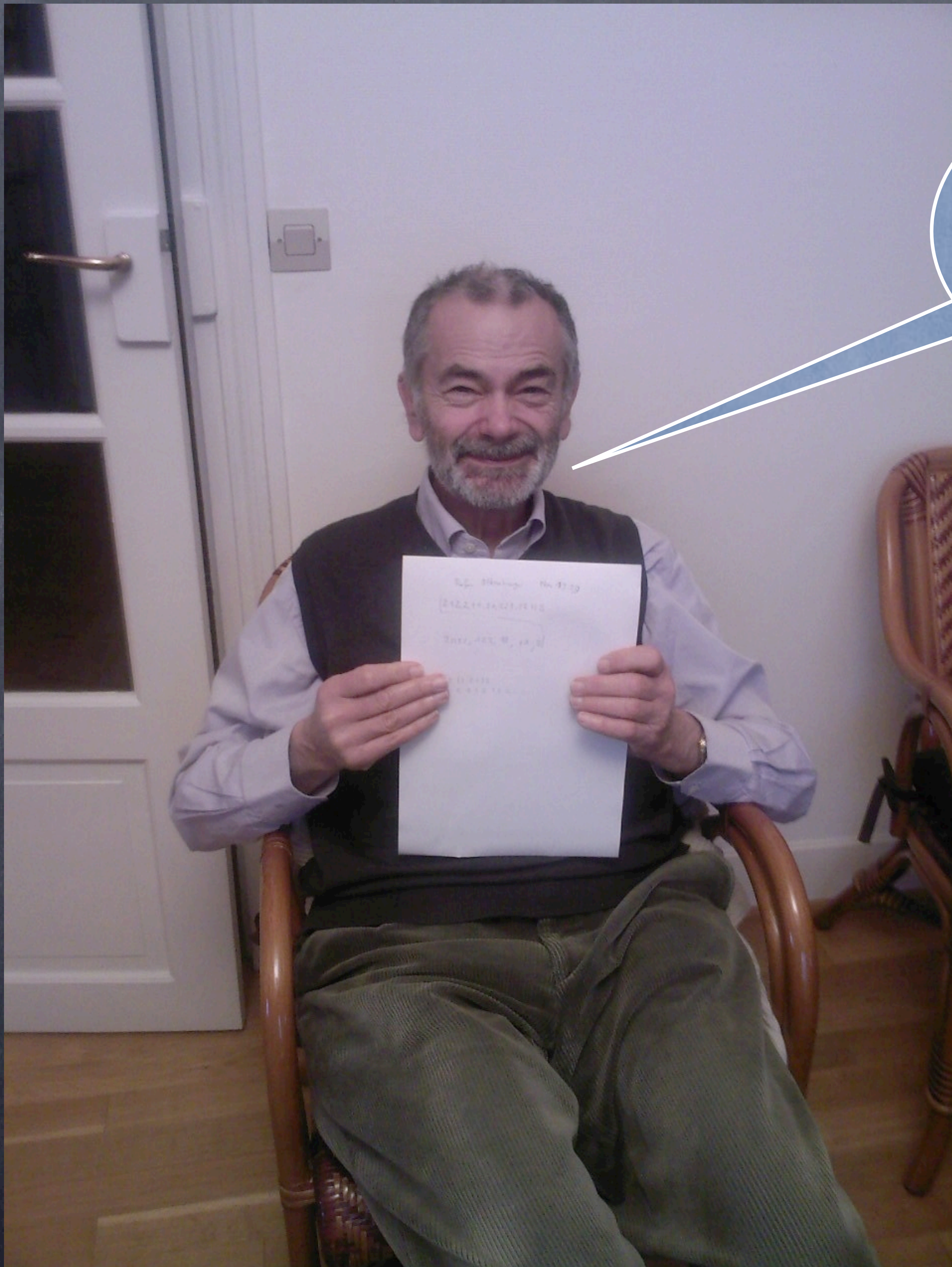


Hello
Everybody

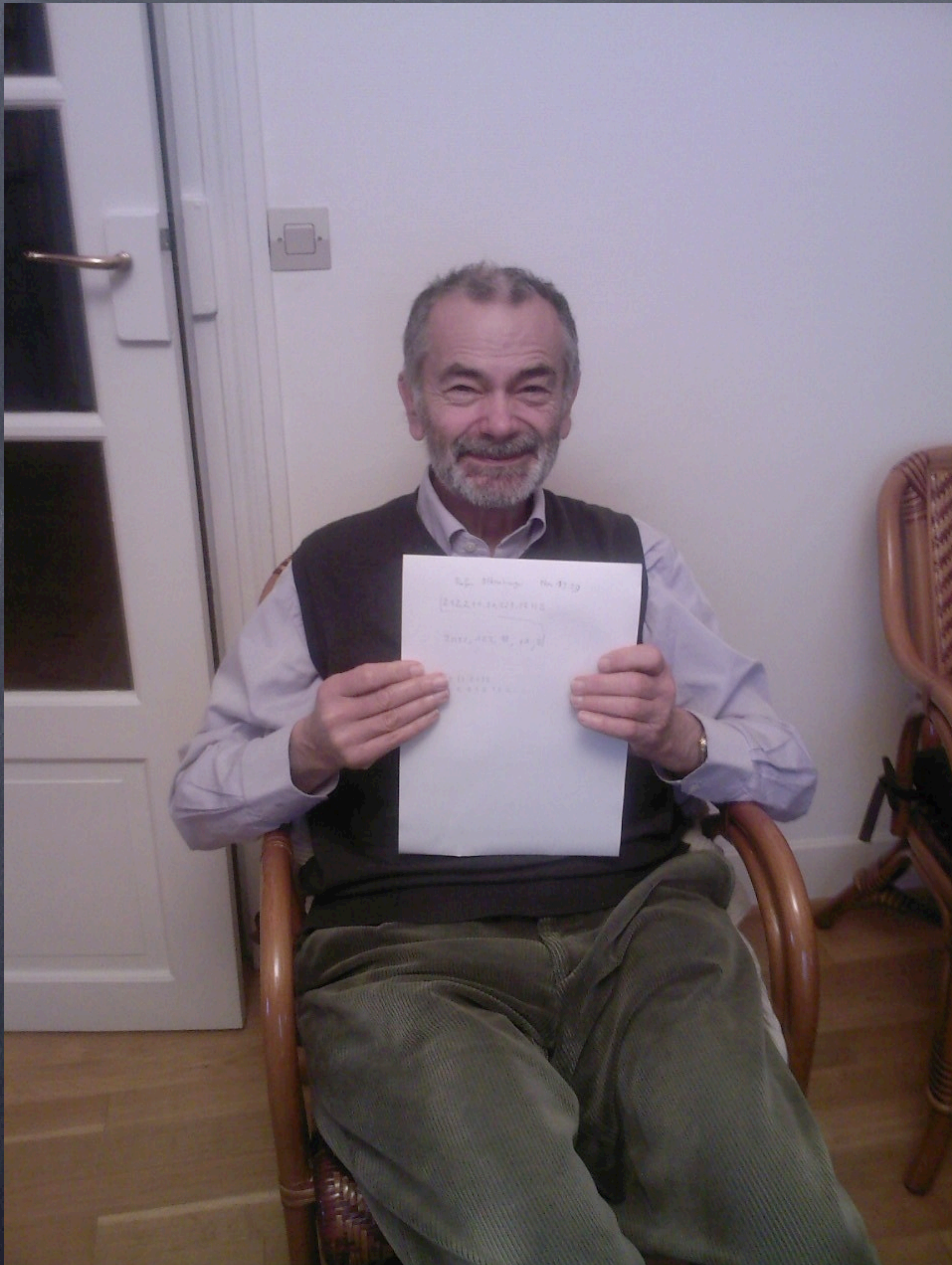
on January 28th, 2012



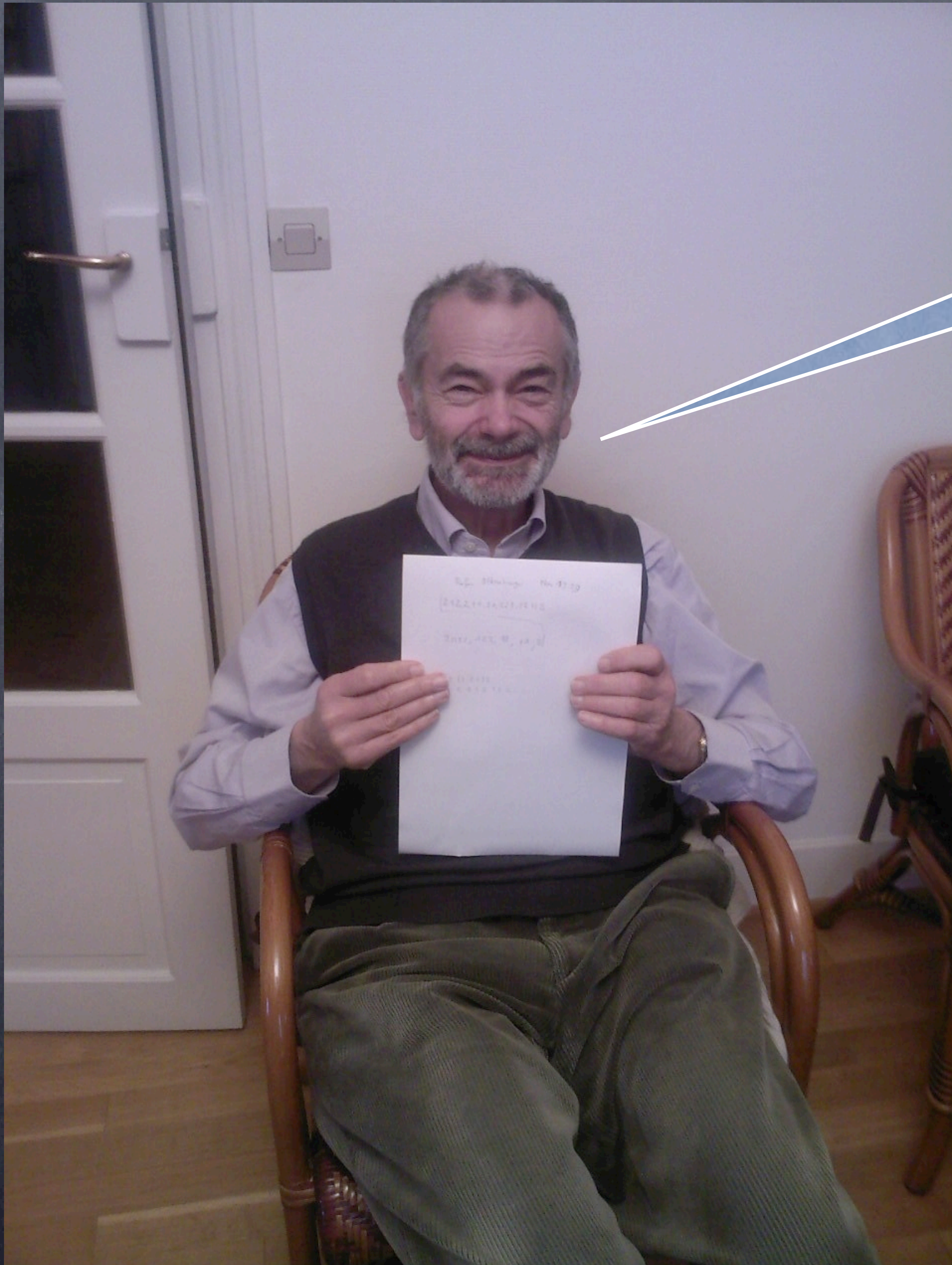
Hello
Everybody

Jean Berstel: Bonjour tout le monde

on January 28th, 2012



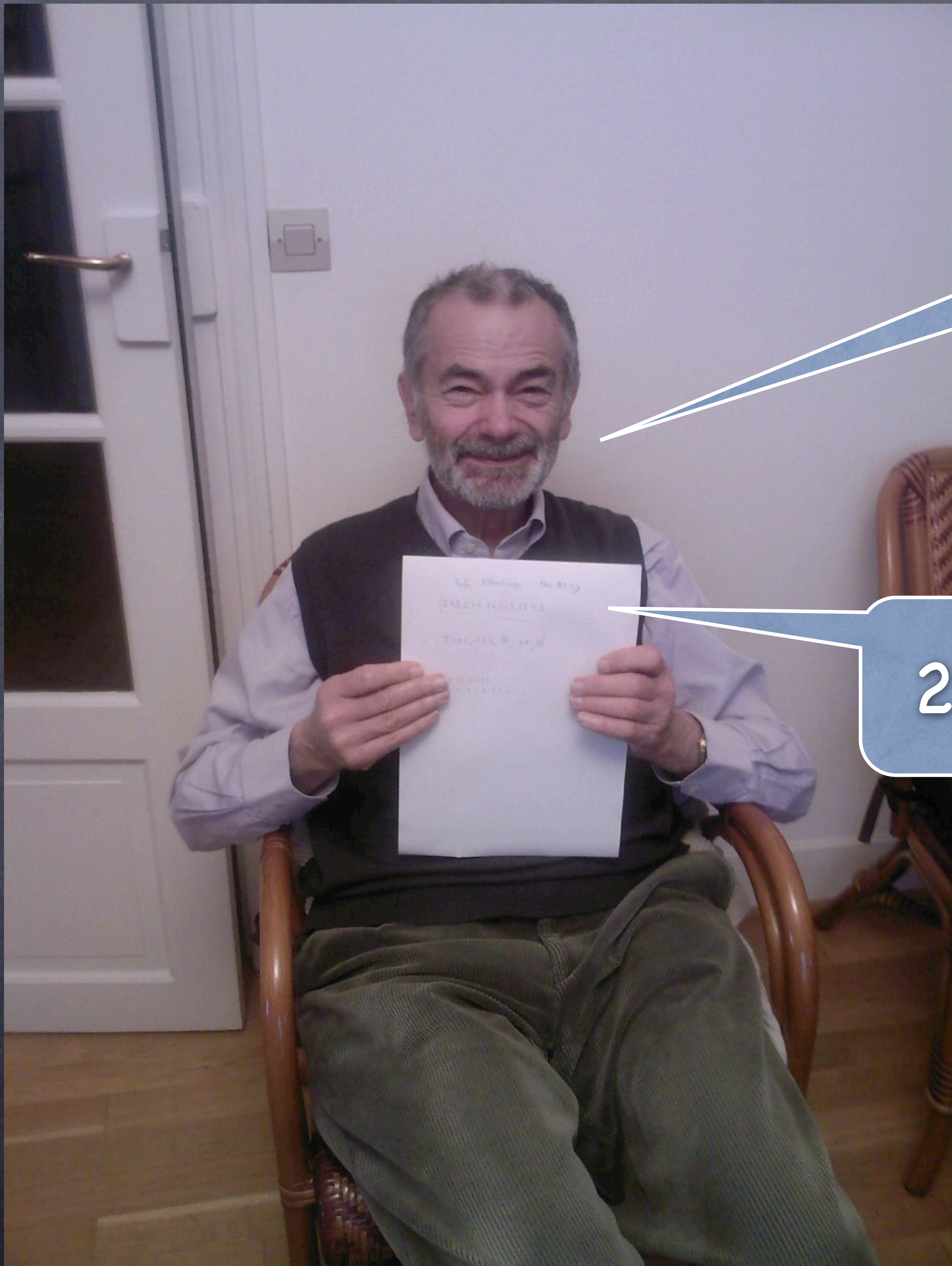
on January 28th, 2012



Est ce que tu connais ça?

Jean Berstel: Do you know this?

on January 28th, 2012

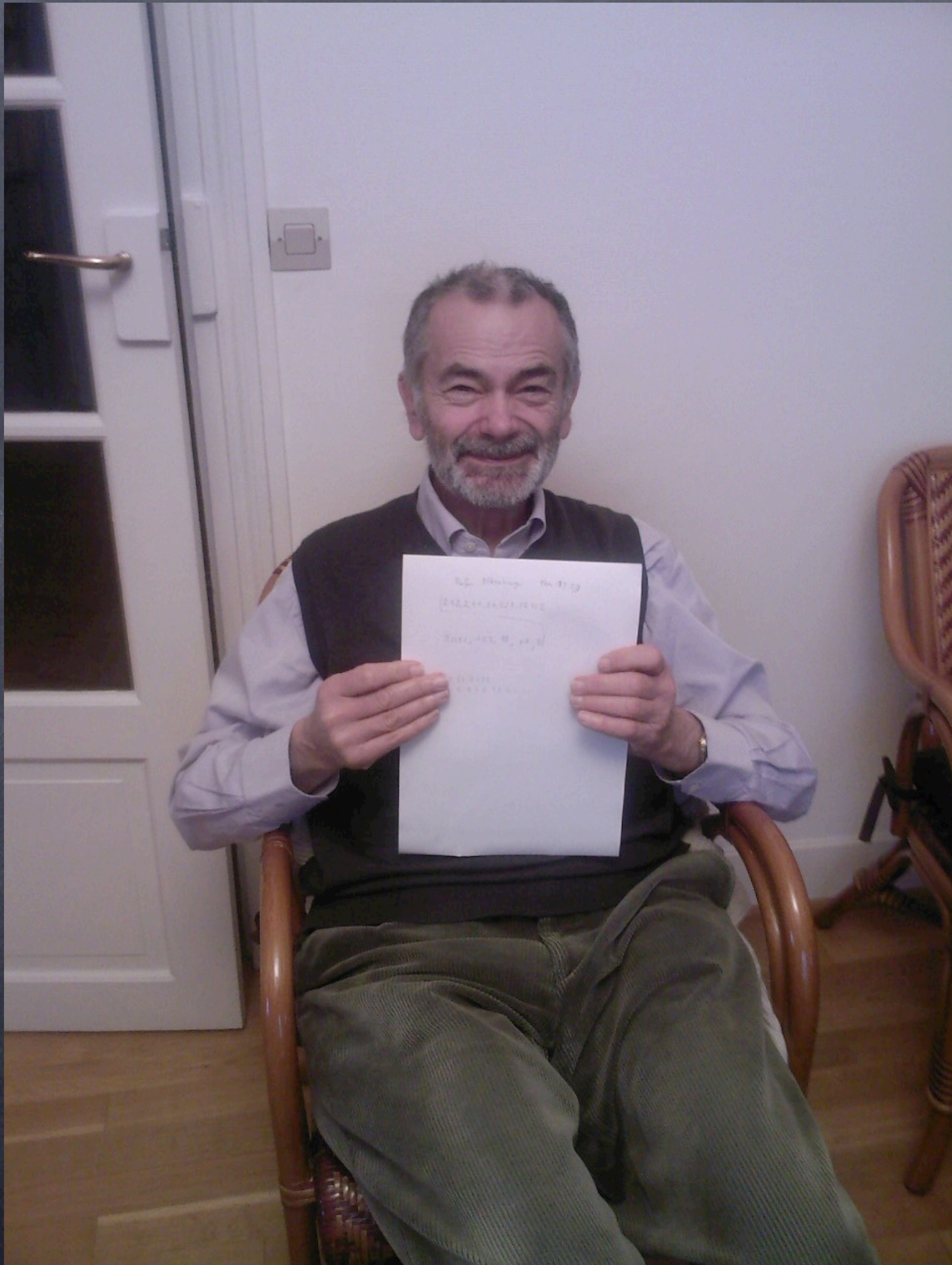


Est ce que tu connais ça?

212,2,11,21,221,22112,11....

Jean Berstel: Do you know this?

on January 28th, 2012



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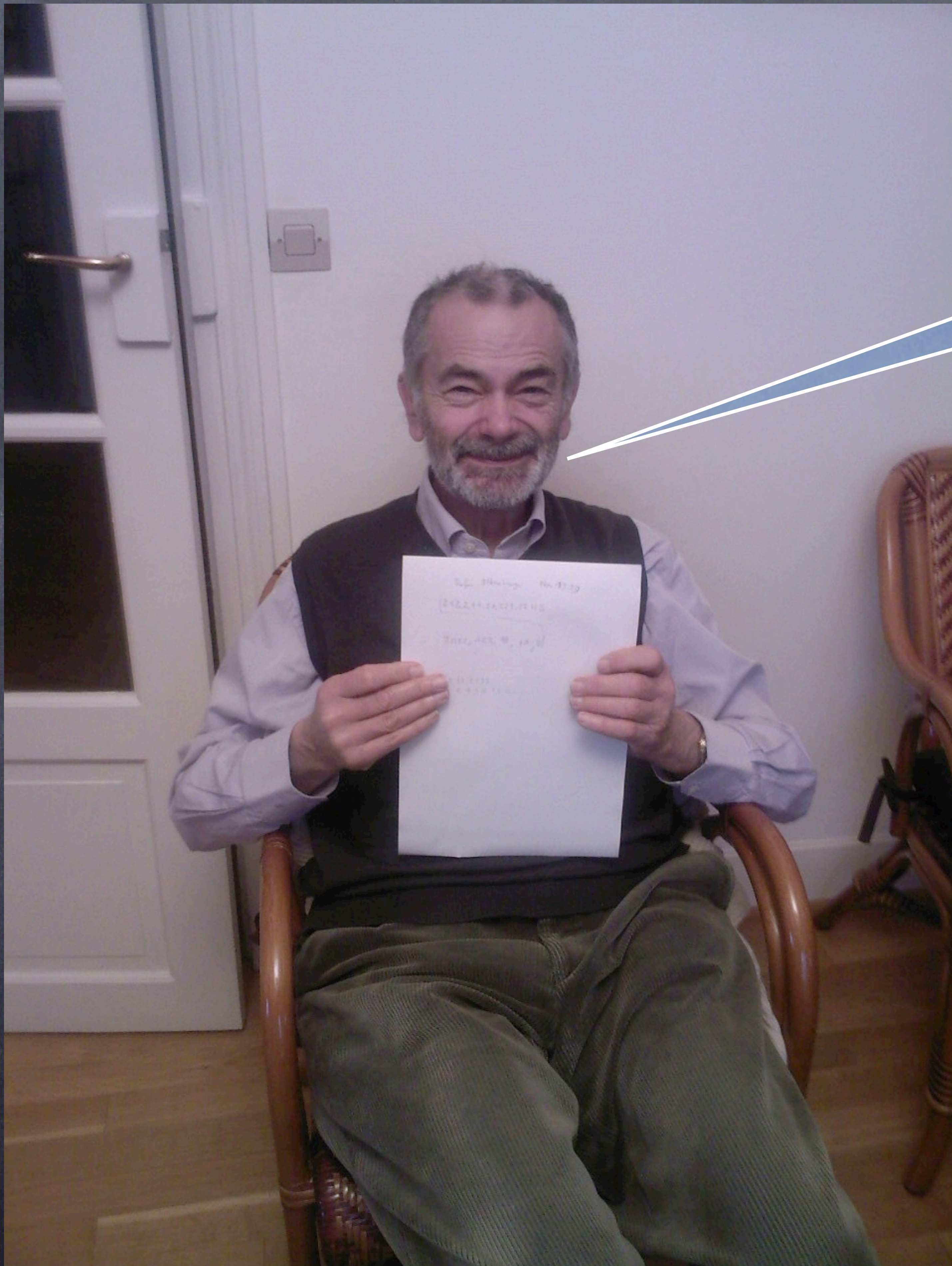

Est ce que tu
connais
Rufus
Oldenburger?

Jean Berstel: Do you know Rufus ....?

on January 28th, 2012

Who is Rufus Oldenburger?

From Wikipedia, the free encyclopedia

The **Rufus Oldenburger Medal** is an award given by the American Society of Mechanical Engineers recognizing significant contributions and outstanding achievements in the field of automatic control. It was established in 1968 in the honor of Rufus Oldenburger.

[edit] Recipients

- 1968: Rufus Oldenburger
- 1969: Nathaniel B. Nichols
- 1970: John R. Ragazzini
- 1971: Charles Stark Draper
- 1972: Albert J. Williams, Jr.
- 1973: Clesson E. Mason
- 1974: Herbert W. Ziebolz
- 1975: Hendrik Wade Bode and Harry Nyquist
- 1976: Rudolf E. Kalman
- 1977: Gordon S. Brown and Harold L. Hazen
- 1978: Yasundo Takahashi
- 1979: Henry M. Paynter

SYMBOLIC DYNAMICS

Lectures by Marston Morse

1937-1938

Notes by Rufus Oldenburger

Edition with preface, 1966

Not previously published

The Institute for Advanced Study

Princeton, New Jersey

Morse and Hedlund (1938) p852

It is conceivable that there might exist a transitive ray for which $\phi(n) = \theta(n)$ for all values of n exceeding some fixed integer m. That this is impossible is shown by the following previously unpublished theorem of R. Oldenburger.

THEOREM 11.1. *If the generating symbols reduce to two free symbols* α *and* β *there exists no transitive ray whose ergodic function* $\phi(n)$ *equals the covering index* $\theta(n)$ *for two successive values of* $n > 1$.

PDF | Clipboard | Journal | Article

MR0000593 (1,99a) Oldenburger, Rufus Factorability of general symmetric matrices. *Compositio Math.* **7,** (1939). 223–228. (Reviewer: C. C. MacDuffee), 15.0X
PDF | Clipboard | Journal | Article

MR0000352 (1,59c) Oldenburger, Rufus Exponent trajectories in symbolic dynamics. *Trans. Amer. Math. Soc.* **46,** (1939), 453–466. (Reviewer: G. A. Hedlund), 46.3X
PDF | Clipboard | Journal | Article

MR1503395 Oldenburger, Rufus Relations between ranks of a general matrix. *Ann. of Math. (2)* 39 (1938), no. 1, 172–177, DML Item
PDF | Clipboard | Journal | Article

MR1501968 Oldenburger, Rufus Rational equivalence of a form to a sum of pth powers. *Trans. Amer. Math. Soc.* 44 (1938), no. 2, 219–249, 11E76
PDF | Clipboard | Journal | Article

MR1563584 Oldenburger, Rufus; Real canonical binary symmetric trilinear forms. *Bull. Amer. Math. Soc.* 43 (1937), no. 8, 546–553, DML Item
PDF | Clipboard | Journal | Article

MR1507256 Oldenburger, Rufus; Real Canonical Binary Trilinear Forms. *Amer. J. Math.* 59 (1937), no. 2, 427–435, DML Item
PDF | Clipboard | Journal | Article

MR1563456 Oldenburger, Rufus; On arithmetic invariants of binary cubic and binary trilinear forms. *Bull. Amer. Math. Soc.* 42 (1936), no. 12, 871–873, DML Item
PDF | Clipboard | Journal | Article

MR1545957 Oldenburger, Rufus; Equivalence of multilinear forms singular on one index. *Duke Math. J.* 2 (1936), no. 4, 671–680, DML Item
PDF | Clipboard | Journal | Article

MR1501856 Oldenburger, Rufus Non-singular multilinear forms and certain p-way matrix factorizations. *Trans. Amer. Math. Soc.* 39 (1936), no. 3, 422–455, 15A23
PDF | Clipboard | Journal | Article

MR2611305 Oldenburger, Rufus; COMPOSITION AND RANK OF N-WAY MATRICES AND MULTILINEAR FORMS. Thesis (Ph.D.)–The University of Chicago. 1934. (no paging), *ProQuest LLC,* Thesis
PDF | Clipboard | Series | Thesis

MR1523115 Oldenburger, Rufus; Transposition of Indices in Multiple-Labeled Determinants. *Amer. Math. Monthly* 41 (1934), no. 6, 350–356, DML Item
PDF | Clipboard | Journal | Article

MR1503184 Oldenburger, Rufus Composition and rank of n-way matrices and multilinear forms—supplement. *Ann. of Math. (2)* 35 (1934), no. 3, 654–657, DML Item
PDF | Clipboard | Journal | Article

MR1503183 Oldenburger, Rufus Composition and rank of n-way matrices and multilinear forms. *Ann. of Math. (2)* 35 (1934), no. 3, 622–653, DML Item
PDF | Clipboard | Journal | Article

MR0000352 (1,59c) 46.3X

Oldenburger, Rufus

Exponent trajectories in symbolic dynamics.

Trans. Amer. Math. Soc. **46,** (1939), 453–466

A symbolic trajectory is a sequence $\cdots a_{-1}a_0a_1a_2\cdots$, where the symbol a_i is chosen from a finite or infinite set of generating symbols. In general T can be considered as a sequence of finite blocks, where the symbols in any one block are the same and the symbols in adjacent blocks are different. The lengths of (number of symbols in) these blocks then form a sequence of integers T_e which is called the exponent trajectory of T. The paper derives a number of relations between T and T_e. If T is periodic or recurrent, T_e is periodic or recurrent, respectively. The converse statements do not hold in general but do hold if T has just two generating symbols. A considerable part of the paper is devoted to a proof that there is one and only one trajectory T in generating symbols 1 and 2 which is identical with its exponent trajectory.

Reviewed by *G. A. Hedlund*

EXPONENT TRAJECTORIES IN SYMBOLIC DYNAMICS*

BY

RUFUS OLDENBURGER

form the *exponent block* B_e of B. Unless a trajectory T contains a subray formed by only one generating symbol, T can be written as a sequence

$$\cdots a^p b^q c^r \cdots , \tag{1}$$

where no two consecutive bases are identical. The exponents in (1) form a trajectory $\cdots pqr \cdots$, which we term the *exponent trajectory* T_e of T. Similarly, if a ray R does not contain a subray formed by one generating symbol, the ray R can be written as $a^p b^q c^r \cdots$, where consecutive bases are distinct. The exponents then form the *exponent ray* R_e of R. A trajectory T (or ray R) will be termed *admissible* if it has an exponent trajectory (or ray); that is, T (or R) does not contain a subray of the form $aaa \cdots$ or $\cdots aaa$.

Periodicity

THEOREM 1. *If a trajectory T in two or more generating symbols is periodic, T is admissible and the exponent trajectory T_e is periodic.*

THEOREM 2. *The exponent trajectory T_e of an admissible periodic trajectory T is distinct from T.*

THEOREM 3. *An admissible trajectory T with two generating symbols is periodic if and only if its exponent trajectory T_e is periodic.*

THEOREM 4. *If the exponent trajectory T_e of a periodic trajectory T in two generating symbols has the period block $a_1a_2 \cdots a_\xi$, the trajectory T has the period ω, where*

$$\omega = \sum_{j=1}^{\xi} a_j, \tag{5}$$

$$\omega = 2\left(\sum_{j=1}^{\xi} a_j\right), \tag{6}$$

according as ξ is even or odd.

Recurrence

Lemma 1. *A recurrent trajectory T with two or more generating symbols is admissible.*

Lemma 2. *If an admissible trajectory T is recurrent, its exponent trajectory contains a finite number of generating symbols.*

Theorem 5. *If an admissible trajectory T is recurrent, the exponent trajectory T_e of T is recurrent.*

Corollary 1. *If T is a recurrent nonperiodic trajectory in two generating symbols, the exponent trajectory T_e of T is a recurrent nonperiodic trajectory.*

Theorem 6. *An admissible trajectory T in two generating symbols is recurrent if and only if its exponent trajectory is strongly recurrent.*

Theorem 7. *A transitive ray in two or more generating symbols is admissible.*

Theorem 8. *The exponent ray R_e of a transitive ray R in two or more generating symbols is transitive.*

4. **A trajectory identical with its exponent trajectory.** In Theorem 2 we noted that a periodic trajectory is distinct from its exponent trajectory. That this is not true for trajectories in general is a consequence of the theorem which follows.

THEOREM 9. *There exists a trajectory identical with its exponent trajectory.*

(7) $$\cdots B_3^{-1} B_2^{-1} B_1^{-1} B_0 B_1 B_2 B_3 \cdots ,$$

where B_i is the exponent block of B_{i+1} for each $i>0$, the last symbol of B_i is distinct from the first symbol of B_{i+1} for each $i>0$, and B_i^{-1} denotes the block obtained from B_i by reversing the symbols in B_i. We illustrate by giving some of the blocks B_i explicitly:

$$B_2 = 11,\ B_3 = 21,\ B_4 = 221,\ B_5 = 22112,\ B_6 = 11221211.$$

$B_0 = 212$; $B_1 = 2$; $B_i = \Delta(B_{i+1})$

B

THEOREM 15. *There is one and only one trajectory T in generating symbols 1, 2 identical with its exponent trajectory.*

The Oldenburger trajectory !!!!

...... 112122,21122,122,12,11,2,212,2,11,21,221,22112,11221211 ......

What's left?

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Coming soon: The first talk on Oldenburger's exponent trajectories