Unrepetitive paths in digraphs

(and the repetition threshold)

Arturo Carpi

Dipartimento di Matematica e Informatica
Università di Perugia

Outstanding Challenges in Combinatorics on Words

Banff, February 19–24, 2012

Unrepetitive graph coloring

- A bi-inifnite word can be viewed as a coloring of edges (or nodes) of $\mathrm{Cay}(\mathbb{Z}; 1)$
- Its factors are the coloring of finite simple paths

Unrepetitive graph coloring

- A bi-inifnite word can be viewed as a coloring of edges (or nodes) of $\mathrm{Cay}(\mathbb{Z}; 1)$
- Its factors are the coloring of finite simple paths
- Generalize to an arbitrary graph G

Unrepetitive graph coloring

- A bi-inifnite word can be viewed as a coloring of edges (or nodes) of $\mathrm{Cay}(\mathbb{Z};1)$
- Its factors are the coloring of finite simple paths
- Generalize to an arbitrary graph G
- E.g., the minimal number of colors to avoid squares is the Thue chromatic number of G (Alon, Grytczuk, Hałuszczak, Riordan, 2002)

Unrepetitive graph coloring

- A bi-inifnite word can be viewed as a coloring of edges (or nodes) of $\mathrm{Cay}(\mathbb{Z}; 1)$
- Its factors are the coloring of finite simple paths
- Generalize to an arbitrary graph G
- E.g., the minimal number of colors to avoid squares is the Thue chromatic number of G (Alon, Grytczuk, Hałuszczak, Riordan, 2002)

We consider a different problem

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E
- Thus we can look for unrepetitive words among walks in G

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E
- Thus we can look for unrepetitive words among walks in G

Motivation

- Symbolic dynamics (ergodicity)

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E
- Thus we can look for unrepetitive words among walks in G

Motivation

- Symbolic dynamics (ergodicity)
- Unrepetitive traces

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E
- Thus we can look for unrepetitive words among walks in G

Motivation

- Symbolic dynamics (ergodicity)
- Unrepetitive traces
- Repetition threshold for words

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E
- Thus we can look for unrepetitive words among walks in G

Motivation

- Symbolic dynamics (ergodicity)
- Unrepetitive traces
- Repetition threshold for words
- Unending chess

Unrepetitive walks

Let $G = (V, E)$ be a finite digraph.

- A walk is a sequence of edges, i.e., a word on the alphabet E
- Thus we can look for unrepetitive words among walks in G

Motivation

- Symbolic dynamics (ergodicity)
- Unrepetitive traces
- Repetition threshold for words
- Unending chess

  A match of chess may be viewed as a walk in a digraph with vertices = positions and edges = moves. With modified rules, infinite square-free walks correspond to unending matches (Morse, Hedlund, 1943)

Preliminaries

Let $G = (V, E)$ be a digraph
A walk in G is any word of $W = E^+ \setminus E^* N E^*$, where

$$N = \{(v_1, v_2)(v_3, v_4) \mid v_1, v_2, v_3, v_4 \in V,\ v_2 \neq v_3\}$$

Preliminaries

Let $G = (V, E)$ be a digraph
A walk in G is any word of $W = E^+ \setminus E^* N E^*$, where

$$N = \{(v_1, v_2)(v_3, v_4) \mid v_1, v_2, v_3, v_4 \in V, \; v_2 \neq v_3\}$$

As any infinite walk terminates in a strongly connected component, we will consider only strongly connected digraphs, w.l.o.g.

Unending square-free walks

Theorem

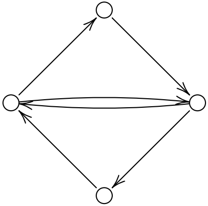
A strongly connected digraph $G = (V, E)$ has an unending square-free walk if and only if

$$\mathrm{Card}(E) \geq \mathrm{Card}(V) + 2$$

Unending square-free walks

Theorem
A strongly connected digraph $G = (V, E)$ has an unending square-free walk if and only if

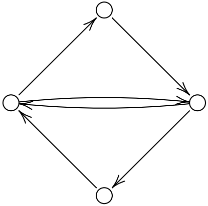
$$\mathrm{Card}(E) \geq \mathrm{Card}(V) + 2$$



Unending square-free walks

Theorem
A strongly connected digraph $G = (V, E)$ has an unending square-free walk if and only if

$$\mathrm{Card}(E) \geq \mathrm{Card}(V) + 2$$



Theorem
Any strongly connected digraph G has an unending cube-free walk unless it is a simple cicle

Vertex sequences

The vertex sequence of a walk

$$w = (v_0, v_1)(v_1, v_2)(v_2, v_3) \cdots$$

is the infinite word

$$v_0 v_1 v_2 v_3 \cdots$$

on the alphabet V
We say that a walk is vertex-square-free if its vertex sequence is square-free

Problem
Effectively characterize digraphs with an infinite vertex-square-free walk

Square-free traces

A alphabet
D symmetric, anti-reflexive relation on A (dependency)
$M(A, D) = A/\approx$ where $\approx$ is the congruence generated by

$$ab \approx ba \text{ for all } (a, b) \in (A \times A) \setminus D$$

(trace monoid)

Theorem (C., de Luca, 1986)

Let $M = M(A, D)$ be a trace monoid. The following propositions are equivalent:

- M has infinitely many square-free traces

Theorem (C., de Luca, 1986)

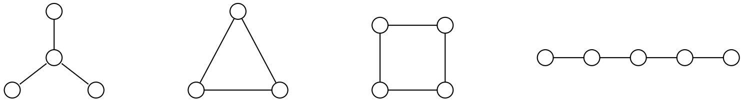
Let $M = M(A, D)$ be a trace monoid. The following propositions are equivalent:

- M has infinitely many square-free traces
- the dependency graph has a vertex-square-free infinite walk

Theorem (C., de Luca, 1986)

Let $M = M(A, D)$ be a trace monoid. The following propositions are equivalent:

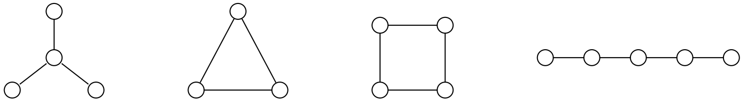
- M has infinitely many square-free traces
- the dependency graph has a vertex-square-free infinite walk
- the dependency graph has one of the following subgraphs



Theorem (C., de Luca, 1986)

Let $M = M(A, D)$ be a trace monoid. The following propositions are equivalent:

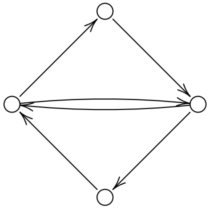
- M has infinitely many square-free traces
- the dependency graph has a vertex-square-free infinite walk
- the dependency graph has one of the following subgraphs



Remark

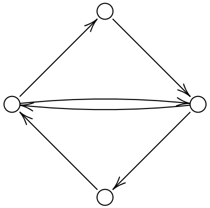
This characterize undirected graphs with a vertex-square-free infinite walk. The problem remains open for digraphs

Example

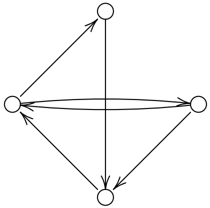


has an infinite
vertex-square-free walk

Example



has an infinite
vertex-square-free walk



has no infinite
vertex-square-free walk

Repetition threshold

- the exponent of a finite word is the ratio of its length and its least period
- the critical exponent of a (possibly infinite) word is the supremum of the exponents of its (finite) factors
- the repetition threshold $\mathrm{RT}(k)$ is the minimal critical exponent of an infinite word on k letters

Repetition threshold

- the exponent of a finite word is the ratio of its length and its least period
- the critical exponent of a (possibly infinite) word is the supremum of the exponents of its (finite) factors
- the repetition threshold $\mathrm{RT}(k)$ is the minimal critical exponent of an infinite word on k letters

Definition

The repetition threshold of a digraph G is the minimal critical exponent $\mathrm{RT}(G)$ of an infinite walk in G

Repetition threshold on n letters

n	$\mathrm{RT}(n)$	
2	2	Thue, 1906
3	$7/4$	Dejean, 1972
4	$7/5$	Pansiot, 1984
$n \geq 5$	n/(n-1)	Moulin-Ollagnier, 1992 for $5 \leq n \leq 11$ Mohammad-Noori, Currie, 2007 for $12 \leq n \leq 14$ C., 2007 for $n \geq 33$ Rao and Currie, Rampersad, 2009 for $15 \leq n \leq 32$

- All conjectured by Dejean, 1972

Generalized repetition threshold

- the k-exponent of a finite word is the ratio of its length and its least period not smaller than k
- the k-critical exponent of a (possibly infinite) word is the supremum of the k-exponents of its (finite) factors
- the generalized repetition threshold $\mathrm{RT}(n, k)$ is the minimal k-critical exponent of an infinite word on n letters (Ilie, Ochem, Shallit, 2004)

Generalized repetition threshold

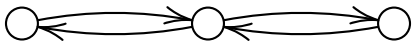
- the k-exponent of a finite word is the ratio of its length and its least period not smaller than k
- the k-critical exponent of a (possibly infinite) word is the supremum of the k-exponents of its (finite) factors
- the generalized repetition threshold $\mathrm{RT}(n, k)$ is the minimal k-critical exponent of an infinite word on n letters (Ilie, Ochem, Shallit, 2004)

Definition

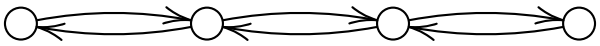
For a digraph G, the generalized repetition threshold $\mathrm{RT}(G, k)$ is the minimal k-critical exponent of an infinite walk in G

Examples

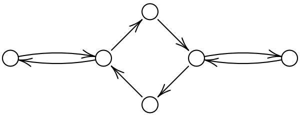
All these graphs have repetition threshold 2:



No square-free infinite walk

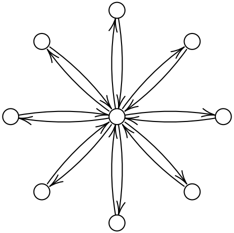


A square-free infinite walk, no vertex-square-free infinite walk



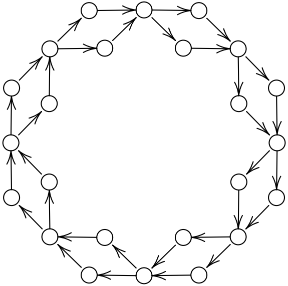
A vertex-square-free infinite walk

Other examples



The n-edge star has repetition threshold $\mathrm{RT}(n)$

Other examples



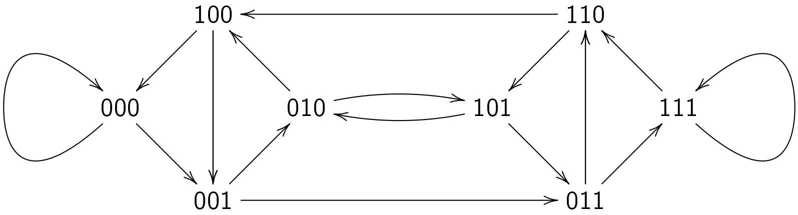
K_n

$3n$ vertices
$4n$ edges
a (2-automatic) infinite walk of critical exponent $1 + 4/n$

$$\frac{n+2}{n} \leq \mathrm{RT}(K_n) \leq \frac{n+4}{n}$$

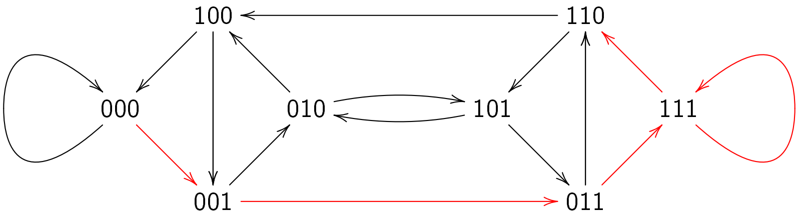
de Bruijn digraph

$B(n,k) = (A^{k-1}, E)$ with $\mathrm{Card}(A) = n$ and
$E = \{(au, ub) \mid a, b \in A,\ u \in A^{k-2}\}$



de Bruijn digraph

$B(n,k) = (A^{k-1}, E)$ with $\mathrm{Card}(A) = n$ and
$E = \{(au, ub) \mid a, b \in A,\ u \in A^{k-2}\}$



00011110

Remark

There is a natural 1-1 correspondence between $A^{\geq k} \cup A^{\omega}$ and the set of finite and infinite walks in $B(n,k)$ which preserves factors and periods (compatibly with length contraction)

Proposition

For $1 \leq m \leq k$,

$$\mathrm{RT}(B(n,m),k) \leq \mathrm{RT}(n,k) \leq \mathrm{RT}(B(n,m),k) + \frac{m-1}{k}$$

Uniform embeddings

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_2 \subseteq V_1$. An embedding of G_2 in G_1 is a monoid morphism a map $\varphi : E_2^* \to E_1^*$ such that

1. for any edge $(v, v') \in E_2$, $\varphi(v, v')$ is a path from v to v' whose internal vertices do not belong to V_2,
2. for any $e_1, e_2 \in E_2$ with $e_1 \neq e_2$, $\varphi(e_1)$ and $\varphi(e_2)$ have no common edges.

Uniform embeddings

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_2 \subseteq V_1$. An embedding of G_2 in G_1 is a monoid morphism a map $\varphi : E_2^* \to E_1^*$ such that

1. for any edge $(v, v') \in E_2$, $\varphi(v, v')$ is a path from v to v' whose internal vertices do not belong to V_2,
2. for any $e_1, e_2 \in E_2$ with $e_1 \neq e_2$, $\varphi(e_1)$ and $\varphi(e_2)$ have no common edges.

Remark

φ maps walks of G_2 into walks of G_1

Uniform embeddings

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_2 \subseteq V_1$. An embedding of G_2 in G_1 is a monoid morphism a map $\varphi : E_2^* \to E_1^*$ such that

1. for any edge $(v, v') \in E_2$, $\varphi(v, v')$ is a path from v to v' whose internal vertices do not belong to V_2,
2. for any $e_1, e_2 \in E_2$ with $e_1 \neq e_2$, $\varphi(e_1)$ and $\varphi(e_2)$ have no common edges.

Remark

φ maps walks of G_2 into walks of G_1

Proposition

If there is a uniform embedding of G_2 in G_1 then

$$\mathrm{RT}(G_1) \leq \mathrm{RT}(G_2)$$

Generalized embeddings

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_2 \subseteq V_1$. A generalized embedding of G_2 in G_1 is a monoid morphism a map $\varphi : E_2^* \to E_1^*$ such that

1. for any edge $(v, v') \in E_2$, $\varphi(v, v')$ is a path from v to v' whose internal vertices do not belong to V_2,
2. for any $e_1, e_2 \in E_2$ with distinct origins and distinct tails, $\varphi(e_1)$ and $\varphi(e_2)$ have no common edges.

Generalized embeddings

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_2 \subseteq V_1$. A generalized embedding of G_2 in G_1 is a monoid morphism a map $\varphi : E_2^* \to E_1^*$ such that

1. for any edge $(v, v') \in E_2$, $\varphi(v, v')$ is a path from v to v' whose internal vertices do not belong to V_2,
2. for any $e_1, e_2 \in E_2$ with distinct origins and distinct tails, $\varphi(e_1)$ and $\varphi(e_2)$ have no common edges.

Proposition

If there is a generalized uniform embedding of G_2 in G_1 then

$$\mathrm{RT}(G_1) \leq \mathrm{RT}(G_2) + \frac{2}{c}$$

where c is the minimal length of cycles in G_2

Embedding in Cayley digraphs

Proposition

Let T be a subtree of a Cayley digraph K, rooted in 1, with leaves $\ell_1, \ell_2, \ldots, \ell_r$, and let $H = \langle \ell_1, \ell_2, \ldots, \ell_r \rangle$, $r \geq 2$. Suppose that the following condition is verified:

- for any pair of distinct internal vertices v_1, v_2 of T such that $v_1^{-1} v_2 \in H$ there exists x such that $v_1 x$ is the unique child of v_1 and $v_2 x$ is the unique child of v_2

Then there is a generalized embedding of $\mathrm{Cay}(H; \ell_1, \ell_2, \ldots, \ell_r)$ in K. Moreover, if all the leaves have the same height in T, then the generalized embedding is uniform

From de Bruijn graph to the symmetric group

Proposition (Moulin-Ollagnier, 1992)

The digraph $\mathrm{Cay}(\mathbb{S}_n; \sigma_0, \sigma_1)$, where

$$\sigma_0 = (1\,2\,\cdots\,n) \text{ and } \sigma_1 = (1\,2\,\cdots\,n-1)$$

is a subgraph of $B(n, n-1)$

From de Bruijn graph to the symmetric group

Proposition (Moulin-Ollagnier, 1992)

The digraph $\mathrm{Cay}(\mathbb{S}_n; \sigma_0, \sigma_1)$, where

$$\sigma_0 = (1\,2\,\cdots\,n) \text{ and } \sigma_1 = (1\,2\,\cdots\,n-1)$$

is a subgraph of $B(n, n-1)$

Fact

Let $n \geq 15$. There is a generalized uniform embedding of $\mathrm{Cay}(G; \tau_0, \tau_1, \tau_2)$ in $\mathrm{Cay}(\mathbb{S}_n; \sigma_0, \sigma_1)$ where $\tau_0 = (7\ 9\ 10\ 8)$, $\tau_1 = (9\ 11\ 12\ 10)$, $\tau_2 = (1\ 5\ 6\ 3\ 4)$, $G = \langle \tau_0, \tau_1, \tau_2 \rangle$.

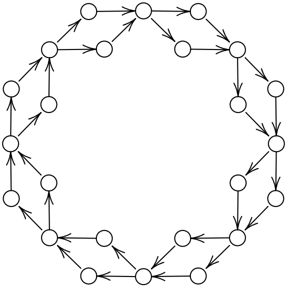
From symmetric group to grid

- Since the orbit of τ_2 does not intersect those of τ_0 and τ_1,

$$\mathrm{Cay}(G;\tau_0,\tau_1,\tau_2) = \mathrm{Cay}(G_1;\tau_0,\tau_1) \times C_5$$

- Computer verification shows that $\mathrm{Cay}(G_1;\tau_0,\tau_1)$ has a simple cycle of length 100
- Thus, $C_{100} \times C_5$ is a subgraph of $\mathrm{Cay}(G;\tau_0,\tau_1,\tau_2)$
- The graph we called K_{100} is a subgraph of $C_{100} \times C_5$

In conclusion, there is a generalized uniform embedding of K_{100} in $B(n, n-1)$



One derives

$$\mathrm{RT}(B(n, n-1)) \leq 1.03 \quad \text{and} \quad RT(n, k) \leq 1.03 + 2/k,\ \ k \geq n-1$$

Actually, K_{100} is embedded in a subgraph of $B(n, n-1)$ where 'short' walks correspond to words of critical exponent $\leq n/(n-1)$.
Thus we have obtained a new infinite word of minimal critical exponent.

Thank you !