Length types of word equations

Word equations with constants

Complexity of the satisfiability problem
(Makanin, Plandowski, Rytter)

$\mathrm{NPTIME}(d \cdot \log \mathtt{N}(d))$

$\mathtt{N}(d)$: upper bound for the length of the shortest solution

$\mathrm{PSPACE}(d)$

Finite representation of all solutions
(Plandowski)

$\mathrm{DEXPTIME}(d)$

Independent systems of equations

- Not infinite (Guba, Albert & Lawrence, 1985)
- Quadratic upper bound in length of the shortest equation for systems with a solution of maximum rank (Saarela, 2011)
- No known upper bound in the number of unknowns
  - even for three unknowns
- Polynomial lower bounds in number of unknowns (Karhumäki, Plandowski, 1994)

Equations in one unknown (with constants)

- $\mathcal{O}(n \log n)$ (S. Eyono Obono, P. Goralčík, M. N. Maksimenko, 1994)
- $\mathcal{O}(n + \#_x \log n)$ (R. Dąbrowski, W. Plandowski, 2011)
- At most two "distinct" solutions? (some hints in Laine, Plandowski, 2011)

Motivational example

$$(x_1 x_2 \cdots x_n)^k = x_1^k x_2^k \cdots x_n^k$$

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Can be simultaneously true for $k = 2$ and $k = 3$?

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$$(x_1 x_2 \cdots x_n)^2 = x_1^2 x_2^2 \cdots x_n^2$$
$$(x_1 x_2 \cdots x_n)^3 = x_1^3 x_2^3 \cdots x_n^3$$

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$$\Updownarrow$$

$$(x_1^2 x_2^2 \cdots x_n^2)^3 = (x_1^3 x_2^3 \cdots x_n^3)^2 = (x_1 x_2 \cdots x_n)^6$$

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$$\Downarrow$$

$$(x_1^2 x_2^2 \cdots x_n^2)^3 = (x_1^3 x_2^3 \cdots x_n^3)^2$$

Motivational example: the length type

$$(x_1^2\, x_2^2\, x_3^2\, x_4^2\, x_5^2\, x_6^2\, x_7^2)^3 = (x_1^3\, x_2^3\, x_3^3\, x_4^3\, x_5^3\, x_6^3\, x_7^3)^2$$

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$$|x_1 x_2 x_3 x_4 x_5 x_6 x_7| = 59$$

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$$|x_1 x_2 x_3 x_4 x_5 x_6 x_7| = 59$$

$$(|x_1|, |x_2|, |x_3|, |x_4|, |x_5|, |x_6|, |x_7|) = (18, 5, 5, 3, 5, 5, 18)$$

Motivational example: the solution

$x_1 \mapsto ababaababaabaababa$

$x_2 \mapsto ababa$

$x_3 \mapsto abaab$

$x_7 \mapsto ababaabaababaababa$

$x_6 \mapsto ababa$

$x_5 \mapsto baaba$

$x_4 \mapsto aba$

Motivational example: one letter type

$$(|x_1|_b, |x_2|_b, |x_3|_b, |x_4|_b, |x_5|_b, |x_6|_b, |x_7|_b) = (7, 2, 2, 1, 2, 2, 7)$$

Working example

$$xy = zxz$$

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$$y \mapsto baab$$
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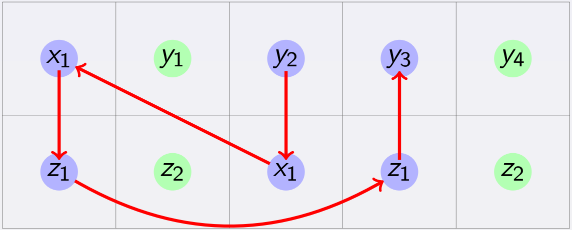
$$(1, 2, 1)$$

x_1	y_2	y_3
z_1	x_1	z_1

Working example

$$xy = zxz$$



Weak equivalence

Last letters of different occurrences of the same variable are not equivalent if they are followed by different variables.

Weak equivalence

$$(1, 2, 1)$$

x_1	y_2	y_3
z_1	x_1	z_1

$$x \mapsto b^A ab^D$$
$$y \mapsto b^B ab^E ab^C$$
$$z \mapsto b^A ab^C$$

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A	x_1	DB	y_2	E	y_3	C
A	z_1	CA	x_1	DA	z_1	C

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$$D + B = C + A$$

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$$A \cdot B = C \cdot D = 0$$

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$$D + B = C + A$$
$$A \cdot B = C \cdot D = 0$$

$$\Downarrow$$

$$A = D \quad B = C$$

A	x_1	DB	y_2	E	y_3	C
A	z_1	CA	x_1	DA	z_1	C

$$x \mapsto b^A a b^D$$
$$y \mapsto b^B a b^E a b^C$$
$$z \mapsto b^A a b^C$$

$$A = D \quad B = C$$

D	x_1	DC	y_2	DD	y_3	C
D	z_1	DC	x_1	DD	z_1	C

$$x \mapsto b^D a b^D$$
$$y \mapsto b^C a b^{2D} a b^C$$
$$z \mapsto b^D a b^C$$

$$(1, 2, 1) + C \cdot (0, 2, 1) + D \cdot (2, 2, 1)$$

A	x_1	DB	y_2	E	y_3	C
A	z_1	CA	x_1	DA	z_1	C

$$x \mapsto b^A ab^D$$
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$$A = D \quad B = C$$

D	x_1	DC	y_2	DD	y_3	C
D	z_1	DC	x_1	DD	z_1	C

$$x \mapsto b^D ab^D$$
$$y \mapsto b^C ab^{2D} ab^C$$
$$z \mapsto b^D ab^C$$

$$(1,2,1) + C \cdot (0,2,1) + D \cdot (2,2,1)$$

$$x \mapsto a \qquad x \mapsto bab$$
$$y \mapsto baab \qquad y \mapsto abba$$
$$z \mapsto ab \qquad z \mapsto ba$$

Application

- Vectors
$$(|x_1|_a, |x_2|_a, \dots, |x_n|_a)$$
are linearly independent for principal solutions.

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- Each equation in three variables having a solution of rank two has also a solution of rank two with x, y or z in a^+ or b^+.