Arithmetization of words and related problems

Yuri Matiyasevich

Steklov Institute of Mathematics at St.Petersburg, Russia

http://logic.pdmi.ras.ru/~yumat

David Hilbert, *Mathematical Problems* [1900]

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist*.

10. Determination of the Solvability of a Diophantine Equation. Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers*.

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Solution: words $V_1, \ldots, V_m \in A_n^*$ such that

$$P^{\upsilon_1,\ldots,\upsilon_m}_{V_1,\ldots,V_m} \equiv Q^{\upsilon_1,\ldots,\upsilon_m}_{V_1,\ldots,V_m}$$

From Words to Numbers
(First Numbering: Matrices)

W.o.l.g $n = 2$, i.e., $A_2 = \{\alpha_1, \alpha_2\}$

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Lemma. Every 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with natural number elements and the determinant equal to 1 can be represented in a unique way as the product of matrices $M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

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$$"\alpha_{i_1} \dots \alpha_{i_k}" \quad \sim \quad M_{i_1} \times \cdots \times M_{i_k}$$

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$$\begin{array}{rcl} "\alpha_{i_1} \ldots \alpha_{i_k}" & \sim & M_{i_1} \times \cdots \times M_{i_k} \\ \upsilon_1, \ldots, \upsilon_m & \sim & a_1, b_1, c_1, d_1, \ldots, a_m, b_m, c_m, d_m \end{array}$$

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\upsilon_1, \dots, \upsilon_m & \sim & a_1, b_1, c_1, d_1, \dots, a_m, b_m, c_m, d_m
\end{array}
$$

$$
P = Q \quad \sim \quad
\begin{array}{l}
P_{11}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{11}(\dots, a_i, b_i, c_i, d_i, \dots,) \\
P_{12}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{12}(\dots, a_i, b_i, c_i, d_i, \dots,) \\
P_{21}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{21}(\dots, a_i, b_i, c_i, d_i, \dots,) \\
P_{22}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{22}(\dots, a_i, b_i, c_i, d_i, \dots,)
\end{array}
$$

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$$\begin{array}{rcl} "\alpha_{i_1} \dots \alpha_{i_k}" & \sim & M_{i_1} \times \cdots \times M_{i_k} \\ \upsilon_1, \dots, \upsilon_m & \sim & a_1, b_1, c_1, d_1, \dots, a_m, b_m, c_m, d_m \end{array}$$

$$P = Q \quad \sim \quad \begin{array}{c} P_{11}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{11}(\dots, a_i, b_i, c_i, d_i, \dots,) \\ P_{12}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{12}(\dots, a_i, b_i, c_i, d_i, \dots,) \\ P_{21}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{21}(\dots, a_i, b_i, c_i, d_i, \dots,) \\ P_{22}(\dots, a_i, b_i, c_i, d_i, \dots,) = Q_{22}(\dots, a_i, b_i, c_i, d_i, \dots,) \\ a_1 d_1 - b_1 c_1 = \cdots = a_m d_m - b_m c_m = 1 \end{array}$$

From Words to Numbers

(Second Numbering: Fibonacci weights)

Every word $X =$ "$\beta_{i_m}\beta_{i_{m-1}}\dots\beta_{i_1}$" in the binary alphabet $B = \{\beta_0, \beta_1\} = \{0, 1\}$ can be viewed as the number

$$x = i_m u_m + i_{m-1} u_{m-1} + \cdots + i_1 u_1$$

written in positional system with weights of digits being the Fibonacci numbers $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $\dots$ (rather than traditional 1, 2, 4, 8, 16, $\dots$).

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"0" $\sim$0	"00" $\sim$0	"000" $\sim$0	"0000" $\sim$0	"00000" $\sim$0
"1" $\sim$1	"01" $\sim$1	"001" $\sim$1	"0001" $\sim$1	"00001" $\sim$1
	"10" $\sim$1	"010" $\sim$1	"0010" $\sim$1	"00010" $\sim$1
	"11" $\sim$2	"011" $\sim$2	"0011" $\sim$2	"00011" $\sim$2
		"100" $\sim$2	"0100" $\sim$2	"00100" $\sim$2
		"101" $\sim$3	"0101" $\sim$3	"00101" $\sim$3
		"110" $\sim$3	"0110" $\sim$3	"00110" $\sim$3
		"111" $\sim$4	"0111" $\sim$4	"00111" $\sim$4

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"1"$\sim$1	"10"$\sim$1	"100"$\sim$2	"1000"$\sim$3	"10000"$\sim$5
	"11"$\sim$2	"101"$\sim$3	"1001"$\sim$4	"10001"$\sim$6
		"110"$\sim$3	"1010"$\sim$4	"10010"$\sim$6
		"111"$\sim$4	"1011"$\sim$5	"10011"$\sim$7
			"1100"$\sim$5	"11100"$\sim$8
			"1101"$\sim$6	"11101"$\sim$9
			"1110"$\sim$6	"11110"$\sim$9
			"1111"$\sim$7	"11111"$\sim$10

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		"1110" $\sim$6	"10110" $\sim$8
			"11000" $\sim$8
			"11010" $\sim$9
			"11100" $\sim$10
			"11110" $\sim$11

From Words to Numbers
(Second Numbering: Zeckendorf's words)

$''' \sim 0$	$''1010'' \sim 4$	$''100000'' \sim 8$	$''101010'' \sim 12$
$''10'' \sim 1$	$''10000'' \sim 5$	$''100010'' \sim 9$	$''1000000'' \sim 13$
$''100'' \sim 2$	$''10010'' \sim 6$	$''100100'' \sim 10$	$''1000010'' \sim 14$
$''1000'' \sim 3$	$''10100'' \sim 7$	$''101000'' \sim 11$	$''1000100'' \sim 15$

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(Second Numbering: Zeckendorf's words)

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Zeckendorf's Theorem. Every positive integer x can be represented in the form

$$x = i_m u_m + i_{m-1} u_{m-1} + \cdots + i_1 u_1$$

with additional restrictions $i_m = 1$, $i_1 = 0$, $i_{k+1} i_k = 0$, and in unique way.

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Zeckendorf's words: they do not begin with "0", do not end with "1", and do not contain "11".

From Words to Numbers
(Third Numbering: Infinite Alphabet)

Every word "$\alpha_{i_k}\alpha_{i_{k-1}}\dots\alpha_{i_1}$" in the infinite alphabet $A_\infty = \{\alpha_1, \alpha_2, \dots\}$ can be presented by the Zeckendorf word

$$"10^{i_k}10^{i_{k-1}}\dots 10^{i_1}".$$

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We get, via Fibonacci numbers, a natural one-to-one correspondence between words in the infinite alphabet A_∞ and natural numbers.

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"α_1"$\sim$1	"α_4"$\sim$5	"$\alpha_3\alpha_1$"$\sim$9	"α_6"$\sim$13
"α_2"$\sim$2	"$\alpha_2\alpha_1$"$\sim$6	"$\alpha_2\alpha_2$"$\sim$10	"$\alpha_4\alpha_1$"$\sim$14
"α_3"$\sim$3	"$\alpha_1\alpha_2$"$\sim$7	"$\alpha_1\alpha_3$"$\sim$11	"$\alpha_3\alpha_2$"$\sim$15

From Words to Numbers
(Concatenation)

$$\begin{array}{rcl} X & = & ''\beta_{i_m}\ldots\beta_{i_1}'' \\ x & = & i_m u_m + \cdots + i_1 u_1 \end{array} \qquad \begin{array}{rcl} Y & = & ''\beta_{j_n}\ldots\beta_{j_1}'' \\ y & = & j_n u_n + \cdots + j_1 u_1 \end{array}$$

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$$\begin{aligned} X &= \text{"}\beta_{i_m}\ldots\beta_{i_1}\text{"} & \qquad Y &= \text{"}\beta_{j_n}\ldots\beta_{j_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 & \qquad y &= j_n u_n + \cdots + j_1 u_1 \end{aligned}$$

$$Z = XY = \text{"}\beta_{i_m}\ldots\beta_{i_1}\beta_{j_n}\ldots\beta_{j_1}\text{"}$$

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$$Z = XY = \text{"}\beta_{i_m} \dots \beta_{i_1}\beta_{j_n} \dots \beta_{j_1}\text{"}$$

$$z = i_m u_{m+n} + \cdots + i_1 u_{1+n} + j_n u_n + \cdots + j_1 u_1$$

From Words to Numbers
(Concatenation)

$$\boxed{u_{k+n} = u_k u_{n+1} + u_{k-1} u_n}$$

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$$\begin{aligned} z &= i_m u_{m+n} + \cdots + i_1 u_{1+n} + j_n u_n + \cdots + j_1 u_1 \\ &= i_m(u_m u_{n+1} + u_{m-1} u_n) + \cdots + i_1(u_1 u_{n+1} + u_0 u_n) + y \end{aligned}$$

From Words to Numbers (Concatenation)

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(Concatenation)

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From Words to Numbers (Concatenation)

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(Concatenation)

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where

$$y_2 = u_n, \quad y_3 = u_{n+1}$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned} \qquad \begin{aligned} Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ y &= j_n u_n + \cdots + j_1 u_1 \\ y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 &= u_n \\ y_3 &= u_{n+1} \end{aligned}$$

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(Concatenation cont.)

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x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\
x_2 & = & u_m \\
x_3 & = & u_{m+1}
\end{array}
\qquad
\begin{array}{rcl}
Y & = & "\beta_{j_n} \ldots \beta_{j_1}" \\
y & = & j_n u_n + \cdots + j_1 u_1 \\
y_1 & = & j_n u_{n-1} + \cdots + j_1 u_0 \\
y_2 & = & u_n \\
y_3 & = & u_{n+1}
\end{array}
$$

$$Z = XY \quad = \quad "\beta_{i_m} \ldots \beta_{i_1} \beta_{j_n} \ldots \beta_{j_1}"$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$\begin{aligned} Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ y &= j_n u_n + \cdots + j_1 u_1 \\ y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 &= u_n \\ y_3 &= u_{n+1} \end{aligned}$$

$$\begin{aligned} Z = XY &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ z &= x_1 y_2 + x y_3 + y \end{aligned}$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned} \qquad \begin{aligned} Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ y &= j_n u_n + \cdots + j_1 u_1 \\ y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 &= u_n \\ y_3 &= u_{n+1} \end{aligned}$$

$$\begin{aligned} Z = XY &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ z &= x_1 y_2 + x y_3 + y \\ z_1 &= x_1(y_3 - y_2) + x y_2 + y_1 \end{aligned}$$

From Words to Numbers
(Concatenation cont.)

$$
\begin{array}{rcl}
X & = & "\beta_{i_m}\ldots\beta_{i_1}" \\
x & = & i_m u_m + \cdots + i_1 u_1 \\
x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\
x_2 & = & u_m \\
x_3 & = & u_{m+1}
\end{array}
\qquad
\begin{array}{rcl}
Y & = & "\beta_{j_n}\ldots\beta_{j_1}" \\
y & = & j_n u_n + \cdots + j_1 u_1 \\
y_1 & = & j_n u_{n-1} + \cdots + j_1 u_0 \\
y_2 & = & u_n \\
y_3 & = & u_{n+1}
\end{array}
$$

$$
\begin{array}{rcl}
Z = XY & = & "\beta_{i_m}\ldots\beta_{i_1}\beta_{j_n}\ldots\beta_{j_1}" \\
z & = & x_1 y_2 + x y_3 + y \\
z_1 & = & x_1(y_3 - y_2) + x y_2 + y_1 \\
z_2 & = & (x_3 - x_2) y_2 + x_2 y_3
\end{array}
$$

From Words to Numbers
(Concatenation cont.)

$$
\begin{aligned}
X &= \text{"}\beta_{i_m}\ldots\beta_{i_1}\text{"} & Y &= \text{"}\beta_{j_n}\ldots\beta_{j_1}\text{"}\\
x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1\\
x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 & y_1 &= j_n u_{n-1} + \cdots + j_1 u_0\\
x_2 &= u_m & y_2 &= u_n\\
x_3 &= u_{m+1} & y_3 &= u_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
Z = XY &= \text{"}\beta_{i_m}\ldots\beta_{i_1}\beta_{j_n}\ldots\beta_{j_1}\text{"}\\
z &= x_1 y_2 + x y_3 + y\\
z_1 &= x_1(y_3 - y_2) + x y_2 + y_1\\
z_2 &= (x_3 - x_2) y_2 + x_2 y_3\\
z_3 &= x_2 y_2 + x_3 y_3
\end{aligned}
$$

Fibonacci Numbers and Diophantine Equations

Theorem (G. D. Cassini [1680]).

$$u_{m+1}^2 - u_{m+1}u_m - u_m^2 = (-1)^m$$

Fibonacci Numbers and Diophantine Equations

Theorem (G. D. Cassini [1680]).

$$u_{m+1}^2 - u_{m+1}u_m - u_m^2 = (-1)^m$$

Theorem (M. J. Wasteels [1902]). If

$$w^2 - wv - v^2 = \pm 1$$

then

$$w = u_{m+1}, \quad v = u_m$$

for some m.

From Words to Numbers
(Second Numbering cont.)

$$\begin{array}{rcc} X & = & "\beta_{i_m} \ldots \beta_{i_1}" \\ x & = & i_m u_m + \cdots + i_1 u_1 \\ x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 & = & u_m \\ x_3 & = & u_{m+1} \end{array}$$

From Words to Numbers
(Second Numbering cont.)

$$\begin{array}{rcl} X & = & "\beta_{i_m} \dots \beta_{i_1}" \\ x & = & i_m u_m + \cdots + i_1 u_1 \\ x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 & = & u_m \\ x_3 & = & u_{m+1} \end{array}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

From Words to Numbers
(Second Numbering cont.)

$$\begin{array}{rcc} X & = & "\beta_{i_m}\dots\beta_{i_1}" \\ x & = & i_m u_m + \cdots + i_1 u_1 \\ x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 & = & u_m \\ x_3 & = & u_{m+1} \end{array}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

$$x_2 \leq x < x_3$$

From Words to Numbers
(Second Numbering cont.)

$$\boxed{\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

$$x_2 \leq x < x_3$$

From Words to Numbers
(Second Numbering cont.)

$$\boxed{\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \dots \beta_{i_1}\text{"} \\ x &= i_m u_m + \dots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \dots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$
$$x_2 \le x < x_3$$

$\frac{x}{x_1} \approx \phi$

From Words to Numbers
(Second Numbering cont.)

$$\boxed{\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \dots \beta_{i_1}\text{"} \\ x &= i_m u_m + \dots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \dots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

$$x_2 \le x < x_3$$

$\frac{x}{x_1} \approx \phi$

$$\phi - 2 < x_1 - x/\phi < \phi - 1$$

From Words to Numbers
(Length of words)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \dots \beta_{i_1}\text{"} & Y &= \text{"}\beta_{j_n} \dots \beta_{j_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 & y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ x_2 &= u_m & y_2 &= u_n \\ x_3 &= u_{m+1} & y_3 &= u_{n+1} \end{aligned}$$

From Words to Numbers

(Length of words)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} & \qquad Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 & y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ x_2 &= u_m & y_2 &= u_n \\ x_3 &= u_{m+1} & y_3 &= u_{n+1} \end{aligned}$$

$$\mathrm{length}(X) = \mathrm{length}(Y) \quad \Leftrightarrow \quad x_2 = y_2$$

From Words to Numbers
(Length of words)

$$\boxed{\mathrm{GCD}(u_m, u_n) = u_{\mathrm{GCD}(m,n)}}$$

$$\begin{aligned}
X &= \text{"}\beta_{i_m}\ldots\beta_{i_1}\text{"} & \qquad Y &= \text{"}\beta_{j_n}\ldots\beta_{j_1}\text{"}\\
x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1\\
x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 & y_1 &= j_n u_{n-1} + \cdots + j_1 u_0\\
x_2 &= u_m & y_2 &= u_n\\
x_3 &= u_{m+1} & y_3 &= u_{n+1}
\end{aligned}$$

$$\mathrm{length}(X) = \mathrm{length}(Y) \quad\Leftrightarrow\quad x_2 = y_2$$

From Words to Numbers

(Length of words)

$$\boxed{\mathrm{GCD}(u_m, u_n) = u_{\mathrm{GCD}(m,n)}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} & Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 & y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ x_2 &= u_m & y_2 &= u_n \\ x_3 &= u_{m+1} & y_3 &= u_{n+1} \end{aligned}$$

$$\begin{aligned} \mathrm{length}(X) = \mathrm{length}(Y) \quad &\Leftrightarrow \quad x_2 = y_2 \\ \mathrm{length}(X) \big| \mathrm{length}(Y) \quad &\Leftrightarrow \quad x_2(2x_3 - x_3) \big| y_2(2y_3 - y_2) \end{aligned}$$

From Words to Numbers

(Length of words)

$$\boxed{\mathrm{GCD}(u_m, u_n) = u_{\mathrm{GCD}(m,n)}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} & \qquad Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 & y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ x_2 &= u_m & y_2 &= u_n \\ x_3 &= u_{m+1} & y_3 &= u_{n+1} \end{aligned}$$

$$\begin{aligned} \mathrm{length}(X) = \mathrm{length}(Y) &\Leftrightarrow x_2 = y_2 \\ \mathrm{length}(X) \big| \mathrm{length}(Y) &\Leftrightarrow x_2(2x_3 - x_3) \big| y_2(2y_3 - y_2) \\ \mathrm{GCD}(\mathrm{length}(X), \mathrm{length}(Y)) = 1 &\Leftrightarrow z_1 x_2(2x_3 - x_2) - \\ &\qquad z_2 y_2(2y_3 - y_2) = 1 \end{aligned}$$

From Words to Numbers

(Length of words)

$$\boxed{\mathrm{GCD}(u_m, u_n) = u_{\mathrm{GCD}(m,n)}}$$

$$\begin{array}{rcl} X & = & "\beta_{i_m} \ldots \beta_{i_1}" \\ x & = & i_m u_m + \cdots + i_1 u_1 \\ x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 & = & u_m \\ x_3 & = & u_{m+1} \end{array} \qquad \begin{array}{rcl} Y & = & "\beta_{j_n} \ldots \beta_{j_1}" \\ y & = & j_n u_n + \cdots + j_1 u_1 \\ y_1 & = & j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 & = & u_n \\ y_3 & = & u_{n+1} \end{array}$$

$$\begin{array}{rcl} \mathrm{length}(X) = \mathrm{length}(Y) & \Leftrightarrow & x_2 = y_2 \\ \mathrm{length}(X) \big| \mathrm{length}(Y) & \Leftrightarrow & x_2(2x_3 - x_3) \big| y_2(2y_3 - y_2) \\ \mathrm{GCD}(\mathrm{length}(X), \mathrm{length}(Y)) = 1 & \Leftrightarrow & z_1 x_2(2x_3 - x_2) - \\ & & \qquad z_2 y_2(2y_3 - y_2) = 1 \end{array}$$

Open Problem. Is there an algorithm for deciding whether a system of word equations with additional restrictions on the lengths of the above types has a solution?

From Words to Numbers

(Length of words cont.)

$$''\alpha_{i_k}\alpha_{i_{k-1}}\dots\alpha_{i_1}''$$

From Words to Numbers

(Length of words cont.)

$$"\alpha_{i_k}\alpha_{i_{k-1}}\ldots\alpha_{i_1}"$$

$$"10^{i_k}10^{i_{k-1}}\ldots10^{i_1}"$$

From Words to Numbers

(Length of words cont.)

$$"\alpha_{i_k}\alpha_{i_{k-1}}\ldots\alpha_{i_1}"$$

$$"10^{i_k}10^{i_{k-1}}\ldots 10^{i_1}"$$

$$\mathrm{length}("10^{i_k}10^{i_{k-1}}\ldots 10^{i_1}") =$$

From Words to Numbers
(Length of words cont.)

$$"\alpha_{i_k}\alpha_{i_{k-1}}\ldots\alpha_{i_1}"$$

$$"10^{i_k}10^{i_{k-1}}\ldots10^{i_1}"$$

$$\mathrm{length}("10^{i_k}10^{i_{k-1}}\ldots10^{i_1}") = k$$

From Words to Numbers

(Length of words cont.)

$$"\alpha_{i_k}\alpha_{i_{k-1}}\ldots\alpha_{i_1}"$$

$$"10^{i_k}10^{i_{k-1}}\ldots 10^{i_1}"$$

$$\mathrm{length}("10^{i_k}10^{i_{k-1}}\ldots 10^{i_1}") = k + i_k + i_{k-1} + \cdots + i_1$$

From Words to Numbers
(Fourth Numbering)

$$X \quad = \quad "\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}"$$

From Words to Numbers
(Fourth Numbering)

$$\boxed{\text{"}1\text{"} \mapsto \text{"}10\text{"}, \ \text{"}0\text{"} \mapsto \text{"}00\text{"}}$$

$$X = \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"}$$

From Words to Numbers
(Fourth Numbering)

$$\boxed{\text{"1"} \mapsto \text{"10"}, \ \text{"0"} \mapsto \text{"00"}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"} \\ &\quad \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0\text{"} \end{aligned}$$

From Words to Numbers

(Fourth Numbering)

$$\boxed{\text{"1"} \mapsto \text{"10"}, \ \text{"0"} \mapsto \text{"00"}}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"} \\ & \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0\text{"} \\ x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \cdots + i_1 u_2 \end{aligned}$$

From Words to Numbers

(Fourth Numbering)

$\boxed{\text{"}1\text{"} \mapsto \text{"}10\text{"},\ \text{"}0\text{"} \mapsto \text{"}00\text{"}}$

$$
\begin{aligned}
X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"} \\
& \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0\text{"} \\
x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \cdots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \cdots + i_1 u_1
\end{aligned}
$$

From Words to Numbers

(Fourth Numbering)

$$\boxed{\text{"1"} \mapsto \text{"10"},\ \text{"0"} \mapsto \text{"00"}}$$

$$\begin{aligned}
X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"} \\
& \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0\text{"} \\
x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \cdots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \cdots + i_1 u_1 \\
x_2 &= u_{2m} \\
x_3 &= u_{2m+1}
\end{aligned}$$

From Words to Numbers
(Fourth Numbering)

$$\boxed{"1" \mapsto "10", \; "0" \mapsto "00"}$$

$$\begin{aligned}
X &= "\beta_{i_m}\beta_{i_{m-1}} \ldots \beta_{i_1}" \\
& "\beta_{i_m}0\beta_{i_{m-1}}0 \ldots \beta_{i_1}0" \\
x &= i_m u_{2m} + i_{m-1} u_{2(m-1)} + \cdots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1} u_{2(m-1)-1} + \cdots + i_1 u_1 \\
x_2 &= u_{2m} \\
x_3 &= u_{2m+1}
\end{aligned}$$

$$x_3^2 - x_3 x_2 - x_2^2 = 1$$

From Words to Numbers

(Fourth Numbering)

$$\boxed{\text{"1"} \mapsto \text{"10"},\ \text{"0"} \mapsto \text{"00"}}$$

$$\begin{aligned}
X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"} \\
& \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0\text{"} \\
x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \cdots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \cdots + i_1 u_1 \\
x_2 &= u_{2m} \\
x_3 &= u_{2m+1}
\end{aligned}$$

$$x_3^2 - x_3x_2 - x_2^2 = 1$$

$$x < x_3$$

From Words to Numbers
(Fourth Numbering)

$\boxed{\text{"1"} \mapsto \text{"10"},\ \text{"0"} \mapsto \text{"00"}}$

$$\begin{aligned} X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\dots\beta_{i_1}\text{"} \\ &\ \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\dots\beta_{i_1}0\text{"} \\ x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \dots + i_1 u_2 \\ x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \dots + i_1 u_1 \\ x_2 &= u_{2m} \\ x_3 &= u_{2m+1} \end{aligned}$$

$$x_3^2 - x_3x_2 - x_2^2 = 1$$

$$x < x_3$$

$$\phi - 2 < x_1 - x/\phi < \phi - 1$$

$\boxed{\text{"10"} \mapsto \text{"1"},\ \text{"00"} \mapsto \text{"0"},\ \text{"01"} \mapsto \text{"1"}}$

From Words to Numbers

(Fourth Numbering)

$$\boxed{"1" \mapsto "10",\ "0" \mapsto "00"}$$

$$\begin{aligned}
X &= "\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}" \\
& \quad "\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0" \\
x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \cdots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \cdots + i_1 u_1 \\
x_2 &= u_{2m} \\
x_3 &= u_{2m+1}
\end{aligned}$$

$$x_3^2 - x_3 x_2 - x_2^2 = 1$$

$$x < x_3$$

$$\phi - 2 < x_1 - x/\phi < \phi - 1$$

$$\boxed{"10" \mapsto "1",\ "00" \mapsto "0",\ "01" \mapsto "1"}$$

From Words to Numbers
(Fourth Numbering)

$$\boxed{\text{"1"} \mapsto \text{"10"}, \ \text{"0"} \mapsto \text{"00"}}$$

$$\begin{aligned}
X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\dots\beta_{i_1}\text{"} \\
& \quad \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\dots\beta_{i_1}0\text{"} \\
x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \dots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \dots + i_1 u_1 \\
x_2 &= u_{2m} \\
x_3 &= u_{2m+1}
\end{aligned}$$

$$x_3^2 - x_3x_2 - x_2^2 = 1$$

$$x < x_3$$

$$\phi - 2 < x_1 - x/\phi < \phi - 1$$

$$\boxed{\text{"10"} \mapsto \text{"1"}, \ \text{"00"} \mapsto \text{"0"}, \ \text{"01"} \mapsto \text{"1"}}$$

Open Problem. Is there an algorithm for deciding whether a system of word equations with additional restrictions on the lengths of words has a solution?