Negative energy modes in some models for plasma physics

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1. Introduction

- **Negative energy modes** (NEMs) refer to spectrally stable modes of oscillations possessing negative energy (more precise definition later)

- NEMs important because can be destabilized by dissipation

  - Intuitively, dissipation makes total energy decay $\rightarrow$ result of amplitude of NEMs increasing

- Also nonlinearity can destabilize equilibria with NEMs

2. NEMs in plasma physics

NEMs in low-dimensional dynamical systems but also in models for continuous media

In astrophysical and laboratory plasmas NEMs occur in several cases, e.g.

- Streaming instabilities (Sturrock, 1958)
- Resistive instabilities in magnetically confined plasmas (Greene and Coppi, 1965)
- Vlasov-Maxwell (Morrison and Pfirsch, 1989/90/92 Correa-Restrepo and Pfirsch 1992/93/97)
- Drift-kinetic equations (Throumoulopoulos and Pfirsch, 1996)
- Two-stream instabilities (Kueny and Morrison, 1995 (a,b), Lashmore-Davies, 2007)
- Ideal magnetohydrodynamics (Hirota and Fukumoto, 2008 (a,b))
- Magnetorotational instability (Ilgisonis et al., 2007, 2009, Khalzov et al., 2008)
- Magnetosonic waves in the solar atmosphere (Joarder et al., 1997)

3. Hamiltonian approach to NEMs

- Hamiltonian framework for NEMs : general and unambiguous definition of energy (Morrison and Kotschenreuther, 1989)
- Based on Hamiltonian normal form

N degree-of-freedom, linear, real Hamiltonian system

$$\dot{z} = J_c A z, \qquad \text{with } z = (q_1, \cdots, q_N, p_1, \cdots, p_N)$$

A constant $2N \times 2N$ matrix

$$H_L = \frac{1}{2} A_{ij} z^i z^j, \qquad \text{quadratic Hamiltonian}$$

$$J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \text{canonical symplectic matrix}$$

4. Normal form for linear Hamiltonian systems

- Consider $z = \tilde{z}\mathrm{e}^{i\omega t} + \tilde{z}^*\mathrm{e}^{-i\omega t}$ (and then drop the tilde)

- $i\omega_\alpha z_\alpha = J_c A z_\alpha, \qquad \alpha = 1, \cdots, N \qquad$ assume N distinct real eigenvalues ω_α

- $-\omega_\alpha$ are also eigenvalues, associated with z_α^*
- Define $h(\alpha, \beta) := i\omega_\alpha z_\beta^T \Omega z_\alpha = z_\beta^T A z_\alpha \quad$ with $\Omega = J_c^{-1}$

- One can show that $h(\alpha, \beta) = 0, \qquad$ if $\quad \beta \neq -\alpha$

- $h(-\alpha, \alpha) = {z_\alpha^*}^T A z_\alpha = i\omega_\alpha {z_\alpha^*}^T \Omega z_\alpha$ is the energy of the mode $(z_\alpha, \omega_\alpha; z_\alpha^*, -\omega_\alpha)$

5. Mode signature

- Can choose normalization constant for the eigenvectors so that

$$z_\alpha^{*T} \Omega z_\alpha = \pm 2i.$$

- Consider z_α eigenvector associated with $\omega_\alpha > 0$
- If $z_\alpha^{*T} \Omega z_\alpha = -2i$ then $(z_\alpha, \omega_\alpha; z_\alpha^*, -\omega_\alpha)$ is a positive energy mode (PEM)

- If $z_\alpha^{*T} \Omega z_\alpha = 2i$ then $(z_\alpha, \omega_\alpha; z_\alpha^*, -\omega_\alpha)$ is a negative energy mode

- Indeed, for a PEM the energy is $h(-\alpha, \alpha) = i\omega_\alpha z_\alpha^{*T} \Omega z_\alpha = 2\omega_\alpha > 0$

6. Mode signature - II

- For stable modes canonical transformation $T : (Q_1, \cdots, Q_N, P_1, \cdots, P_N) \to (q_1, \cdots q_N, p_1, \cdots p_N)$ leading to normal form of the Hamiltonian:

$$H_L = \frac{1}{2} \sum_{\alpha=1}^{N} \sigma_\alpha \, \omega_\alpha (P_\alpha^2 + Q_\alpha^2) \, ,$$

  with $\sigma_i \in \{-1, 1\}$ and ω_α positive eigenvalues of the system

- In the normal form stable modes of $H_L \to$ sum of harmonic oscillators with different frequencies

- Modes with $\sigma = -1$ give negative contribution: these are NEMs

- If eigenvalues ω_α and eigenvectors z_α are known, the procedure for determining the transformation T is algorithmic

7. Model for electron temperature gradient driven turbulence

- Turbulence and formation of structures ("streamers") observed in tokamak fusion devices can be due to instabilities driven by gradients in electron temperature (ETG)

$$\frac{\partial}{\partial t}(1-\nabla^2)\phi = [\phi, \nabla^2\phi + x] + \left[\frac{p}{\sqrt{r}}, \sqrt{r}x\right], \quad (1)$$

$$\frac{\partial}{\partial t}\frac{p}{\sqrt{r}} = \left[\frac{p}{\sqrt{r}}, \phi\right] + [\sqrt{r}x, \phi], \quad (2)$$

- Slab model (Gürcan and Diamond, 2004) for evolution of pressure fluctuations $p(x,y)$ and electrostatic potential $\phi(x,y)$

- Coupling of advection equation for p and Charney-Hasegawa-Mima type equation for ϕ

- $r \propto \nabla T_{e0}$ provides instability

- $[f,g] := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$

8. Hamiltonian structure for the ETG model

- ETG model possesses Hamiltonian structure (Gürcan and Diamond, 2005)
- Thus it can be cast in the form

$$\frac{\partial \chi^i}{\partial t} = \{\chi^i, H\}, \qquad i = 1, \cdots, n$$

with $n = 2$ field variables and $H[\chi^1, \cdots, \chi^n]$ Hamiltonian functional

- $\{,\}$ Poisson bracket: antisymmetric bilinear operator satisfying Leibniz and Jacobi identity
- Field variables $\chi^1 = \Lambda := \phi - \nabla^2 \phi$ and $\chi^2 = \mathcal{P} := \frac{p}{\sqrt{r}} + \sqrt{r}x$
- Hamiltonian functional and Poisson bracket:

$$\begin{aligned} H(\Lambda, \mathcal{P}) &= \frac{1}{2} \int d^2x \left(\Lambda \mathcal{L}^{-1} \Lambda - \mathcal{P}^2 + 2\sqrt{r}\mathcal{P}x \right), \\ \{F, G\} &= \int d^2x (x - \Lambda)[F_\Lambda, G_\Lambda] - \mathcal{P}([F_\Lambda, G_\mathcal{P}] + [F_\mathcal{P}, G_\Lambda]). \end{aligned}$$

with $\mathcal{L} f = f - \nabla^2 f$

9. Linearized ETG model

- Poisson bracket for the ETG model possesses Casimirs

$$C_1 = \int d^2x \mathcal{H}(\mathcal{P}), \qquad C_2 = \int d^2x (\Lambda - x)\mathcal{F}(\mathcal{P})$$

with arbitrary $\mathcal{H}$ and $\mathcal{F}$

- Casimir C: $\{C, F\} = 0, \quad \forall F \Rightarrow$ Casimirs are invariants for the dynamics
- Linearize the model around equilibria (no flow - linear pressure gradient)

$$\Lambda_{eq} = \mathcal{L}^{-1}\Lambda_{eq} = 0, \qquad \mathcal{P}_{eq} = \alpha_{\mathcal{P}} x \tag{3}$$

with constant $\alpha_{\mathcal{P}}$, yields

$$\begin{aligned} \dot{\tilde{\Lambda}} &= -\frac{\partial}{\partial y}\mathcal{L}^{-1}\tilde{\Lambda} - \sqrt{r}\frac{\partial}{\partial y}\tilde{\mathcal{P}}, \\ \dot{\tilde{\mathcal{P}}} &= \alpha_{\mathcal{P}}\frac{\partial}{\partial y}\mathcal{L}^{-1}\tilde{\Lambda}\,. \end{aligned}$$

- Equilibria (3) are critical points of free energy functional $F := H + C_1 + C_2$ for $\mathcal{F}(\mathcal{P}) = 0$ and $\mathcal{H}(\mathcal{P}) = (1 - \sqrt{r}/\alpha_{\mathcal{P}})\mathcal{P}^2/2$

10. Hamiltonian structure for the linearized ETG system

- Linearized system still Hamiltonian
- $\tilde{\Lambda} = \sum_{\mathbf{k}=-\infty}^{+\infty} \tilde{\Lambda}_{\mathbf{k}}(t)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}$, $\quad \tilde{\mathcal{P}} = \sum_{\mathbf{k}=-\infty}^{+\infty} \tilde{\mathcal{P}}_{\mathbf{k}}(t)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}$ yields Hamiltonian system for Fourier amplitudes:

$$\begin{aligned}
\dot{\tilde{\Lambda}}_{\mathbf{k}} &= i\frac{k_y}{1+k_\perp^2}\tilde{\Lambda}_{\mathbf{k}} + i\sqrt{r}k_y\tilde{\mathcal{P}}_{\mathbf{k}}, \qquad \text{where} \quad k = k_y \quad \text{and} \quad k_\perp^2 = k_x^2 + k^2 \\
\dot{\tilde{\mathcal{P}}}_{\mathbf{k}} &= -i\alpha_{\mathcal{P}}\frac{k_y}{1+k_\perp^2}\tilde{\Lambda}_{\mathbf{k}}\,,
\end{aligned}$$

$$\text{with} \quad H_L = \sum_{k=1}^{+\infty} H_L^k = 2\pi \sum_{k=1}^{+\infty} \left(\frac{|\tilde{\Lambda}_k|^2}{1+k_\perp^2} - \frac{\sqrt{r}}{\alpha_{\mathcal{P}}}|\tilde{\mathcal{P}}_k|^2 \right),$$

$$\text{and bracket} \quad \{F,G\} = \sum_{k=1}^{+\infty} \frac{ik}{2\pi}\left[\left(\frac{\partial F}{\partial \tilde{\Lambda}_k}\frac{\partial G}{\partial \tilde{\Lambda}_{-k}} - \frac{\partial F}{\partial \tilde{\Lambda}_{-k}}\frac{\partial G}{\partial \tilde{\Lambda}_k} \right) \right.$$

$$\left. -\alpha_{\mathcal{P}}^2 \left(\frac{\partial F}{\partial \tilde{\Lambda}_k}\frac{\partial G}{\partial \tilde{\mathcal{P}}_{-k}} + \frac{\partial F}{\partial \tilde{\mathcal{P}}_k}\frac{\partial G}{\partial \tilde{\Lambda}_{-k}} - \frac{\partial F}{\partial \tilde{\mathcal{P}}_{-k}}\frac{\partial G}{\partial \tilde{\Lambda}_k} - \frac{\partial F}{\partial \tilde{\Lambda}_{-k}}\frac{\partial G}{\partial \tilde{\mathcal{P}}_k} \right) \right].$$

11. Canonical form

- Change of variables:

$$
\begin{aligned}
q_k^1 &= \sqrt{\frac{\pi}{k\alpha_{\mathcal{P}}^2}}(\tilde{\mathcal{P}}_k + \alpha_{\mathcal{P}}\tilde{\Lambda}_k + \tilde{\mathcal{P}}_{-k} + \alpha_{\mathcal{P}}\tilde{\Lambda}_{-k}) \,, \\
p_k^1 &= -i\sqrt{\frac{\pi}{k\alpha_{\mathcal{P}}^2}}(\tilde{\mathcal{P}}_k + \alpha_{\mathcal{P}}\tilde{\Lambda}_k - \tilde{\mathcal{P}}_{-k} - \alpha_{\mathcal{P}}\tilde{\Lambda}_{-k}) \,, \\
q_k^2 &= \sqrt{\frac{\pi}{k}}(\tilde{\Lambda}_k + \tilde{\Lambda}_{-k}), \qquad p_k^2 = i\sqrt{\frac{\pi}{k}}(\tilde{\Lambda}_k - \tilde{\Lambda}_{-k}) \,,
\end{aligned}
$$

- In the real variables $z^k = (q_1^k, q_2^k, p_1^k, p_2^k)$ the system becomes canonical:

$$\dot{z}^k = J_c A^k z^k.$$

$$A^k = \begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -c & b \end{pmatrix}, a = -\sqrt{r}\alpha_{\mathcal{P}}k, \quad b = \frac{k}{1+k_\perp^2} - k\sqrt{r}\alpha_{\mathcal{P}}, c = \sqrt{r}|\alpha_{\mathcal{P}}|k.$$

- $\Rightarrow$ Framework of the general theory previously described

12. Mode signature for ETG model

$$\omega_s^k = \frac{k}{2(1+k_\perp^2)}\left[1 - \sqrt{1 - 4(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r}}\right], \qquad \text{slow mode}$$

$$\omega_f^k = \frac{k}{2(1+k_\perp^2)}\left[1 + \sqrt{1 - 4(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r}}\right], \qquad \text{fast mode}$$

- The system possesses also the eigenvalues $\omega^k_{-s,-f} = -\omega^k_{s,f}$
- Equilibria spectrally stable iff $\alpha_{\mathcal{P}} < \frac{1}{4(1+k_\perp^2)\sqrt{r}}$
- If $r \to 0$ or $\alpha_{\mathcal{P}} \to 0$ then stable drift wave
- Eigenvectors

$$z^k_{s,f} = q^k_{1\,s,f}\begin{pmatrix} 1 \\ -B_\mp \\ -i \\ -iB_\mp \end{pmatrix}, \qquad z^k_{-s,-f} = {q^k_{1\,s,f}}^* \begin{pmatrix} 1 \\ -B_\mp \\ i \\ iB_\mp \end{pmatrix},$$

where $B_\pm = \frac{b+a\pm\sqrt{(b+a)^2-4c^2}}{2c}$,

13. Mode signature for ETG model (Tassi and Morrison, 2011)

- Consider slow modes:

$$ {z_{-s}^{k}}^{T} \Omega z_s^k = 2i(1 - B_-^2) {q_{1s}^k}^* q_{1s}^k . $$

- For stable modes one finds $1 - B_-^2 > 0$
- Choose normalization constant $q_{1s}^k = {q_{1s}^k}^* = \frac{1}{\sqrt{1-B_-^2}}$
- Energy of the kth slow mode: $h^k(-s, s) = i\omega_s^k {z_{-s}^k}^T \Omega z_s^k = -2\omega_s^k$
- If $\omega_s^k > 0$ the slow mode is a stable but negative energy mode
- This occurs for $0 < \alpha_{\mathcal{P}} < \frac{1}{4(1+k_\perp^2)\sqrt{r}}$
- If pressure gradient is negative ($\alpha_{\mathcal{P}} < 0$) $\Rightarrow$ spectral stability without NEM
- Fast modes : $h^k(-f, f) = i\omega_f^k {z_{-f}^k}^T \Omega z_f^k = 2\omega_f^k > 0 \quad \forall k_\perp, k, r, \alpha_{\mathcal{P}}$

  $\Rightarrow$ Fast modes always PEMs

14. Normal form for linearized ETG model

- $T^k : (q_1^k, q_2^k, p_1^k, p_2^k) \rightarrow (Q_1^k, Q_2^k, P_1^k, P_2^k)$ with

$$T^k = \begin{pmatrix} \frac{1}{D_-} & \frac{1}{D_+} & 0 & 0 \\ -\frac{B_-}{D_-} & -\frac{B_+}{D_+} & 0 & 0 \\ 0 & 0 & \frac{1}{D_-} & -\frac{1}{D_+} \\ 0 & 0 & \frac{B_-}{D_-} & -\frac{B_+}{D_+} \end{pmatrix}, \qquad D_\pm = \sqrt{B_\pm^2 - 1}$$

puts the Hamiltonian (for stable modes) into its normal form

$$H'_L = \frac{1}{2}\sum_k{}' \omega_f^k \left({Q_2^k}^2 + {P_2^k}^2 \right) - \omega_s^k \left({Q_1^k}^2 + {P_1^k}^2 \right),$$

- Slow modes give negative contribution to the energy when $\omega_s^k > 0$

15. Dispersion relation for ETG model

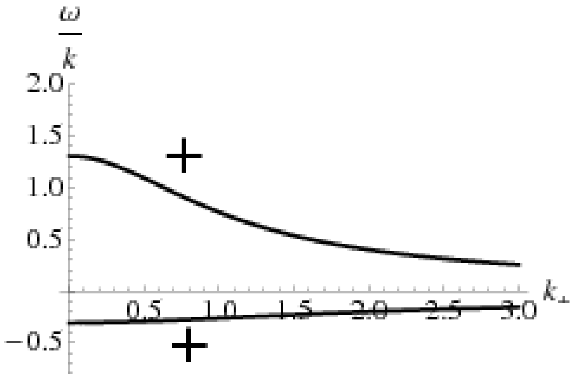

Figure 1: $\alpha_{\mathcal{P}} = -0.3$, $\sqrt{r} = 0.2$

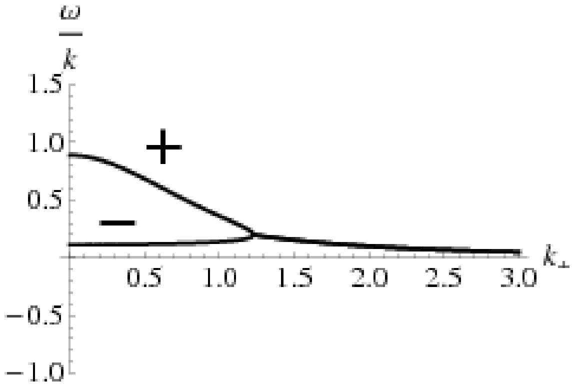

Figure 2: $\alpha_{\mathcal{P}} = 0.5$, $\sqrt{r} = 0.2$

- Instability occurs at $k_\perp = 1.22$ (Fig. 2)
- Collision of eigenvalues of a PEM with a NEM (Kreĭn bifurcation)
- Presence of NEMs reflects in undefiniteness of $\delta^2 F(\Lambda_{eq}, \mathcal{P}_{eq})$, where $F = H + C_1 + C_2$
- Energy-Casimir method predicts formal stability for $\mathcal{F}(\mathcal{P}_{eq}) = 0$, $\mathcal{H}''(\mathcal{P}_{eq}) > 1 \Rightarrow \alpha_{\mathcal{P}} < 0$ i.e. no NEMs

16. Magnetic reconnection in collisionless plasmas

- Magnetic reconnection: modification of the way infinitesimal plasma volumes are connected by means of magnetic field lines
- Involved in e.g., solar flares, magnetic substorms, sawtooth oscillations in tokamaks

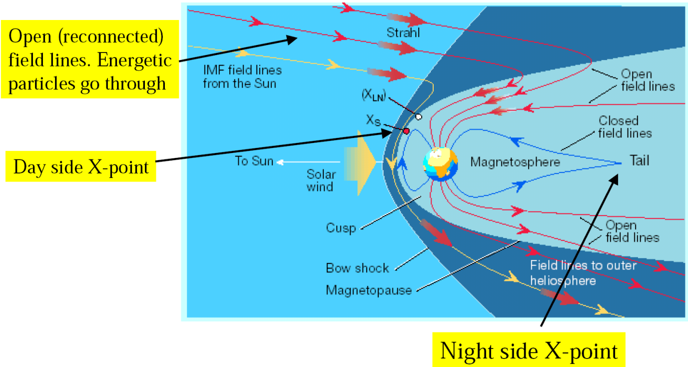

Figure 3: From M. Lockwood, Nature, 409, 677 (2001).

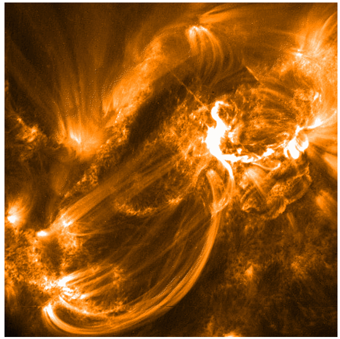

Figure 4: Image taken by TRACE.

- Electron inertia can cause reconnection in high-temperature tokamak plasmas

17. Fitzpatrick-Porcelli model (Fitzpatrick and Porcelli, 2004, 2007)

$$\frac{\partial(\psi - d_e^2\nabla^2\psi)}{\partial t} + [\varphi, \psi - d_e^2\nabla^2\psi] - d_\beta[\psi, Z] = 0 \tag{4}$$

$$\frac{\partial Z}{\partial t} + [\varphi, Z] - c_\beta[v, \psi] - d_\beta[\nabla^2\psi, \psi] = 0 \tag{5}$$

$$\frac{\partial \nabla^2\varphi}{\partial t} + [\varphi, \nabla^2\varphi] + [\nabla^2\psi, \psi] = 0 \tag{6}$$

$$\frac{\partial v}{\partial t} + [\varphi, v] - c_\beta[Z, \psi] = 0. \tag{7}$$

- Slab model with β dependence $\left(\beta = \frac{\text{internal pressure}}{\text{magnetic pressure}}\right)$
- $\mathbf{B} = \nabla\psi \times \hat{\mathbf{z}} + (B_0 + c_\beta Z)\hat{\mathbf{z}}, \qquad \mathbf{v} = \hat{\mathbf{z}} \times \nabla\varphi + v\hat{\mathbf{z}}$
- $c_\beta = \sqrt{\frac{\beta}{1+\beta}}$, $d_\beta = d_i c_\beta$, d_e electron skin depth ($d_e^2 \propto m_e$ causes reconnection)
- (4) from electron momentum equation, (5) from electron vorticity equation
- (6) from vorticity equation, (7) from momentum equation

18. Noncanonical Hamiltonian formulation for the FP model

- Noncanonical Hamiltonian structure of the FP model (Tassi et al. (2008))
- Field variables and Hamiltonian:

$$\chi^1 = \psi_e = \psi - d_e^2 \nabla^2 \psi, \qquad \chi^2 = Z, \qquad \chi^3 = U = \nabla^2 \varphi, \qquad \chi^4 = v \quad (8)$$

$$H = \frac{1}{2} \int_{\mathcal{D}} d^2x [d_e^2 (\nabla^2 \psi)^2 + |\nabla \varphi|^2 + v^2 + |\nabla \psi|^2 + Z^2]. \quad (9)$$

- kinetic energy magnetic+internal energy
- Poisson bracket of Lie-Poisson type:

$$\{F, G\} = \int d^2x \, (U[F_U, G_U] + \psi_e([F_{\psi_e}, G_U]$$
$$+[F_U, G_{\psi_e}] - d_\beta([F_Z, G_{\psi_e}] + [F_{\psi_e}, G_Z]) + c_\beta([F_v, G_Z] + [F_Z, G_v]))$$
$$+Z([F_Z, G_U] + [F_U, G_Z] - d_\beta {d_e}^2 [F_{\psi_e}, G_{\psi_e}] + c_\beta {d_e}^2 ([F_v, G_{\psi_e}] + [F_{\psi_e}, G_v])$$
$$-\alpha[F_Z, G_Z] - c_\beta \gamma [F_v, G_v]) + v([F_v, G_U] + [F_U, G_v]$$
$$+c_\beta {d_e}^2 ([F_Z, G_{\psi_e}] + [F_{\psi_e}, G_Z]) - c_\beta \gamma ([F_v, G_Z] + [F_Z, G_v]))\big) .$$

where $\alpha = d_\beta + c_\beta \frac{{d_e}^2}{d_i}, \gamma = \frac{{d_e}^2}{d_i}$

19. Casimir invariants

- The FP model has four independent infinite families of Casimirs:

$$C_1 = \int d^2x \mathcal{H}(D) \qquad \text{with } D = \psi_e + d_i v \text{ and arbitrary } \mathcal{H}$$

$$C_2 = \int d^2x \zeta \mathcal{F}(D) \qquad \text{with } \zeta = U + \frac{d_i}{c_\beta (d_e^2 + d_i^2)} Z \text{ and arbitrary } \mathcal{F}$$

$$C_{3,4} = \int d^2x g_\pm(\bar{T}_\pm) \quad \text{with } \bar{T}_\pm = \psi_e - \frac{d_e^2}{d_i} v \mp d_e \sqrt{1 + \frac{d_e^2}{d_i^2}} Z \text{ and arbitrary } g_\pm$$

20. Lagrangian invariants

- FP model can be reformulated as

$$\frac{\partial D}{\partial t} = -[\varphi, D],$$

$$\frac{\partial \zeta}{\partial t} = -[\varphi, \zeta] + \frac{1}{d_e^2 + d_i^2}[D, \psi],$$

$$\frac{\partial \bar{T}_\pm}{\partial t} = -\left[\varphi \pm c_\beta \sqrt{1 + \frac{d_i^2}{d_e^2}}\psi, \bar{T}_\pm\right],$$

- 3 out 4 Casimir families associated to Lagrangian invariants $(D,\ \bar{T}_\pm)$ advected with appropriate "stream functions" $(\varphi,\ \bar{\varphi}_\pm \equiv \varphi \pm\ c_\beta \sqrt{1 + \frac{d_i^2}{d_e^2}}\psi)$
- Poloidal magnetic flux ψ is not a Lagrangian invariant (frozen-in condition violated by electron inertia!)

21. Linearization around homogeneous equilibria

- Linearization around homogeneous equilibria (no poloidal flow - constant poloidal magnetic field)

$$\psi_{eq} = \alpha_\psi x, \quad \varphi_{eq} = 0, \quad Z_{eq} = \alpha_Z x, \quad v_{eq} = \alpha_v x.$$

- Linearized model still admits a canonical Hamiltonian formulation

$$q_k^{(1)} = -\sqrt{\frac{\pi}{k|\alpha_{\bar{D}}|}}\left(\bar{D}_k + \bar{D}_{-k}\right), \qquad p_k^{(1)} = i\sqrt{\frac{\pi}{k|\alpha_{\bar{D}}|}}\left(\bar{D}_k - \bar{D}_{-k}\right),$$

$$q_k^{(2)} = \sqrt{\frac{\pi}{k\alpha_\zeta}}\left(\zeta_k + \zeta_{-k}\right), \qquad p_k^{(2)} = i\sqrt{\frac{\pi}{k\alpha_\zeta}}\left(\zeta_k - \zeta_{-k}\right),$$

$$q_k^{(3)} = \sqrt{\frac{\pi}{k\alpha_+}}\left(T_{+k} + T_{+-k}\right), \qquad p_k^{(3)} = i\sqrt{\frac{\pi}{k\alpha_+}}\left(T_{+k} - T_{+-k}\right),$$

$$q_k^{(4)} = \sqrt{\frac{\pi}{k\alpha_-}}\left(T_{-k} + T_{--k}\right), \qquad p_k^{(4)} = i\sqrt{\frac{\pi}{k\alpha_-}}\left(T_{-k} - T_{--k}\right),$$

for $k = k_y = 1, \cdots, +\infty$ and where $\bar{D}_{\pm k} = -\alpha_\zeta D_{\pm k} + \alpha_D \zeta_{\pm k}$.

22. Dispersion relation

- Dispersion relation $\kappa_\perp^2 = -\frac{(N^2-N_r^2)(N-N_+)(N-N_-)}{N^2(N^2-N_\delta^2)}$ where

  $$N = \frac{\omega d_e}{k v_A d_\beta} \quad N_\pm = \frac{\nu \pm \sqrt{\nu^2+4\delta(\delta+s)}}{2}, \quad N_r = \frac{d_e}{d_\beta}, \quad N_\delta = \sqrt{1+\frac{d_e^2}{d_i^2}}$$

  $$v_A = \alpha_\psi, \quad \kappa_\perp = k_\perp d_e, \quad s = \frac{\alpha_v d_e}{v_A}, \quad \nu = \frac{\alpha_Z d_e}{v_A}, \quad \delta = \frac{d_e}{d_i}$$

- 4th degree dispersion relation but nevertheless derived a spectral stability criterion for FP model:

  Given $\mathcal{C}_\delta := \{N : |N| < N_\delta\}$ and $\mathcal{N} := \{N_+, N_-, N_r, -N_r\}$ and $\alpha_\psi > 0$

  - Case 1) : $\delta + s > 0 \rightarrow$ absent or positive parallel velocity gradient:
    If at least two elements of $\mathcal{N}$ belong to $\mathcal{C}_\delta$ then the equilibrium is stable

  - Case 2) : $\delta + s < 0 \rightarrow$ negative parallel velocity gradient.
    - If $\nu^2+4\delta(\delta+s) > 0$ (moderate parallel velocity gradient) equilibrium unstable for large enough $\kappa_\perp^2$
    - If $\nu^2 + 4\delta(\delta + s) < 0$ (strong parallel velocity gradient) then equilibrium unstable for all $\kappa_\perp$ but always two stable branches

23. Mode signature for the FP model

- NEMs and PEMs can be identified even without solving the 4th degree dispersion relation
- NEMs do not change signature unless: instability occurs or eigenvalue crosses zero
- mode signature independent on coordinates

  $\Rightarrow$ sufficient to look at the limits $k_\perp \to 0$ and $k_\perp \to +\infty$
- $k_\perp \to 0$: $\omega_{1,2} = \pm k v_A$, Alfvén waves

  $\omega_{3,4} = \frac{k\alpha_Z d_\beta}{2}\left(1 \pm \sqrt{1 + \frac{4v_A^2}{d_e d_i \alpha_Z^2}\left(\frac{d_e}{d_i} + \frac{\alpha_v d_e}{v_A}\right)}\right)$, modified drift wave
- $k_\perp \to \infty$: $\omega_{1,2} = 0$, $\omega_{3,4} \to \pm k v_{the}$ if $d_\beta \simeq \rho_s$ (waves at the electron thermal speed)
- Inserting eigenvalues and eigenvectors for $k_\perp \to 0$ and $k_\perp \to +\infty$ gives energy signature in those limits

24. Negative energy modes

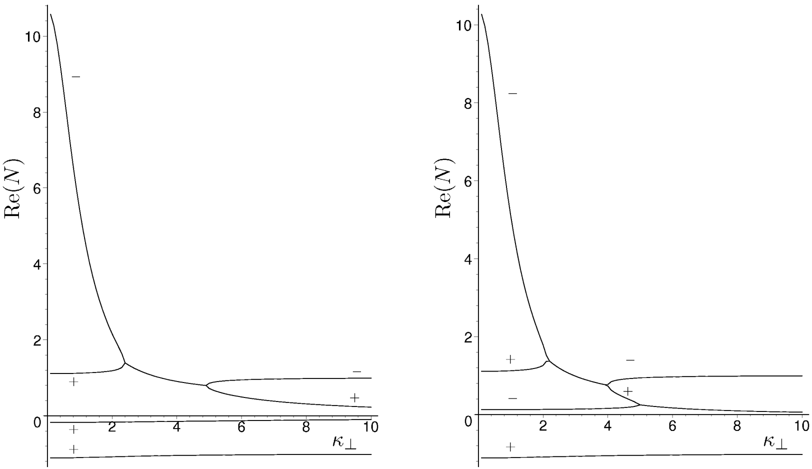

Figure 5: Examples of solutions of the dispersion relation for Case 1 (left) and Case 2 (right).

- In Case 1 Alfvén waves are PEMs but one drift-shear mode is a NEM
- Kreĭn bifurcation between Alfvén and drift shear mode
- In case 2 Alfvén waves still PEMs but two drift-shear modes are NEMs
- A drift shear mode involved in a second Kreĭn bifurcation at larger $k_\perp$

25. Conclusions

- Reviewed NEMs and PEMs unambiguously defined using Hamiltonian structure for general equilibria
- **ETG model:**
- Formal stability (no NEMs) when pressure gradient negative
- For positive pressure gradient spectral stability with NEM (slow mode) for long wavelength
- Instability due to collision between eigenvalues of NEM and PEM
- **Magnetic reconnection model:**
- Spectral stability criterion and mode signature (even without explicitly solving dispersion relation)
- At small $k_\perp$ Alfvén waves are PEMs whereas drift-shear waves are NEMs or PEMs depending on parallel velocity gradient
- Instability due to collision between Alfvén (PEM) and drift-shear (NEM) waves with positive frequencies