Spectral Analysis, Stability and Bifurcation in Modern Nonlinear Physical Systems (12w5073)

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1 Overview of the Field

Linearised stability analysis of stationary and periodic solutions of both finite and infinite dimensional dynamical systems is a central issue in many (physical) applications. Such systems usually depend on parameters, so an important question is what happens to stability when the parameters are varied. This implies that one has to study the spectrum of a linear operator and its dependence on parameters. Moreover, systems arising in physics and other applications often possess special structure, for example Hamiltonian systems. Therefore spectrum and Jordan structure no longer suffice to characterize equivalent systems (under smooth coordinate transformations) but additional invariants are needed. Identifying and interpreting these in infinite dimensional systems seems more involved than in finite dimensional situations. For example, one may consider the symplectic or Krein signature for imaginary eigenvalues in linear finite dimensional Hamiltonian systems. When such eigenvalues meet as the parameters vary, the existence of additional invariants causes non-generic behaviour. In particular, a collision of eigenvalues on the imaginary axis may have dynamical consequences since the stability may change, depending on the additional invariants. At such a collision the boundary of the so called stability domain in parameter space may have singularities. This phenomenon occurs in numerous applications and it may have various physical consequences and interpretations. On the other hand stability questions can also be studied by index theory (Morse index, Maslov index). These approaches are not unrelated; for example the symplectic or Krein signature is connected to the Morse index.

2 Recent Developments and Open Problems

Already established (e.g., by Floquet, Lyapunov and Poincare by the end of the XIXth century) and greatly benefitting from the mathematical achievements of the XXth century, classical stability analysis of stationary and periodic solutions of finite- and infinite-dimensional dynamical systems has experienced a rebirth during the last two decades. One of the reasons for this new active phase is the significant progress in nonlinear PDEs where solitary waves were discovered for a wide class of equations related to modern physical applications. Another motivating source is the remarkable success of the methods of geometrical optics in the analysis of three-dimensional destabilization of two-dimensional flows of ideal and viscous fluid. This provides a universal mechanism whereby complex three-dimensional motion can arise directly from large-scale two-dimensional coherent structures, which is important for the theory of turbulence. The geometrical optics stability analysis was extended by Eckhoff to general systems of symmetric hyperbolic PDEs including the case of multiple roots of the dispersion relation that determines bi-characteristics along which a localized perturbation evolves according to the amplitude equation that serves for stability analysis. The third research area that stimulates development of such stability theory nowadays is dissipation-induced instabilities and their relation to non-Hermitian degeneracies of the spectrum. Between these three subject areas there exist numerous connections that we shall discuss briefly.

Geometrical optics stability analysis is an example of separation of fast and slow variables in the adiabatic approximation. The WKB solution may contain an additional term that expresses the cumulative change of the complex phase as the wave propagates along a ray (geometric phase). There exists a link between geometric phase and strong stability of Hamiltonian systems.

On the other hand multiple roots of the dispersion relation are connected to non-trivial physical effects such as conical refraction discovered by Hamilton in crystal optics and found in hydromagnetics by Ludwig in 1960s. The presence of a double semi-simple eigenvalue determines a singularity of the dispersion surface - Hamilton's diabolical point (DP) - that yields a conical ray surface, which is observable in experiments with birefringent crystals. In the presence of absorption and optical activity the conical singularities of the dispersion surface can transform into branch points that correspond to double eigenvalues with the Jordan block (exceptional points, EPs). This happens because the matrix determining the dispersion relation becomes a non-Hermitian one, for which an EP has a lower codimension than for DP.

Dispersion surfaces with the same Hermitian and non-Hermitian singularities are characteristic of travelling waves in rotating fluids and structures with frictional contact such as a rotating glass bowl of a glass harmonica that is touched by the fingers of a musician. The glass harmonica is a gyroscopic system perturbed by dissipative and non-conservative positional forces acting at the contact. Its industrial counterpart is rotating machinery like brake disks, paper calendars, and turbine shafts. The vibrations of a glass harmonica provide an audible example of dissipation-induced instability in a Hamiltonian system.

The effect of dissipation on Hamiltonian systems can be visualized by its action on pure imaginary eigenvalues. Landahl discovered that in an incompressible laminar boundary layer over a flexible surface the wall damping destabilizes waves of the Tollmien-Schlichting type that have a negative energy sign. MacKay studied movement of eigenvalues of Hamiltonian equilibria under non-Hamiltonian perturbation and found that the way in which simple pure imaginary eigenvalues of an equilibrium of a Hamiltonian system migrate under non-Hamiltonian perturbation is related to the energy and dissipation rate for the corresponding modes. Maddocks and Overton initiated the study of multiple pure imaginary eigenvalues and showed that for an appropriate class of dissipatively perturbed Hamiltonian systems, the number of unstable modes of the dynamics linearized at a nondegenerate equilibrium is determined solely by the index of the equilibrium regarded as a critical point of the Hamiltonian.

In the general case of non-Hamiltonian vector fields, the occurrence of double imaginary eigenvalues (1:1 resonant Hopf-Hopf bifurcation) has codimension three, whereas the codimension is one for Hamiltonian-Hopf bifurcation. Langford had shown that the interface between these two cases possesses the Whitney umbrella singularity in the parameter space; the Hamiltonian systems lie on its handle (for periodic systems this has been shown by Hoveijn and Ruijgrok). The Whitney umbrella singularity on the stability boundary of a near-Hamiltonian system corresponds to a double pure imaginary eigenvalue with a Jordan block (EP). Bottema discovered that this singularity explains a discrepancy between the stability domain of an undamped system and that in the limit of vanishing dissipation - Ziegler's destabilization paradox - which is a typical phenomenon in structural and contact mechanics, atmospheric physics and fluid-structure interactions.

The role of the energy sign of the mode associated with an eigenvalue of the Hamiltonian system is visible not only in case of dissipative perturbations. The Hamiltonian-Hopf bifurcation in which two pairs of complex conjugate eigenvalues approach the imaginary axis symmetrically from the left and right, then merge in double purely imaginary eigenvalues and separate along the imaginary axis (or the reverse) is caused by the interaction of eigenvalues with the opposite sign of the energy.

The concepts of signature and positive and negative energy modes were influenced by work of Weierstrass, Rayleigh and Thompson and Tait. Precise formulations in finite dimensions were available by the 1930s and were later developed by Sobolev (1943), Pontryagin (1944), Krein (1950) and Burgoyne & Cushman (1977). In Hamiltonian systems, a sign of the energy which is associated with purely imaginary eigenvalues (in the linearization about an equilibrium) or Floquet multipliers (in the linearization about a periodic orbit) is called symplectic (or Krein) signature whereas the Hamiltonian-Hopf bifurcation is frequently called the Krein collision.

Krein defines the signature as the sign of the square norm induced by the indefinite inner product where the metric operator J is symplectic in case of Hamiltonian systems. In general, the indefinite metric is given by an indefinite bilinear form on the underlying Hilbert space, and selfadjointness is defined by analogy with the Euclidean inner product. J-selfadjoint operators were first introduced by Sobolev (1943) in connection with rotating shallow water. This line of inquiry was continued by Pontryagin (1944) in his pioneering article. Pontryagin's theorem on invariant subspaces started a new branch of Functional Analysis, dedicated to the theory of linear operators in indefinite metric space. The spectral properties and the geometry of sign-definite invariant subspaces of dissipative and contractive operators acting on indefinite metric spaces were studied by Azizov and Iohvidov (1980-1985). In many cases, linearization of a nonlinear wave equation at a spatially localized solution such as a bound state or a solitary wave results in a generalized eigenvalue problem that can be studied using the spectral theory of a J-self-adjoint operator acting in some indefinite metric space. A mean-field α^2-dynamo of magnetohydrodynamics provides another example of a J-self-adjoint operator, where the metric operator J is not symplectic.

Krein (1950-1970) took an axiomatic approach to the spectral theory of unitary and self-adjoint operators acting in Pontryagin space. In addition to establishing the connection between signature and instability (e.g., collision of modes of opposite signature is a necessary condition for complex instability), an important consequence of Krein's work was a signature for linear Hamiltonian systems with periodic coefficients, where the energy sign is time dependent. In other words, the use of energy sign is of limited use whereas the concept of signature generalizes to the case of non-constant coefficients.

Many interesting non-autonomous periodic and non-periodic ODEs arise in the geometrical optics stability analysis. For example, the amplitude equation for elliptic instability reduces to the Schrödinger equation with periodic potential. Another source is linearization of nonlinear PDEs about solitary wave solutions. Indeed, Hamiltonian evolution PDEs in one space dimension, such as the nonlinear Schrödinger equation, reaction-diffusion equation, long-wave short-wave resonance equations, the fifth-order Korteweg de Vries equation (the latter arises, e.g., in beam buckling, pattern formation, thin-film flows and in the theory of capillary-gravity water waves), have the property that their steady part is a finite dimensional Hamiltonian system. For such systems, solitary wave solutions can be characterized as homoclinic orbits of the Hamiltonian ordinary differential equation (ODE).

The spectral problem associated with the linearization about a given homoclinic orbit, in the time dependent equations, yields a parameter-dependent family of linear Hamiltonian systems where the spatial variable plays the role of time. The Hamiltonian matrix depends on the spatial variable as well as on the parameter which can be interpreted both as the control parameter and as the spectral parameter of the linearization that determines the stability of the solitary wave. Since the latter vanishes when the spatial variable tends to infinity, the limit yields an autonomous Hamiltonian system "at infinity" whose eigenvalues depend on the control parameter. Note that this is a situation of the theory of multiparameter eigenvalue problems due to Atkinson, Volkmer and Binding where eigencurves in the plane of the two spectral parameters is a natural object of investigation.

Solutions of the system at infinity form a fixed reference stable Lagrangian subspace. The signed count of its non-trivial intersections with an image of a path of unstable Lagrangian subspaces of the non-autonomous Hamiltonian system gives the Maslov index of this path which is a function of the control parameter. The system at infinity is also used in the construction of the Evans function of the control parameter whose zeros determine the discrete spectrum of the linearization. The Evans function is related to an exterior product of the paths of stable and unstable Lagrangian subspaces. With the variation of the control parameter the Maslov index jumps by one at each eigenvalue of the linearization and thus counts the number of discrete eigenvalues. This intriguing and non-obvious property is used both for calculating the Maslov index of a homoclinic orbit and for determining the number of eigenvalues in the right half of the complex plane, i.e., the instability index of the solitary wave.

The Maslov index applies to a wide range of other physical applications: semi-classical quantization, quantum chaology, classical mechanics, etc. Providing a count of eigenvalues of some self-adjoint operators, it is also used in the stability analysis of traveling waves. When a solitary wave is a limit of periodic waves one can obtain its Maslov index by using the classical definition of the index for periodic orbits and increasing the period to infinity. In this procedure the formula for the Maslov index requires knowledge of the Krein indices of the Floquet multipliers on the unit circle.

The count of eigenvalues in generalized eigenvalue problems and operator polynomials by means of the eigenvalues of the operators at powers of the spectral parameter goes back to results of Kelvin and Tait (1869). This is related both to the classical topics of gyroscopic stabilization and its destruction in the presence of dissipation and non-conservative positional forces (dissipation-induced instabilities), and to the modern question of instabilities of traveling and solitary waves. Pontryagin and Krein space decomposition for establishing sharp bounds on the number of unstable eigenvalues of the original operator recently led to a significant generalization of the classical Kelvin-Tait-Chetaev theorem on gyroscopic stabilization and clarification of its topological meaning in terms of the Euler characteristic of a surface constructed by means of the first integral of the dynamical system. Combination of these results with singularity theory, group-theoretic and perturbation approaches is expected to result in a rather complete and unified constructive theory of dissipation-induced instabilities.

3 Presentation Highlights

3.1 Index theorems and count of eigenvalues

- In the opening talk **Peter Lancaster** discussed new algebraic arguments for a classical Kelvin-Tait-Chetaev theorem and its generalizations, emphasizing the links between linear algebra and mechanics on the example of gyroscopic stabilization and its behavior under dissipative perturbations.

- **Richard Kollar** presented a graphical Krein signature theory that combines a graphical interpretation of the Krein signature well-known in the spectral theory of polynomial operator pencils as well as in the theory of multiparameter eigenvalue problems with the generalization of the Evans function, the Evans-Krein function, that allows the calculation of Krein signatures in a way that is easy to incorporate into existing Evans function evaluation codes at virtually no additional computational cost. The graphical Krein signature makes extremely elegant the proofs of index theorems for linearized Hamiltonians in the finite dimensional setting: a general result implying as a corollary Vakhitov-Kolokolov criterion (or Grillakis-Shatah-Strauss criterion) generalized to problems with arbitrary kernels, and a count of real eigenvalues for linearized Hamiltonian systems in canonical form. Finally it was demonstrated how the graphical approach can be used to derive new types of criteria prohibiting Hamiltonian-Hopf bifurcations under collisions of two eigenvalues of opposite signature. The talk was a unique and comprehensive survey of the index theorems motivated by very different physical, algebraic, and control theory applications.

- **Jussi Behrndt** discussed the spectral properties of a class of ordinary and partial differential operators with indefinite weight functions. These operators are not symmetric or self-adjoint with respect to a Hilbert space scalar product but they can still be viewed to be symmetric with respect to a suitably chosen Krein space inner product. A general approach via decomposition and perturbation methods were presented to obtain results on the structure of the real and non-real spectrum, as well as quantitative bounds on the non-real spectrum.

- The root radius and root abscissa of a monic polynomial are respectively the maximum modulus and the maximum real part of its roots; both these functions are nonconvex and are non-Lipschitz near polynomials with multiple roots. **Michael Overton** presented constructive methods for efficient minimization of these nonconvex functions in the case that there is just one affine constraint on the polynomial's coefficients. Then he turned to the spectral radius and spectral abscissa functions of a matrix, which are analogously defined in terms of eigenvalues. He explained how to use nonsmooth optimization methods to find local minimizers of these quantities for parameterized matrices and how to use nonsmooth analysis to study local optimality conditions for these nonconvex, non-Lipschitz functions. The pseudospectral radius and abscissa of a matrix A that are respectively the maximum modulus or maximum real part of elements of its pseudospectrum (the union of eigenvalues of all matrices within a specified distance of A), are also nonconvex functions but locally Lipschitz, although the pseudospectrum itself is not a Lipschitz set-valued map. A new method to compute these quantities efficiently for a large sparse matrix A was discussed at the end of the talk.

3.2 Challenging stability and instability problems in physical applications

- **Davide Bigoni** demonstrated counterintuitive examples of structures buckling in tension, where no compressed elements are present. These simple structures exhibit interesting postcritical behaviors, for instance, multiple configurations of vanishing external force. An experimental realization of the flutter instability in the Ziegler pendulum induced by dry friction was demonstrated with the destabilizing effect of dissipation.

- **Edgar Knobloch** reviewed some of the essential properties of the magnetorotational instability, i.e. a magnetic field induced instability of differential rotation that is likely to be of fundamental importance in astrophysics because of its angular momentum transport properties, both in the dissipationless regime and in the dissipative regime, emphasizing the role played by magnetic cross-helicity in determining the nature of this instability. Applications to transport of angular momentum require an understanding of the amplitude of the instability. Its evolution is complex, however, because it involves three radically different timescales: the rotation frequency, the inverse Alfven travel time and the dissipation rate. An asymptotically reduced model was presented that sheds light on the equilibration process both in an intermediate, nominally dissipationless regime, and in the ultimate regime where dissipation takes over, showing how phase mixing can saturate Maxwell and Reynolds stresses even when the instability is still evolving.

- **Emmanuele Tassi** discussed the negative energy modes that are an important issue for the stability properties of continuous media, such as plasmas or fluids. These are spectrally stable modes, possessing negative energy. Their identification is important, because negative energy modes can be destabilized by small perturbations, induced, for instance, by dissipation. A general and effective framework for the study of negative energy modes is the Hamiltonian one. The knowledge of the Hamiltonian structure of a system, allows to unambiguously identify the presence of negative energy modes, through the reduction of the Hamiltonian for the linearized system to its normal form. Two examples of Hamiltonian (in particular Lie-Poisson) systems of interest for plasma physics, were presented, that possess negative energy modes, when linearized about homogeneous equilibria. The two models describe the phenomena of magnetic reconnection and of electron temperature gradient driven turbulence, respectively. Both systems exhibit Krein bifurcations when negative energy modes merge with positive energy modes for critical values of the wavelength of the perturbations.

- **Panayotis Kevrekidis** presented an overview of recent theoretical, numerical and experimental work concerning the static, stability, bifurcation and dynamic properties of coherent structures that can emerge in one- and higher-dimensional settings within Bose-Einstein condensates at the coldest temperatures in the universe (i.e. at the nanoKelvin scale). It was discussed how this ultracold quantum mechanical setting can be approximated at a mean-field level by a deterministic PDE of the nonlinear Schrödinger type and what the fundamental nonlinear waves of the latter are, such as dark solitons and vortices. A further layer of simplified description via nonlinear ODEs encompassing the dynamics of the waves within the traps that confine them, and the interactions between them, was then presented. Finally, an attempt was taken to compare the analytical and numerical implementation of these reduced descriptions to recent experimental results and speculate towards a number of interesting future directions within this field.

- **William Langford** presented a mathematical model of convection in a rotating hemispherical shell of fluid, with radial gravity and a pole-to-equator temperature gradient on the inner boundary. The fluid in the model satisfies the Navier-Stokes Boussinesq PDE and the heat equation. For moderately strong values of the temperature gradient, convection cells appear that resemble the Hadley, Ferrel and polar cells of the present day climate of the Earth. The model reproduces the trade winds, westerlies, jet stream and polar easterlies of today's climate. As the temperature gradient is decreased, the Hadley cell slows in circulation velocity and expands poleward; also the jet stream moves poleward. All these changes have been observed recently in the atmosphere of Earth. Eventually, for still smaller values of the temperature gradient in the model, the Ferrel and polar cells disappear. Furthermore, the model exhibits bistability and hysteresis. One of these two stable states resembles today's climate; the other is more like the "equable" paleoclimate that existed on Earth for much of geological time.

- **Stephane Le Dizes** analyzed the characteristics of the linear waves living on a vortex in an incompressible inviscid homogeneous fluid, showing that a large axial wavenumber asymptotic analysis can be used to provide information on their spatial structure and dispersion relation. The stabilizing role of critical point singularities was discussed and analysed in this framework. Asymptotic results were illustrated and compared to numerical results for a family of vortices ranging from the Rankine vortex (disk of uniform vorticity) to the Lamb-Oseen vortex (gaussian vorticity profile). Then, the waves on similar vortices but in a fluid uniformly stratified in the direction of the vortex axis were considered. It was shown that stratification is a source of instability. Using the large axial wavenumber asymptotic analysis, it was demonstrated that the instability mechanism is associated with the radiative character of the waves. Connections with similar instability in shallow water or in a compressible fluid was made. Experimental evidence of the radiative instability was also provided.

- High-Reynolds number flows are dominated by vortical structures. Vortex filaments are unstable to a number of instabilities: the long wavelength Crow instability, the short wavelength Moore-Saffman-Tsai- Widnall (MSTW) instability and the ultra-short wavelength elliptical instability. The MSTW instability concerns a vortex in strain and was first examined by Moore and Saffman in a general context but with asymptotically small strain. Most of the actual studies since have concentrated on the case of a piecewise continuous profile of vorticity (a Rankine vortex) which supports discrete normal modes. **Stefan Llewellyn Smith** considered in his talk the more general case of non-infinitesimal strain using exact solutions of the Euler equations called hollow vortices and smooth vorticity profiles by looking at an initial-value problem.

- **Paolo Luzzatto-Fegiz** presented the conditions for the development of a Hamiltonian-Hopf instability in vortex arrays. By building on the theory of Krein signatures for Hamiltonian systems, and considering constraints owing to impulse conservation, it was demonstrated that a resonant instability (developing through coalescence of two eigenvalues) cannot occur for one or two vortices. This deduction was illustrated by examining available linear stability results for one or two vortices. It was indicated that a resonant instability may, however, occur for three or more vortices. For these more complex flows, a simple model was proposed, based on an elliptical vortex representation, to detect the onset of a resonant instability. An example was given in support of the theory by examining three co-rotating vortices, for which a linear stability analysis has been performed. The stability boundary in this model is in a good agreement with the full stability calculation. In addition, it was shown that eigenmodes associated with an overall rotation or an overall displacement of the vortices always have eigenvalues equal to zero and $\pm i\Omega$, respectively, where Ω is the angular velocity of the array.

- **Dmitry Pelinovsky** presented a sharp criterion of transverse stability and instability of line solitons in the discrete nonlinear Schrodinger (dNLS) equation on a square two-dimensional lattice near the anti-continuum limit. The fundamental (single-site) line soliton is proved to be transversely stable (unstable) when it bifurcates from the hyperbolic (elliptic) point of the dispersion surface. The results hold for both focusing and defocusing dNLS equation via a staggering transformation. The one-dimensional dNLS equation with the continuous diffraction term was also considered and it was proven that the fundamental line soliton is transversely unstable in both cases when it bifurcates from the hyperbolic and elliptic points of the dispersion surface. In the former case, the instability is caused by the resonance between eigenvalues of negative energy (Krein signature) and the continuous spectrum of positive energy. Analytical results were illustrated numerically.

- In Hamiltonian systems, the Hamiltonian Hopf (HH) bifurcation occurs when two pairs of stable eigenvalues collide at some parameter value and bifurcate to the quartet. According to the Krein-Moser theorem, this bifurcation can only happen if the colliding eigenvalue pairs have opposite signature, which can be determined by evaluating the energy on the eigenfunction. Such a transition to instability (overstability) is seen in the discrete spectrum of PDEs that describe many physical systems, such as the fluid plasma two-stream instability, top-hat distribution description of Jean's instability, and the contour dynamics description of shear flow or Kelvin-Helmholtz instability. The continuum Hamiltonian Hopf (CHH) bifurcation is a similar, but mathematically more challenging, bifurcation that occurs in Hamiltonian PDEs with a continuous spectrum. Examples include the Vlasov equation, Euler's fluid equation, MHD, etc. To understand this bifurcation it is necessary to first attach a signature to the continuous spectrum of Hamiltonian PDEs, a nontrivial task since eigenfunctions of the continuous spectrum are non-normalizable. Having the signature, a version of Krein-Moser theorem is possible provided an appropriate definition of structural stability and parameter variation are given. Thus, in the CHH bifurcation, the continuous spectrum plays the role of one of the eigenvalue pairs of the HH bifurcation. In his talk **Phil Morrison** reviewed the HH bifurcation in the PDE context, gave examples, and described how the CHH bifurcation appears in a variety of physical systems. Rigorous aspects of CHH were described in a companion talk by **George Hagstrom**.

3.3 Dissipation-induced instabilities

- **Gianne Derks** considered the infinite time behaviour of a family of stationary solutions of Euler's equation, which can be described as constrained minima of energy on level sets of enstrophy. For free boundary conditions, this family shadows solutions of 2D Navier-Stokes equations. However, under the no-slip and under the Navier-slip boundary conditions and in a circular domain, the infinite time Navier-Stokes evolution orbit of a starting point on the family of constrained minima has order 1 distance to the family, however small the viscosity is. The viscosity in the Navier-Stokes equations is a singular perturbation for Euler's equation and one might suspect that the viscosity-induced instability is related to this singularity. This is not the case: we show that the same phenomenon can be observed for the averaged Euler equations and second grade fluids with Navier-slip boundary conditions in a circular domain.

- In 1952 Ziegler observed (I) that viscous dissipation can move pure imaginary eigenvalues of a Lyapunov stable time-reversible non-conservative mechanical system (Ziegler's pendulum loaded by a follower force) to the right half of the complex plane and (II) that the threshold of asymptotic stability generically does not converge to the threshold of the Lyapunov stability of the non-damped system when dissipation coefficient tends to zero. In 1956 Bottema related the structurally unstable situation (II) to the Whitney umbrella singularity of the stability boundary. **Oleg Kirillov** has shown the examples of Hamiltonian, reversible and $\mathcal{PT}$-symmetric systems of physics and mechanics with the similar effects of dissipation-induced instabilities and non-commuting limits of vanishing dissipation. The relation of these effects to the multiple non-derogatory eigenvalues occurring both on the stability boundary and inside the domain of asymptotic stability was discussed, the connection to the spectral abscissa minimization was shown and in the Hamiltonian case it was demonstrated that a suitable combination of damping and nonconservative positional forces can destabilize the eigenvalues with both positive and negative Krein (symplectic) signature of the unperturbed system.

- The talk of **Olivier Doare** was devoted to the influence of dissipation on local and global instabilities in the media of infinite length. The waves that are neutral in absence of dissipation become temporally amplified when damping terms are added in the wave equation. The concept of wave energy, introduced in plasma physics, represents a considerable value to the discussion of this effect. The energy of a wave is defined as the work done on the system to generate the neutral wave from $t = -\infty$ to $t = 0$. Consequently, a wave is of negative energy if its establishment lowers the total energy of the system. It was then found that negative energy waves are destabilized by addition of damping. Studies considering waves propagating in an infinite medium are referred to as *local*. Additionally to plasma physics, negative energy waves have been studied in mechanics, in the context of compliant panels interaction with inviscid flows and the instabilities of the surface between two non-miscible fluids. It was found that the presence of gyroscopic terms in the wave equation is necessary to have negative energy waves in the system and that the existence of negative energy waves is a necessary condition to observe destabilization by dissipation in the finite length system. Some simple local criteria, based on characteristic length of rigidity and damping forces were used to develop simple criteria that predict global instability. Finally, destabilization by damping in the context of recent works on energy harvesting using fluttering piezoelectric flexible plates was discussed.

3.4 Integrable systems and bifurcations

- In families of isoperimetrically constrained variational principles the signs of the eigenvalues of the Hessian of the energy with respect to the Lagrange multipliers enters into the second order necessary conditions. **John Maddocks** explained in his talk how the computation of looping probability for an elastic polymer, such as DNA, can be cast in terms of path integrals where the leading order approximation involves an isoperimetric variational principle, and the first, or semi-classical, correction involves the determinant of the same Hessian of the energy with respect to the Lagrange multipliers.

- **Pietro-Luciano Buono** discussed recent results about the Hip-Hop orbit of the Newtonian 2N-body problem. The Hip-Hop orbit (in reduced space) is a periodic solution with time-reversing and spatio-temporal symmetries and in fact, it is a brake orbit. The analytical proof of linear instability of the Hip-Hop orbit was presented using Maslov index methods. Numerical simulations were shown of the Hip-Hop orbit as the energy is varied which exhibits a sequence of symmetry-breaking bifurcations and avenues for classifying those bifurcations were discussed.

- **Richard Cushman** in his talk treated in detail an example of a one parameter family of Hamiltonian systems, which exhibits an S^1-equivariant sign exchange bifurcation in its linearization about an equilibrium point.

- The uncovering of the role of monodromy in integrable Hamiltonian fibrations has been one of the major advances in the study of integrable Hamiltonian systems in the past few decades: on one hand monodromy turned out to be the most fundamental obstruction to the existence of global action-angle coordinates while, on the other hand, it provided the correct classical analogue for the interpretation of the structure of quantum joint spectra. Fractional monodromy is a generalization of the concept of monodromy: instead of restricting our attention to the toric part of the fibration we extend our scope to also consider singular fibres. In his talk **Konstantinos Efstathiou** analyzed fractional monodromy for n_1:$(-n_2)$ resonant Hamiltonian systems with n_1, n_2 coprime natural numbers. In particular, systems that for $n_1, n_2 > 1$ contain one-parameter families of singular fibres which are 'curled tori', were considered. The geometry of the fibration was simplified by passing to an appropriate branched covering. In the branched covering the curled tori and their neighborhood become untwisted thus simplifying the geometry of the fibration: essentially the same type of generalized monodromy was obtained independently of n_1, n_2. Fractional monodromy was then recovered by pushing the results obtained in the branched covering back to the original system.

- **Jeroen Lamb** discussed a system with a deterministic pitchfork bifurcation with the additive noise. It was shown that there is qualitative change in the random dynamics at the bifurcation point in the sense that after the bifurcation, the Lyapunov exponent cannot be observed almost surely in finite time. This bifurcation was associated with a breakdown of both uniform attraction and equivalence under uniformly continuous topological conjugacies, and with non-hyperbolicity of the dichotomy spectrum at the bifurcation point.

- **Zensho Yoshida** considered bifurcation of equilibrium points in fluids or plasmas using the notion of Casimir foliation that occurs in a noncanonical Hamiltonian formulation of an ideal fluid or plasma. The nonlinearity of the system makes the Poisson operator inhomogeneous on phase space (the function space of state variables), resulting in a nontrivial center of the Poisson algebra; the center elements are called Casimirs. Orbits are constrained on level-sets of Casimirs, i.e. Casimir leaves. Even if a Hamiltonian is simple (typically a fluid/plasma Hamiltonian is just the "norm" of phase space, unlike bumpy Hamiltonians modeling strongly coupled systems), energy contours on a Casimir leaf may have considerably complicated shapes. Invoking a simple model of plasma, it was shown that the equilibrium points on Casimir leaves bifurcate as Casimir parameters change. The energies of bifurcated equilibrium points can be compared to estimate the stability. In ideal dynamics, however, a higher-energy state may sustain stably by other Casimir constraints; in fact "resonant singularities" generate infinite number of "singular Casimir elements" which foliate the phase space and separate different equilibrium points. A singular perturbation (introduced by finite dissipation) destroys the Casimir leaves, removing the topological constraint and allowing the state vector to move towards lower-energy state in unconstrained phase space. An extended Hamiltonian mechanical representation of such an instability caused by a singular perturbation was proposed.

- **Yasuhide Fukumoto** considered a steady Euler flow of an inviscid incompressible fluid characterized as an extremum of the total kinetic energy (=the Hamiltonian) with respect to perturbations constrained to an isovortical sheet (=coadjoint orbits). The criticality in the Hamiltonian was used to calculate the energy of three-dimensional waves on a steady vortical flow, and, as a by-product, to calculate the mean flow, induced by nonlinear interaction of waves with themselves. These formulas were applied to study the linear and weakly nonlinear stability of a rotating flow confined in a cylinder of elliptic cross-section. The linear instability, parametric resonance between a pair of Kelvin waves, is known as the Moore-Saffman-Tsai-Widnall (MSTW) instability. The linear stability characteristics is well captured from the viewpoint of Krein's theory of Hamiltonian spectra. Furthermore, with the mean flow induced by the Kelvin waves, a hybrid method of combining the Eulerian and the Lagrangian approaches was shown to be effective to deduce the amplitude equations to third order.

- Stability problems of various fluid models (PDEs) are widely addressed in the studies of complex fluids, geophysical fluids, astrophysical and laboratory plasmas and so on. If the model is physically well-posed, it is promising to find its Hamiltonian structure and Casimir invariants in the dissipationless limit, which can yield a priori estimates for Lyapunov stability. More detailed stability analysis is often facilitated by restoring the Lagrangian description of fluid, especially when there are many Lagrangian invariants (i.e., frozen-in fields). The Lagrangian viewpoint is advantageous in that it enjoys an effective use of variational principle. For example, in linear stability analysis, the variational principle renders the eigenvalue problem being composed of Hermitian and anti-Hermitian operators. The concepts of action-angle variables and adiabatic invariance can be formulated for not only discrete spectrum but also continuous one. By invoking the Lie series expansion, weakly nonlinear analysis is also performed systematically, and the normal forms for mode-mode couplings are extracted by least algebraic manipulations. Even for strongly nonlinear problem such as explosive instability, the variational principle enables us to infer its mechanism in a heuristic manner. **Makoto Hirota** in his talk gave an overview of recent advancements in this Lagrangian approach.

3.5 Semi-classical approximation

- **Michael Berry** presented a study of the evolution of optical polarization in a stratified nontransparent dielectric medium twisted cyclically along the propagation direction. The twist is chosen to encircle a degeneracy (branch-point) in the plane of parameters describing the medium. Polarization evolutions are determined analytically and illustrated as tracks on the Poincaré sphere and the stereographic plane. Even when the twist is slow, the exact evolutions differ sharply from those of the local eigenpolarizations and can display extreme sensitivity to initial conditions with the tracks exhibiting elaborate coilings and loopings that would be very interesting to explore experimentally. Underlying these dramatic violations of adiabatic intuition are the disparity of exponentials and the Stokes phenomenon of asymptotics.

- **Sergey Dobrokhotov** discussed a multi-dimensional analogue of the problem of the level splitting for a Schrödinger-particle placed into a symmetric double-well potential. The splitting is described by the formula $\triangle E = A\exp(-\frac{J}{h})$ with the phase J based on a certain classical trajectories known as instanton. The constructive but complicated formula of the amplitude A is also connected with the instanton. In the talk it was shown that the splitting formula takes more natural and simple form if one changes the instanton by so-called libration (unstable closed trajectories) and use the normal forms coming from classical mechanics. Finally, a non-trivial question on the level splitting in the presence of magnetic field was discussed. It was demonstrated that in 2-D case using the partial Fourier transform and mixed momentum-position coordinates one can reduce the quantum double-well problem with magnetic field to the standard quantum double-well problem and to study the splitting in this situation too.

- **Setsuro Fujiié** considered the semi-classical Schrödinger operator $P := -h^2\Delta + V(x)$ in R^n, where h is a small positive (semi-classical) parameter and $V(x)$ is a real-valued smooth potential decaying at infinity. If the classical dynamics for the corresponding classical Hamiltonian $p(x,\xi) := |\xi|^2 + V(x)$ has *trapped* trajectories on $p^{-1}(z_0)$ for a real positive energy z_0, it is expected that there exist the so-called *resonances* close to z_0 in the lower half complex plane. The imaginary part of resonances, called *width*, means the reciprocal of the exponential decay rate of the corresponding states for the evolution as time tends to $+\infty$. In the talk the width of resonances associated with an unstable equilibrium of the potential was discussed. If $x = x_0$ is an unstable equilibrium, i.e. a local maximum of the potential, then the point $(x,\xi) = (x_0, 0)$ in the phase space is a hyperbolic fixed point of the Hamilton vector field, and it is itself a trapped trajectory. Hence resonances may appear near the energy $E_0 := V(x_0)$. Contrary to the case of a stable equilibrium, the trap by an unstable equilibrium is much weaker, and, as consequence, the resonance width should be large. Additional higher bumps of potential create *homoclinic* trajectories converging to the hyperbolic fixed point as time tends to $+\infty$ and $-\infty$. Assuming that the trapped set at the energy level E_0 consists of the hyperbolic fixed point and these homoclinic trajectories, lower bounds for the resonance width were obtained.

4 Scientific Progress Made

Many participants commented during the meeting as well as after it that they benefited a lot from the cross-disciplinary nature of the talks. In particular,

- **Michael Berry** found that the workshop was well conceived, bringing together people in fields that are different yet similar enough for meaningful and indeed useful connections to develop.
- **Almut Burchard** and **Marina Chugunova** continued their collaboration and exchanged recent ideas obtained after working apart from each other for a while.
- **Sergey Dobrokhotov** said that the workshop turned out to be very useful because many results impact his and his colleagues research; the new contacts he made could lead to joint works in the future.
- **Panos Kevrekidis** and **Dmitry Pelinovsky** have used the workshop space to enhance their collaboration and to complete a manuscript, which is now published in Europhysics Letters [4].
- **Oleg Kirillov** was included in almost every discussion. As a feedback during the workshop, he improved his manuscript, which is now accepted for Phil. Trans. R. Soc. A [3].
- **Richard Kollar** benefited from many discussions with other speakers and said that the main highlight of this workshop is bringing together people from different groups and research background but specializing on the same range of stability and bifurcation problems. These discussions contributed to his recent review paper for SIAM Reviews [1].
- **Michael Overton** received very good feedback on his talk and got to know many colleagues whom he had not met before.
- **Carsten Trunk** had admitted that his personal research activities were greatly influenced by the workshop, so much that even his PhD students will feel it.
- **Charles Williamson** made new contacts and said the meeting had helped to one of his students in developing his own future and scientific contacts.
- **Zensho Yoshida** had benefited by the workshop very much and started new interactions.

About twenty specialized articles are in preparation for the volume [2] that will be published by Wiley-ISTE with the aim to collect the highlights of the meeting.

5 Outcome of the Meeting

The BIRS Workshop on Spectral Analysis, Stability and Bifurcations in Modern Nonlinear Physical Systems collected together a unique combination of experts in modern dynamical systems, mathematical physics, PDEs, numerical analysis, operator theory, and applications.

One of the immediate outcomes of the meeting that makes its materials available for a broader audience is a post-conference volume of papers [2] from the participants of the workshop. This project is aimed to collect unique viewpoints of the participants on the history, current state of the art, and prospects of research in their fields contributing to the progress of stability theory. Our vision of this book is a collection of essays - mathematical, physical, and mechanical. The contributions will show connections between different approaches, applications, and ideas. We believe that such a book could set up the benchmarks and goals for the next generation of researchers and be a true event in modern stability theory.

The other outcomes will be seen over a long range of time, when the ideas formulated and discussed during the workshop, as well as new collaborations made, will lead to new scientific publications and new research discoveries.

References

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[4] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Frantzeskakis, PT-symmetric lattices with spatially extended gain/loss are generically unstable, Eur. Phys. Lett. **101** (2013), 11002 (6 pages).