Long-term behavior of subcritical contact processes

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Random measures and measure valued processes
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Outline

1. Introduction-the contact process
2. Eigenmeasures - main result and applications
3. Proof outlines
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1. Introduction - the contact process
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The contact process - definition

Ingredients:
- Finite or countable group $\Lambda$
- Infection kernel $a(i, j), i, j \in \Lambda$
  - translation invariant, irreducible, $|a| := \sum_{i \in \Lambda} a(0, i) < \infty$
- Recovery rate $\delta \geq 0$

$(\Lambda, a, \delta)$-contact process with states in $\{0, 1\}^\Lambda$:
- type 1 at site $i$ induces a type 1 at site $j$ with rate $a(i, j)$
- type 1 at site $i$ becomes a type 0 at rate $\delta$

Remark: Equip $\{0, 1\}^\Lambda$ with the product metric.
The contact process - definition

As a process $\eta = (\eta_t)_{t \geq 0}$ taking values in $\mathcal{P} := \{ A : A \subset \Lambda \}$ (set of 1’s) it has the formal generator

$$Gf(A) := \sum_{i,j \in \Lambda} a(i,j) 1_{\{i \in A\}} 1_{\{j \notin A\}} \{ f(A \cup \{j\}) - f(A) \} + \delta \sum_{i \in \Lambda} 1_{\{i \in A\}} \{ f(A \setminus \{i\}) - f(A) \}.$$

We write $\eta^A$ if $\eta^A_0 = A$ a.s.
Duality:

Consider **reversed infection rates**: \( a^\dagger(i, j) := a(j, i) \)

- \((\eta_t^A)_{t \geq 0} : (\Lambda, a, \delta)\)-contact process
- \((\eta^\dagger_t B)_{t \geq 0} : (\Lambda, a^\dagger, \delta)\)-contact process

Then

\[
P[\eta_t^A \cap B \neq \emptyset] = P[A \cap \eta^\dagger_t B \neq \emptyset] \quad A, B \in \mathcal{P}(\Lambda), \quad t \geq 0.
\]
The contact process - elementary properties

Survival probability:

We say that the \((\Lambda, a, \delta)\)-contact process \textbf{survives} if

\[ \rho(A) := \mathbb{P}[\eta_t^A \neq \emptyset \ \forall t \geq 0] > 0 \]

for some, and hence for all nonempty \(A\) of finite cardinality \(|A|\).

We set \(\theta := \rho(\{0\})\) and call

\[ \delta_c := \sup\{\delta \geq 0 : \theta > 0\} \]

the \textbf{critical recovery rate}.

\[ \rightarrow \delta > \delta_c \ \text{subcritical} \]
The contact process - elementary properties

Long term behavior:

$$\mathbb{P}[\eta^\Lambda_t \in \cdot] \xrightarrow{t \to \infty} \nu,$$

$\nu$ : upper invariant law.

- $\nu$ is concentrated on the nonempty subsets of $\Lambda$ if the $(\Lambda, a^+, \delta)$-contact process survives.
- $\nu = \delta_\emptyset$ if the $(\Lambda, a^+, \delta)$-contact process dies out.
The contact process: elementary properties

Exponential growth:

There exists a constant $r = r(\Lambda, a, \delta)$ with $-\delta \leq r \leq |a| - \delta$ such that

$$r = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|\eta_t^A|] \quad A \in \mathcal{P}_{\text{fin},+}.$$ 

Notation:

$$\mathcal{P}_{\text{fin},+} := \mathcal{P}_{\text{fin}} \cap \mathcal{P}_+$$

$$\mathcal{P}_{\text{fin}} := \{A \subset \Lambda : |A| < \infty\}$$

$$\mathcal{P}_+ := \{A \subset \Lambda : |A| > 0\}$$
The contact process: elementary properties

Known properties of the exponential growth rate:

- $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$.
- $\delta \mapsto r(\delta)$ is nonincreasing and Lipschitz continuous on $[0, \infty)$ with Lipschitz constant 1.
- If $r > 0$, then the contact process survives.
- $\{\delta \geq 0 : r(\delta) < 0\} = (\delta_c, \infty)$. 
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1. Introduction-the contact process

2. Eigenmeasures - main result and applications

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Definition of eigenmeasures

A measure $\mu$ on $\mathcal{P}_+$ is an **eigenmeasure** of the $(\Lambda, a, \delta)$-contact process if $\mu$ is nonzero, locally finite, and there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int \mu(dA) \mathbb{P}[\eta^A_t \in \cdot] |_{\mathcal{P}_+} = e^{\lambda t} \mu \quad t \geq 0.$$ 

We call $\lambda$ the associated **eigenvalue**.
Known properties of eigenmeasures

- **Existence:**
  Each \((\Lambda, a, \delta)\)-contact process has a (spatially) homogeneous eigenmeasure \(\hat{\nu}\) with eigenvalue \(r = r(\Lambda, a, \delta)\).

- **Scaling and normalization:**
  If \(\hat{\nu}\) is an eigenmeasure, then also \(c\hat{\nu}\) for \(c > 0\).
  We normalize: \(\int \hat{\nu}(dA)1_{\{0 \in A\}} = 1\)

- **Uniqueness:**
  In general not known if \(\hat{\nu}\) is unique.
  Special case: a irreducible, \(\bar{\nu}\) nontrivial and \(r(\Lambda, a, \delta) = 0\), then \(\hat{\nu}\) is unique and \(\hat{\nu} = c\bar{\nu}\).
Notions of convergence

Let $\mu_n, \mu$ be locally finite measures on $\mathcal{P}$.

- $\mu_n \rightarrow \mu$ vaguely ($\mu_n \Rightarrow \mu$) $\iff$

  $$\int \mu_n(dA)f(A) \rightarrow \int \mu(dA)f(A)$$

  for $f$ continuous, compactly supported.

- For $\mu_n, \mu$ concentrated on $\mathcal{P}_{\text{fin, +}}$, $\mu_n \rightarrow \mu$ locally $\iff$

  $$\mu_n|_{\mathcal{P}_{\text{fin, i}}} \rightarrow \mu|_{\mathcal{P}_{\text{fin, i}}} \text{ weakly}$$

  where $\mathcal{P}_{\text{fin, i}} := \mathcal{P}_{\text{fin}} \cap \mathcal{P}_i$ with $\mathcal{P}_i := \{A \in \mathcal{P} : i \in A\}$.

  (Local convergence implies vague convergence.)
Uniqueness of and convergence to eigenmeasures

**Theorem 1: Sturm, Swart**

Let $a$ be irreducible and $r < 0$. Then:

- There exists a **unique** homogeneous eigenmeasure $\hat{\nu}$ of the $(\Lambda, a, \delta)$-contact process such that $\int \hat{\nu}(dA) 1_{\{0 \in A\}} = 1$.
- $\hat{\nu}$ has eigenvalue $r$ and is **concentrated on** $\mathcal{P}_{\text{fin}}$.
- If $\mu$ is any nonzero, homogeneous, locally finite measure on $\mathcal{P}_+$, then

$$e^{-rt} \int \mu(dA) \mathbb{P}[\eta^A_t \in \cdot] \rvert_{\mathcal{P}_+(\Lambda)} \xrightarrow{t \to \infty} c(\mu) \hat{\nu}.$$ 

If $\mu$ is concentrated on $\mathcal{P}_{\text{fin, +}}$ this holds in the sense of local convergence.
Process modulo shifts

Identify sets modulo shifts:

\[ \tilde{\mathcal{P}}_{\text{fin}} := \{ \tilde{A} : A \in \mathcal{P}_{\text{fin}} \} \]

\[ \tilde{A} := \{ iA : i \in \Lambda \} \]

Let \( \tilde{\eta} \) be the on \( \tilde{\mathcal{P}}_{\text{fin}} \) induced Markov process:

\((\Lambda, a, \delta)\)-contact process modulo shifts.

Transition probabilities:

\[ \tilde{P}_t(\tilde{A}, \tilde{B}) = \sum_{C \in \tilde{\mathcal{P}}_{\text{fin}}, C + \tilde{C} = \tilde{B}} P_t(A, C) = m(B)^{-1} \sum_{i \in \Lambda} P_t(A, iB) \]
Connection to quasi-invariance

Let $\Delta$ be a $\mathcal{P}_{\text{fin},+}$-valued random variable with

$$\hat{\nu} = c \sum_{i \in \Lambda} \mathbb{P}[i\Delta \in \cdot]$$

and $\hat{\nu}$ the law of $\Delta$ modulo shifts.

**Theorem 2: Sturm, Swart**

Under the assumptions of Theorem 1 the law $\hat{\nu}$ is a quasi-invariant law for the $(\Lambda, a, \delta)$-contact process modulo shifts. For any $A \in \mathcal{P}_{\text{fin},+}$

$$\mathbb{P}[\tilde{\eta}_t^A \in \cdot \mid \eta_t^A \neq \emptyset] \xrightarrow{t \to \infty} \hat{\nu},$$

where $\Rightarrow$ denotes weak convergence on $\tilde{\mathcal{P}}_{\text{fin},+}$.

$\to$ Ferrari, Kesten, Martinez ’96
Application

As an application we derive an expression for the derivative of the exponential growth rate

$$r = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|\eta_t^0|].$$

Let

$$\mu_t := \sum_i \mathbb{P}[\eta_t^i \in \cdot | \mathcal{P}_+]$$

$$\pi_t := \mu_t(\{A : 0 \in A\}) = \sum_i \mathbb{P}[0 \in \eta_t^i] = \mathbb{E}[|\eta_t^0|].$$

We use Theorem 1 implying $e^{-rt} \mu_t \overset{t \to \infty}{\longrightarrow} c^{\circ}$ and

$$\frac{\partial r}{\partial \delta} = \frac{\partial}{\partial \delta} \lim_{t \to \infty} \frac{1}{t} \log \pi_t.$$
Theorem 3: Sturmf, Swart

Under the assumptions of Theorem 1 the function

$$\delta \mapsto r(\Lambda, a, \delta)$$

is continuously differentiable on \((\delta_c, \infty)\) and satisfies

$$\frac{\partial}{\partial \delta} r(\Lambda, a, \delta) < 0 \text{ on } (\delta_c, \infty).$$

Moreover,

$$\frac{\partial}{\partial \delta} r(\Lambda, a, \delta) = -\frac{\int \hat{\nu}(dA) \int \hat{\nu}^\dagger(dB) 1_{\{A \cap B = \{0\}}}{\int \hat{\nu}(dA) \int \hat{\nu}^\dagger(dB)|A \cap B|^{-1} 1_{\{0 \in A \cap B\}}}$$

where \(\hat{\nu}\) and \(\hat{\nu}^\dagger\) are the eigenmeasures of the \((\Lambda, a, \delta)\)- and \((\Lambda, a^\dagger, \delta)\)-contact processes, respectively.
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Proof outline Theorem 3

One can show that

$$\frac{\partial}{\partial \delta} r(\delta) = \frac{\partial}{\partial \delta} \lim_{t \to \infty} \frac{1}{t} \log \pi_t(\delta)$$

$$= \lim_{t \to \infty} \frac{1}{t} \frac{\partial}{\partial \delta} \log \pi_t(\delta)$$

$$= \lim_{t \to \infty} \frac{1}{t} \frac{\partial}{\partial \delta} \frac{\pi_t(\delta)}{\pi_t(\delta)}.$$

Use local convergence of Theorem 1 for the following expressions:
Proof outline Theorem 3

\[
\pi_t(\delta) = \sum_j \mathbb{P}[(j, 0) \rightsquigarrow (0, t)] = \sum_i \mathbb{P}[(0, 0) \rightsquigarrow (i, t)]
\]

\[
= \sum_i \mathbb{P}[\eta_s^{\{0\}} \cap \eta_{t-s}^{\{i\}} \neq \emptyset]
\]

\[
= \sum_{i,j} \mathbb{E}[|\eta_s^{\{0\}} \cap \eta_{t-s}^{\{i\}}|^{-1} 1_{\{j \in \eta_s^{\{0\}} \cap \eta_{t-s}^{\{i\}}\}}]
\]

\[
= \sum_{i,j} \mathbb{E}[|\eta_s^{\{j^{-1}\}} \cap \eta_{t-s}^{\{j^{-1}i\}}|^{-1} 1_{\{0 \in \eta_s^{\{j^{-1}\}} \cap \eta_{t-s}^{\{j^{-1}i\}}\}}]
\]

\[
= \int \mu_{s,\delta}(dA) \int \mu_{t-s,\delta}(dB)|A \cap B|^{-1} 1_{\{0 \in A \cap B\}}
\]
Proof outline Theorem 3

With \((0, 0) \rightsquigarrow (j, s) (i, t)\) denoting the event of an open path from \((0, 0)\) to \((i, t)\) with \((j, s)\) pivotal

\[
\frac{1}{t} \frac{\partial}{\partial \delta} \pi_t(\delta) = - \sum_{i,j} \frac{1}{t} \int_0^t ds \, \mathbb{P}[\((0, 0) \rightsquigarrow (j, s) (i, t)\)]
\]

\[
= - \sum_{i,j} \frac{1}{t} \int_0^t ds \, \mathbb{P}[(j^{-1}, -s) \rightsquigarrow (0, 0) (j^{-1} i, t - s)]
\]

\[
= - \sum_{i,j} \frac{1}{t} \int_0^t ds \, \mathbb{P}[\eta_s^{(i)} \cap \eta_t^{\dagger (j)} = \{0\}]
\]

\[
= - \frac{1}{t} \int_0^t ds \int \mu_{s, \delta}(dA) \int \mu_{t-s, \delta}^{\dagger}(dB) 1_{\{A \cap B = \{0\}\}}
\]
Proof outline Theorem 1 - Step 1

Existence of an eigenmeasure concentrated on $\mathcal{P}_{\text{fin}}$:

**Proposition 1**

Let $r < 0$. Then there exists a homogeneous eigenmeasure $\hat{\nu}$ with eigenvalue $r$ of the $(\Lambda, a, \delta)$-contact process such that

$$\int \hat{\nu}(dA)|A|1_{\{i \in A\}} < \infty \quad (i \in \Lambda).$$

In particular, $\hat{\nu}$ is concentrated on $\mathcal{P}_{\text{fin}}$. 
Step 1: Eigenmeasure concentrated on $\mathcal{P}_{\text{fin}}$

Proof outline of Proposition 1:
Let $\hat{\mu}_\lambda := \int_0^\infty \mu_t \, e^{-\lambda t} \, dt$ and $\hat{\pi}_\lambda := \int_0^\infty \pi_t \, e^{-\lambda t} \, dt$.

- Swart '09: The measures $\frac{1}{\hat{\pi}_\lambda} \hat{\mu}_\lambda$ ($\lambda > r$) are relatively compact.

Each subsequential limit as $\lambda \downarrow r$ is a homogeneous eigenmeasure of the $(\Lambda, a, \delta)$-contact process, with eigenvalue $r(\Lambda, a, \delta)$.

- We have

$$\limsup_{\lambda \downarrow r} \frac{1}{\hat{\pi}_\lambda} \int \hat{\mu}_\lambda (\, dA) 1_{\{0 \in A\}} |A| < \infty.$$
Step 2: Uniqueness and vague convergence

For uniqueness and vague convergence it suffices to show for $B \in \mathcal{P}_{\text{fin}}$,+

$$e^{-rt} \int \mu P_t(dA) 1_{\{A \cap B \neq \emptyset\}} \xrightarrow{t \to \infty} c(\mu) \int \nu(dA) 1_{\{A \cap B \neq \emptyset\}}.$$

In order to rewrite the right hand side define

$$h_\mu(A) := \int \mu(dB) 1_{\{A \cap B \neq \emptyset\}} \quad A \in \mathcal{P}_{\text{fin}}.$$
Step 2: Uniqueness and vague convergence

For $B \in \mathcal{P}_{\text{fin, +}}$ the right hand side is

$$e^{-rt} h_\mu P_t(B) = e^{-rt} P_t^\dagger h_\mu(B) = e^{-rt} P_t^\dagger \tilde{h}_\mu(\tilde{B})$$

$$= \tilde{h}_\nu(\tilde{B}) \sum_{\tilde{A} \in \tilde{\mathcal{P}}_{\text{fin, +}}} Q_t^\dagger(\tilde{B}, \tilde{A}) \frac{\tilde{h}_\mu(\tilde{A})}{\tilde{h}_\nu(\tilde{A})}$$

with

$$Q_t^\dagger(\tilde{B}, \tilde{A}) := e^{-rt} \frac{\tilde{h}_\nu(\tilde{A})}{\tilde{h}_\nu(\tilde{B})} P_t^\dagger(\tilde{B}, \tilde{A})$$
Step 2: h-transformed Markov process

Proposition 2

$Q_t^\dagger(\tilde{A}, \tilde{B})$ are the transition probabilities of an irreducible, positively recurrent Markov process with state space $\tilde{\mathcal{P}}_{\text{fin}, +}$.

Because $\frac{\tilde{h}_\mu}{\tilde{h}_\nu}$ can be shown to be bounded we obtain

$$\frac{\tilde{h}_\mu}{\tilde{h}_\nu}(\tilde{B}) \sum_{\tilde{A} \in \tilde{\mathcal{P}}_{\text{fin}, +}} Q_t^\dagger(\tilde{B}, \tilde{A}) \frac{\tilde{h}_\mu(\tilde{A})}{\tilde{h}_\nu(\tilde{A})}$$

$$\xrightarrow{t \to \infty} h_\nu(B) \ c(\mu) = \int_{\mathcal{A}} \check{\nu}(dA) 1_{\{A \cap B \neq \emptyset\}} \ c(\mu)$$
Step 3: Local convergence

It suffices to show "pointwise convergence":
For \( B \in \mathcal{P}_{\text{fin}} \), we have

\[
e^{-rt} \mu P_t(\{B\}) \xrightarrow{t \to \infty} c(\mu) \circ \nu(\{B\})
\]

using the positively recurrent Markov chain.
Open problems:
Analogous results in the critical and supercritical regime.

Thank you for your attention!