Applications of Iwasawa Algebras

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1 Overview of the Field

Let $\mathbb{Q}$ be the field of rational numbers and p denote an odd prime number. An elliptic curve E over a base field F is a nonsingular projective curve of genus one which has an F-rational point. The study of rational points on elliptic curves over $\mathbb{Q}$ is an ancient problem that attracted Fermat's interest in the 17th century. Its deep connections with modern number theory and arithmetic were well-understood by the early second half of the last century. Given an elliptic curve $E/\mathbb{Q}$, the set of points $E(\mathbb{Q})$ is known (Mordell-Weil Theorem) to be a finitely generated abelian group and its rank is the *Mordell-Weil rank.*. Attached to E is the Hasse-Weil L-function, $L(E/\mathbb{Q}, s)$, a function of a complex variable and a generalisation of the classical Riemann-zeta function. The order of vanishing of the L-function at the integer 1 is the *analytic rank* and the celebrated Birch and Swinnerton-Dyer conjecture (BSD Conjecture), one of the Clay Millennium problems, asserts that it equals the Mordell-Weil rank.

Iwasawa theory had its origins in Iwasawa's ground breaking work in the 1960's [19]on the growth of class numbers in infinite $\mathbb{Z}_p$-extensions of number fields. It was realised fairly early on that techniques from Iwasawa theory could be used to study the growth of Mordell-Weil rank in infinite p-adic Lie extensions of number fields. Classical Iwasawa theory achieved important results for abelian extensions such as the cyclotomic $\mathbb{Z}_p$-extension and $\mathbb{Z}_p \times \mathbb{Z}_p$-extensions associated to elliptic curves with complex multiplication. The centre piece of Iwasawa theory of elliptic curves is the formulation of the 'main conjecture' which provides a philosophical reason for the phenomena underlying the BSD conjecture, viz. a relation between an algebraic-arithmetic object (the Mordell-Weil rank) and an analytic-arithmetic object (the analytic rank). In the last decade of the last century, the first steps towards formulating the main conjecture for elliptic curves over nonabelian p-adic Lie extensions were taken,and this culminated in the eventual formulation of the main conjecture [8] around a decade ago. The dual Selmer group of the elliptic curve over the infinite p-adic Lie extensions is the algebraic object of study here. It is a finitely generated module over the **Iwasawa algebra** $\mathbb{Z}_p[[G]]$ where G is the p-adic Lie group which is the Galois group of the infinite p-adic Lie extension. Understanding the structure of the Selmer group, along with descent arguments then yields information on the points on the elliptic curve over the base field and also the growth of the Mordell-Weil rank for elliptic curves in the finite layers of the infinite extension. While classical Iwasawa theory affords a structure theorem when G is abelian, the study of modules over Iwasawa algebras when G is nonabelian is a vigorous area of study, thanks to its applications in arithmetic.

In the theory of local Galois representations Iwasawa algebras come in through (φ, Γ)-modules, a theory originally due to J.M. Fontaine. In [10] he describes an equivalence between the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of finite dimensional $\mathbb{Q}_p$-linear representations of the absolute Galois group G_K of a local field K and the category $(\varphi,\Gamma)-\mathrm{Mod}_B^{\acute{e}t}$ of étale (φ,Γ)-modules over B:

$$D : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \longleftrightarrow (\varphi,\Gamma)-\mathrm{Mod}_B^{\acute{e}t}.$$

Here, Γ can be chosen to be the Galois group $G(K_\infty/K)$ of any arithmetically profinite (Galois) extension K_∞ of K. Such extensions admit the field of norms construction of Fontaine and Wintenberger [12, 11, 28], i.e., there is a field E in characteristic p such that $G_{K_\infty} \cong G_E$. If A denotes a Cohen ring for E, i.e., a complete discrete valuation ring such that $A/p \cong E$, then B can be chosen as the field $A[\frac{1}{p}]$. The rings A and B come equipped with an operator φ and an action by Γ which commutes with φ. Then a (φ,Γ)-module (M,φ_M) over C in $\{E,A,B\}$ is a finitely generated free C-module M together with a φ_C-semi-linear operator φ_M and a continuous Γ-operation, which commutes with φ_M. (M,φ_M) is called étale, if the map $C_{\varphi_C}\otimes_C M \to M, c\otimes m \mapsto c\varphi_M(m)$, where in the tensor product C is considered as C-module via φ_C, is an isomorphisms (and if the pair arises from one over A in case $C = B$). In the following we shall write $D(V)$ for the (φ,Γ)-module attached to a p-adic representation V of G_K.

Only in the case $K = \mathbb{Q}_p$ (or at least unramified) and $K_\infty = K(\mu(p))$ all the involved rings become very explicit:

$$E = \mathbb{F}_p((\pi)), \qquad A = (\mathbb{Z}_p[[\pi]][\frac{1}{\pi}])^\wedge \qquad (p\text{-adic completion}),$$

where $\varphi(1+\pi) = (1+\pi)^p$ and $\gamma(1+\pi) = (1+\pi)^{\chi(\gamma)}$ for all $\gamma \in \Gamma$ denoting by $\chi : \Gamma \to \mathbb{Z}_p^\times$ the cyclotomic character. Note that the ring A contains the ring $A^+ = \mathbb{Z}_p[[\pi]]$ which is isomorphic to the **Iwasawa algebra** $\Lambda(\mathbb{Z}_p)$.

Due to the thesis of Herr [15, 16] one knows how to explicitly calculate the Galois cohomology $H^i(K,V)$ in terms of the associated (φ,Γ)-module $D(V)$. Roughly speaking the Herr complex can be obtained as

$$\mathit{cone}\left(C^\bullet(\Gamma,M) \xrightarrow{1-\varphi} C^\bullet(\Gamma,M)\right)$$

where $C^\bullet(\Gamma,M)$ denotes any complex calculating continuous group cohomology of M with respect to Γ. If T denotes a Galois stable lattice of V, then we write $H^i_{Iw}(K_\infty,V) = \left(\mathrm{proj}\lim_n H^i(K_n,T)\right)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ for the Iwasawa cohomology of V. Again, as observed by Fontaine, it can be computed in terms of the (φ,Γ)-module, viz as the cohomology of the complex

$$D(V) \xrightarrow{1-\psi} D(V)$$

concentrated in degree 1 and 2, supposed φ has a left inverse operator ψ. Note that $\mathcal{D}_{cont}(\mathbb{Z}_p,\mathbb{Z}_p)^{\psi=0} \cong \mathcal{D}_{cont}(\mathbb{Z}_p^\times,\mathbb{Z}_p) \cong \mathcal{D}_{cont}(\Gamma,\mathbb{Z}_p) \cong \Lambda(\Gamma)$ which is no accident!

In particular, the theory of (φ,Γ)-modules provides the link between p-adic Hodge Theory and Galois/Iwasawa cohomology, from which its importance in (local) Iwasawa theory arises. On the other hand the above mentioned equivalence of categories can be effectively used (at least for $GL_2(\mathbb{Q}_p)$ as already done by Colmez in [9]) in the context of the p-adic local Langlands program.

It is a very basic principle of representation theory to make group representations into modules over some appropriate group ring. The rationale behind this is that the wealth of methods from (non)commutative algebra becomes applicable. For finite groups G one uses the naive algebraic group ring $K[G]$ over the respective field of coefficients K. For topological groups like Lie groups the right objects are algebras of distributions of different kinds on the group.

The deep interrelations between number theory and representation theory come about through the Langlands program. It predicts an array of correspondences between the finite dimensional representations of the absolute Galois group of a local or global number field and the representations of the local or adelic point groups of reductive algebraic groups over the respective field. The workshop focussed on the local situation. Let L by a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers and $\mathcal{G}_L$ its absolute Galois group. The classical local Langlands programm is concerned with representations of $\mathcal{G}_L$ in vector spaces over the complex numbers $\mathbb{C}$ or over an algebraically closed l-adic field $\overline{\mathbb{Q}}_l$ for some prime number $l \neq p$. The category of finite dimensional p-adic $\mathcal{G}_L$-representations is much more complicated. But starting in the 1980's with the seminal work of Fontaine it is now rather well understood in terms of various linear algebra data. For example, by a theorem of Fontaine, it is naturally equivalent to the category of etale (φ,Γ)-modules. This is one of the places where representation theory connects to the topic of our workshop.

Developments over the past twenty years in the fundamental problem of modularity of Galois representations strongly suggests to study the p-adic variation of the objects under consideration. This in turn leads to the necessity to extend the Langlands program to p-adic coefficients. Recent work of Colmez has achieved exactly this for the field $L = \mathbb{Q}_p$, two-dimensional p-adic $\mathcal{G}_{\mathbb{Q}_p}$-representations, and the continuous p-adic representation theory of the reductive group $\mathrm{GL}_2(\mathbb{Q}_p)$. Beyond this case almost everything is still shrouded in mystery and the open problems abound. There is not even a conjectural formulation yet of a p-adic local Langlands correspondence in general.

2 Recent Developments and Open Problems

The BSD conjecture itself is now accompanied by refinements such as the Equivariant Birch and Swinnerton-Dyer conjecture, which in turn is linked to equivariant versions of the main conjecture. Closely linked to the Main Conjecture is the Tamagawa Number conjecture [6] which postulates exact links between special values of L-functions and certain measures associated to arithmetic groups. There is also an equivariant version of the Tamagawa number conjecture [4]. A common thread in all important results in these intertwined areas is the use of Iwasawa algebras. The algebraic side of the main conjecture relates the G-Euler characteristic of the dual Selmer groups (which are in turn expressed by Tor groups of modules associated to Iwasawa algebras) to the special values of the L-function. The bridge from results pertaining to the main conjecture to extracting exact arithmetic information on rational points needs an other important, technical ingredient, namely the ‘Euler systems’. These Euler systems also play a key role in deducing global information via the local information obtained from Fontaine’s theory, and in constructing the p-adic L-functions, which are p-adic avatars of the complex L-function and interpolate special values of the latter.

One of the novelties in the noncommutative main conjecture is the description of the p-adic L-function as an element in the K-group K_1 of certain localization of the non commutative Iwasawa algebras. This has led to Iwasawa theory and Iwasawa algebras making natural connections with algebraic K-theory and the study of classical localization exact sequences in algebraic K-theory has thus an important bearing on Iwasawa theory. Understanding the explicit algebraic structure of these localized K-groups has thus become a question occupying the attention of number theorists. The proof of the noncommutative main conjecture in the simplest case of the trivial motive in the last few years has further demonstrated the importance of exploring these questions [20].

The noncommutative main conjecture for elliptic curves has a natural generalization to other motives and their associated Galois representations, notably in the case of elliptic modular forms. The geometric aspects of the study of the arithmetic of elliptic curves are overtly lacking in this set-up, but nevertheless there is a remarkable commonality in phenomena between these two contexts, reinforced by the profound modularity results of the Galois representations associated to elliptic curves. Hida theory [18] provides a framework where these Galois representations can be assembled and interpolated in a p-adic family and helps one to propagate known results in the Iwasawa theory of elliptic curves to the Iwasawa theory of Galois representations associated to modular forms. This has opened up a whole new area of ‘Big’ Galois representations and associated Iwasawa algebras $R[[G]]$, where R is a more general regular local ring and has already proved to be fruitful with diverse applications. Finally, congruences between modular forms or Galois representations (i.e. where the residual representations are isomorphic) have yielded interesting congruences between algebraic special values of complex L-functions and values of p-adic L-functions twisted by Artin characters in the classical case where G is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension. The noncommutative main conjecture in turn predicts congruences between twisted p-adic L-values associated to Artin representations that arise purely in the non commutative Iwasawa theoretic setting, and a theoretical proof or extensive numerical verifications would provide evidence for the veracity of the non commutative main conjecture. Finally, we mention the Stark-Tate conjecture in number theory which is close in spirit to the BSD Conjecture in that it postulates information on the coefficient of the leading term of the Taylor expansion associated to the Artin L-function of a Galois extension of number fields, and generalizes the analytic class number formula. In 1990, Bloch and Kato applied p-adic Hodge theory to the study of arithmetic of algebraic varieties and formulated their famous conjectures about special values of L-functions. One of the key objects introduced in their paper is the so called exponential map of Bloch-Kato

$$\exp_{K,V} : D_{dR}(V) \to H^1(K, V)$$

which is a vast generalisation of the Kummer map for abelian varieties to p-adic de Rham representations V. Few years after, Perrin-Riou [22] discovered that this map has nice p-adic interpolation properties. For any absolutely crystalline representation V she constructed an interpolation of this map (usually called Perrin-Riou's exponential or large exponential map) for different twists $V(j)$ of V along the cyclotomic p- extension. The properties of the large exponential map are closely related to the following important topics:

1. Computation of Tamagawa numbers or more generally the local ϵ-conjecture (called $C_{EP}(L/K, V)$ in [2]) which can be viewed as a local analog of the Iwasawa Main Conjecture (or global Tamagawa number conjecture).

2. General theory of p-adic L-functions.

Thanks to the work of many people we have now a satisfactory picture which includes the proofs of Perrin-Riou's reciprocity law and local ϵ-conjecture as well as the formulation of the Iwasawa Main Conjecture for global p-adic representations which are crystalline at p and the conjectural relationship with the theory of Euler systems. On the over hand for more general representations the results are not so complete. Perrin-Riou extended a part of her theory to absolutely semistable representations but the integral properties of the large exponential map in this case are not known. We are going to explain the first topic in some more detail now:

The significance of (local) ϵ-factors à la Deligne and Tate or more general of the (conjectural) ϵ-isomorphism suggested by Fukaya and Kato in [13, §3] is at least twofold: First of all they are important ingredients to obtain a precise functional equation for L-functions or more generally for (conjectural) ζ-isomorphism (loc. cit., §2) of motives in the context of equivariant or non-commutative Tamagawa number conjectures. If M denotes a motive over $\mathbb{Q}$ and $M^*(1)$ its Kummer dual, then for the leading terms $L^*(M)$ of the complex L-function $L(M, s)$ attached to M there is conjecturally the following functional equation

$$L^*(M) = (-1)^\eta \prod \epsilon_v(M) \frac{L^*_\infty(M^*(1))}{L^*_\infty(M)} L^*(M^*(1))$$

where η denotes the order of vanishing at $s = 0$ of the completed L-function $L_\infty(M^*(1), s)L(M^*(1), s)$; here the factors L_∞ at infinity are built up by certain Γ-factors and certain powers of 2 and π depending on the Hodge structure of M. While the global Tamagawa Number Conjecture (TNC) relates the leading term $L^*(M)$ to (the determinant of) global Galois/étale cohomology $R\Gamma_c(spec(\mathbb{Z}[\frac{1}{S}]), T)$ with compact support, the local Tamagawa number conjecture relates the ϵ-constants $\epsilon(\mathbb{Q}_\ell, V)$ to (the determinant of) local cohomology $R\Gamma(\mathbb{Q}_l, T)$ for all ℓ; roughly speaking the validity of the latter conjecture asserts that the global TNC is valid for M if and only if it is valid for its Kummer dual $M^*(1)$ taking into account Artin-Verdier/Poitou-Tate duality and assuming the validity of the functional equation. Technically (in the language of Fukaya-Kato) the local conjecture is expressed by an integral isomorphism

$$\epsilon_{\mathbb{Z}_p}(\mathbb{Q}_\ell, T) : \mathrm{Det}(0) \to \mathrm{Det}(R\Gamma(\mathbb{Q}_l, T))\mathrm{Det}(T)$$

whose base change from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ contains the Bloch-Kato exponential map as well as the ϵ- and Γ-factors attached to V. Fukaya and Kato conjecture that such an isomorphism exists for pairs $(\Lambda, \mathbb{T})$ consisting of certain p-adic rings Λ and finitely generated projective Λ-modules $\mathbb{T}$ with a commuting continuous $G_{\mathbb{Q}_l}$-action instead of the pair $(\mathbb{Z}_p, T)$. The most important examples are Iwasawa deformations where Λ is the Iwasawa algebra of some p-adic Lie group G and $\mathbb{T} = \Lambda(G)^\sharp \otimes_{\mathbb{Z}_p} T$. Such "big" ϵ-isomorphism are supposed to interpolate the $\epsilon_{\mathbb{Z}_p}(\mathbb{Q}_\ell, T)$.

Secondly ϵ-factors and -isomorphisms are essential in interpolation formulae of (actual) p-adic L-functions and for the relation between ζ-isomorphisms and (conjectural, not necessarily commutative) p-adic L-functions as discussed in (loc. cit., §4). Of course the two occurrences are closely related, for a survey on these ideas see also [26].

Yasuda [29] proved the ϵ-isomorphism-conjecture for $l \neq p$. Benois and Berger [2] have proved the conjecture $C_{EP}(L/K, V)$, i.e., the existence of $\epsilon_{\mathbb{Z}_p}(\mathbb{Q}_p, T)$ in the above setting, for arbitrary crystalline representations V of G_K, where K is an unramified extension of $\mathbb{Q}_p$ and L a subextension of $K_\infty = K(\mu(p))$ over K. In the special case $V = \mathbb{Q}_p(r)$, $r \in \mathbb{Z}$, Burns and Flach [5] proved a local Equivariant Tamagawa Number Conjecture using global ingredients in a semi-local setting. Kato (unpublished) had already constructed $\epsilon_{\Lambda(G)}(\mathbb{Q}_p\mathbb{T})$ for G any abelian p-adic Lie quotient of $G_{\mathbb{Q}_p}$ and any rank one $\Lambda(G)$-module $\mathbb{T}$, see

[27] for details. To show that these ϵ-isomorphism satisfy the right interpolation property with respect to Artin characters of G, usually Berger's explicit formulae in [3] based on the theory of (ϕ, Γ)-modules is used.

Of course it would be most desirable to extend the existence of ϵ-isomorphism also to non-abelian local p-adic Lie extensions (with Galois group G). This seems to require completely new ideas and to be out of reach at present. In particular it would require a convenient description of Iwasawa cohomology in terms of adequate (φ, G)-modules over *ramified* base fields K over $\mathbb{Q}_p$. Another missing extension is that to semi-stable representations.

The p-adic representation theory of groups of L-points $\mathbf{G}(L)$ of p-adic reductive groups $\mathbf{G}$ over L has been developed by Emerton and Schneider/Teitelbaum. In particular the work of the latter relies on translating this representation theory into the module theory over certain distribution algebras associated either with $\mathbf{G}(L)$ or with its compact open subgroups. Corresponding to the fact that two different types of representations, Banach space as well as locally analytic representations, play an important role, also the distribution algebras come in two different forms. To fix ideas let G denote a compact p-adic Lie group, and let K be another p-adic field which is the coefficient field of the representations. We then have the function spaces

$$C(G, K) := K\text{-valued continuous functions on } G$$

and

$$C^{an}(G, K) := K\text{-valued locally analytic functions on } G.$$

The former is a K-Banach space and the latter a locally convex K-vector space of compact type. Their continuous duals

$$D(G, K) \qquad \text{and} \qquad D^{an}(G, K)$$

are naturally endowed with a convolution product and are a Banach algebra and a nuclear Fréchet algebra, respectively. They are the distribution algebras in question. They can be viewed as a Banach and a Fréchet completion, respectively, of the abstract algebraic group ring $K[G]$ (which does not reflect the topology of G and therefore is completely insufficient for the present purposes). Of course, the group G is, via the Dirac distributions, a subgroup of the group of units in any of these algebras. In fact, if o_K denotes the ring of integers in K, then the embedding $G \longrightarrow D(G, K)$ naturally extends to an algebra isomorphism

$$K \otimes_{o_K} o_K[[G]] \xrightarrow{\cong} D(G, K) .$$

In other words, $D(G, K)$ is a reinterpretation, in the language of harmonic analysis, of the Iwasawa algebra of G over K. This is the other basic place where p-adic representation theory connects to the topic of our workshop.

As classical Iwasawa theory relies very much on structural results about Iwasawa algebras and their modules it is, from an algebraic point of view, crucial for p-adic representation theory to understand the distribution algebras $D(G, K)$, $D^{an}(G, K)$, and their modules. Since $D(G, K)$ is noetherian by a result of Lazard the finitely generated $D(G, K)$-modules form the right abelian category to work with. In contrast $D^{an}(G, K)$ is no longer noetherian. But Schneider/Teitelbaum have shown that it is a so called Fréchet-Stein algebra which allows to construct an abelian category of "locally finitely generated" modules. This then leads to the categories of admissible Banach space and admissible locally analytic representations which are the good framework in which a reasonable theory can be developed. But apart from these basic results very little is known about the structural properties of these distribution algebras.

3 Presentation Highlights

Among the highlights of the conference are certainly the survey talks: long before the workshop took place we had asked some colleagues to present an overview and give an introduction to one specific application of Iwasawa algebras.

R. Greenberg gave a survey of modules over commutative Iwasawa algebras, with special reference to the representations and multiplicities of the Selmer groups of elliptic curves extended to the algebraic closure $\bar{\mathbb{Q}}$. This is relevant for important conjectures in Iwasawa theory such as the Parity conjecture and the equivariant

BSD conjecture. The parity conjecture asserts that the $\mathbb{Z}_p$-corank of the Selmer group over the cyclotomic $\mathbb{Z}_p$-extension has the same parity as that of the Selmer group over the base. If one grants the finiteness of the Tate-Shafarevich group, then this would be a weak but preliminary evidence for the BSD Conjecture. In his lecture, Greenberg posed a few open questions that are related to representation theory and modules over Iwasawa algebras which would have arithmetic applications in the study of the Mordell-Weil rank of elliptic curves in a p-adic Lie extension.

It has been mentioned that the p-adic L-function in the noncommutative main conjecture is an element of the K_1-group of the completion of a localized non commutative Iwasawa algebra, and that the classical exact localization sequence plays an important role in the definition of this p-adic L-function. The non commutative main conjecture for the trivial motive was one of the important results proved in non commutative Iwasawa theory, due to Ritter-Weiss and Kakde independently. M. Kakde gave a survey lecture on the K-theory for Iwasawa algebras and its related localisations and completions that occur in Iwasawa theory.

J. Pottharst gave a very impressive and comprehensive *Review of* (φ, Γ)*-modules*, which was extremely effectively organised leading through this highly technical matters. After recalling the basic definitions and constructions he continued to discuss Galois representations of *finite height*, i.e., such that $D(V)$ descends to $B^+ := A^+[\frac{1}{p}]$, Wach modules, which are nice models over B^+, and *overconvergent* Galois representations, i.e., those which descend to

$$B^\dagger = \{\sum_{n\in\mathbb{Z}} a_n\pi^n | a_n \in \mathbb{Q}_p, \text{ bounded for } n \to +\infty, \text{converging "fast" to 0 for } n \to -\infty\}.$$

For (étale) (φ, Γ)-modules over the Robba ring

$$B^\dagger_{rig} = \{f = \sum_{n\in\mathbb{Z}} a_n\pi^n | a_n \in \mathbb{Q}_p, \ f \text{ converges on some annulus } r \leq |\pi| < 1\},$$

one again obtains an equivalence of categories with $Rep_{\mathbb{Q}_p}(G_K)$, moreover, one can apply Kedlaya's theory of slopes. Successive extensions of rank 1 objects in the category of (not necessarily étale) (φ, Γ)-modules over the Robba ring are called trianguline. 'Leaving' the category of étale (φ, Γ)-modules - if necessary - has become an important technique even if one is mainly interested in p-adic representations! Finally Pottharst discusses these constructions in settings where the coefficients $\mathbb{Q}_p$ are replaced by either complete noetherian local rings with finite residue field (formal case, by Dee) or by affinoid algebras over $\mathbb{Q}_p$ (rigid case, by Berger and Colmez). The Herr complex in the latter generality also satisfies comparison, perfectness and base change properties.

The survey lecture presented by G. Zabradi explained in a concise manner why and how Iwasawa algebras in the form of distribution algebras are needed to construct reasonable categories of p-adic representations of p-adic reductive groups. He went on to describe that certain rings like Fontaine's ring $\mathcal{O}_\mathcal{E}$ and the Robba ring have counterparts for general compact p-adic Lie groups. These generalizations can be understood as topological localizations of distribution algebras. He indicated how these localizations might play a crucial role in relating p-adic reductive group representations to (φ, Γ)-modules (and hence, by Fontaine's theorem, to p-adic local Galois representations). This would lead eventually to a extension of Colmez's approach to p-adic local Langlands. The lecture summarized in a very clear form the basics from which the theory has to start but also highlighted some recent ideas on a path towards a general p-adic local Langlands program.

The final survey talk in this area was delivered by T. Chinburg on equivariant Euler characteristics and Iwasawa theory. In this interesting talk, he set out a dictionary between the non commutative Iwasawa Main Conjecture and Thomason's approach to the Riemann-Roch theorem. In particular, his talk highlighted several common ideas and techniques between the study of equivariant coherent Euler characteristics and Iwasawa theory. Several aspects of the classical study of the Galois module structure of rings of integers of number fields are relevant in this new set-up and opens up new areas of attack for the study of modules over Iwasawa algebras as well as the K-theory of Iwasawa algebras.

4 Scientific Progress Made

As is clear from the presentation highlights in the above paragraph, there were several results made in a plethora of research directions in the area of Iwasawa algebras. There were several concrete advancements made in this broad area, ranging from the study of modules over Iwasawa algebras, new directions in the K-theory and algebraic geometry by way of connections to noncommutative Riemann-Roch theorems. The study of the nonabelian Euler characteristics and Iwasawa theory of supersingular elliptic curves points to a variety of new problems that need to be studied.

It has been remarked earlier that Euler systems play an important role in obtaining results related to the BSD conjecture, from results proved around the main conjecture. The Euler systems are norm compatible elements in certain cohomology groups associated to the Galois representation. Their main application is in bounding the size of the Selmer group which is then used to obtain information on the Mordell-Weil ranks. In her talk, S. Zerbes described the construction of an Euler system over the cyclotomic tower for the product of the Galois representations attached to the Rankin product of two weight two modular forms. This generalizes earlier work of Beilinson-Flach and Bertolini-Darmon-Rotger. The Stark-Tate conjecture in number theory is close in spirit to the BSD Conjecture in that it postulates information on the coefficient of the leading term of the Taylor expansion associated to the Artin L-function of a Galois extension of number fields, and generalizes the analytic class number formula.

C. Popescu's lecture focused on the construction a new class of Iwasawa modules, which are the number field analogues of the p-adic realizations of the Picard 1-motives constructed by Deligne in the 1970s and studied extensively from a Galois module structure point of view in some other recent work of his with C. Greither. In the abelian case, this leads to a proof of an Equivariant Main Conjecture, identifying the first Fitting ideal of the Iwasawa module in question over the appropriate profinite group ring with the principal ideal generated by a certain equivariant p-adic L-function. This is an integral, an equivariant refinement of the classical Main Conjecture over totally real number fields proved by Wiles in 1990. Finally, under additional hypotheses, these results and Iwasawa co-descent arguments prove refinements of the (imprimitive) Brumer-Stark Conjecture and the Coates-Sinnott Conjecture, away from their 2-primary components, in the most general number field setting. In his talk, A. Nickel discussed different formulations of the equivariant Iwasawa main conjecture attached to a Galois extension of totally real number fields whose Galois group is a p-adic Lie group of dimension one. The corresponding versions in the abelian case was formulated by Greither and Popescu. Under additional hypothesis, he outlined how these nonabelian versions of the Stark-Tate conjectures can be proved, as well as a stronger form of the Coates-Sinnott conjecture.

M. Witte's lecture focused on the proof of the surjectivity of the connecting map in the localization sequence which, while highly relevant for the non commutative main conjecture, is also of independent interest. In particular, the proof uses Waldhausen's theory and opens up possibilities of a better understanding of the K-theory of Iwasawa algebras.

Since the study of Euler characteristics of the modules associated to Iwasawa algebras form an important part of the algebraic side of the main conjecture, the Galois (co)homology groups of the modules naturally gain prominence. For commutative Iwasawa algebras, the gamut of results from commutative ring theory (Gorenstein, Cohen-Macaulay, dimension theory, regularity, etc) come in useful in computing the Galois cohomology groups. One of the fundamental results used in the study of noncommutative Iwasawa algebras is that they are Auslander regular (left and right) noetherian rings which is a strong and useful property. R. Sharifi's talk generalised certain quality results proved by Nekovar for commutative Iwasawa algebras, to the nonabelian setting. This intern makes available several useful techniques that can be used to study the cohomology groups for modules over non commutative Iwasawa algebras.

There were short presentations made by younger researchers (graduate students and postdocs) in this theme. S. Shekhar spoke on the congruence between special p-adic twisted L-values for modular forms of weight 2 that are congruent mod p. As part of the theoretical evidence for the main conjecture, he also presented congruence between Euler characteristics of Selmer groups for such congruent elliptic curves over the noncommutative False Tate extension and also dihedral extensions. One of the applications of these results is to conclude that BSD for one of the congruent curves would imply it also for the other. S. Jha spoke on the study of the fine Selmer group for Hida families. The fine Selmer group over an extension of the base field is a subgroup of the Selmer group and the main conjecture in the classical setting can be rephrased in terms of an exact sequence involving the fine Selmer group. In this talk, Somnath focused on how information on the fine Selmer group of an elliptic curve can be propagated in a Hida family to obtain information on the fine Selmer groups of Galois representations associated to ordinary modular forms. Finally, F. Sprung spoke about some intriguing results for the special values of p-adic L-functions associated to elliptic curves with super singular reduction at the prime p. His result presents interesting information about the growth of the

Tate-Shafarevich group for elliptic curves, one of the most mysterious groups in arithmetic that is conjectured to be finite, and occurs in the exact formulae of the BSD conjecture.

D. Loeffler reported in his talk about a joint project with O. Venjakob and S. Zerbes, which was actually finished during the workshop. They extend the work of Benois and Berger by constructing ϵ-isomorphisms $\epsilon_{\Lambda(G)}(\mathbb{Q}_p\mathbb{T})$ for G any abelian p-adic Lie quotient of $G_{\mathbb{Q}_p}$ (in particular those of dimension 2) and any $\Lambda(G)$-module $\mathbb{T}$, which arises as Iwasawa deformation of some crystalline representation. The key ingredient in their construction of the local ϵ-isomorphism is the two-variable regulator map introduced by Loeffler and Zerbes. Essentially, the local ϵ-isomorphism is obtained by taking the determinant of the two-variable regulator, after dividing out by certain correction factors determined purely by the Hodge-Tate weights of the Galois representation.

A vast generalisation of Kato's local ϵ-conjecture has been achieved by K. Nakamura, who extends both its formulation (in general) and its proof (for trianguline ones) to (φ, Γ)-modules over the (formal or rigid) Robba ring using his extension (i) of the Bloch-Kato exponential map and (ii) of Perrin-Riou's big exponential map to his larger category. He first described the new fundamental line (generalising the previous one $\mathrm{Det}(R\Gamma(\mathbb{Q}_l, T))\mathrm{Det}(T))$ satisfying the properties: base change, duality, multiplicativity on short exact sequences and compatibility with the étale case. For the proof he uses explicit calculations for the rank one case using Coleman maps; then the general trianguline case follows by the multiplicativity supposed one can show independence of the choice of triangulation.

Concerning the big exponential map for de Rham (φ, Γ)-modules A. Riedel obtained independently similar results as Nakamura, on which he reported in an evening session. Moreover he is able to prove reciprocity laws for certain (φ, Γ)-modules in that context. In the same session U. Schmitt reported on his work towards Fukaya and Kato's twist conjecture, which relates the p-adic L-function of an elliptic curve to twists of the p-adic L-function associated to the Tate Motive. In particular, in the case of CM elliptic curves, he showed that the local ingredient, i.e., the interpolation part of a local main conjecture, can be derived form Kato's local ϵ-conjecture in the rank one case.

L. Berger presented his new approach for attacking p-adic local Langlands for $GL_2(F)$, F a finite extension of $\mathbb{Q}_p$. He suggests to replace the cyclotomic $\mathbb{Z}_p$-extension by the extension arising by a standard Lubin-Tate formal group for F. To this aim he has developed a theory of multivariable Lubin-Tate (φ, Γ)-modules with which he is able to partly describe the relationship between overconvergent and F-analytic G_F-representations. Finally, he discussed the possible role of the p-adic Fourier theory of Schneider and Teitelbaum for the intended connection to the p-adic local Langlands program.

T. Ochiai generalized the Coleman map, i.e., the "inverse" of the big exponential map for $V = \mathbb{Q}_p(1)$, to families of Galois representations, e.g. to Hida deformations associated with GL_2 or GSp_4.

K. Ardakov spoke about recent joint work with S. Wadsley in which they address the problem of the existence of two-sided ideals in Iwasawa algebras. This certainly is one of the basic structural question about any kind of specific class of algebras. In the case of Iwasawa algebras it is notoriously difficult. Experience suggests that the Iwasawa algebra of a compact p-adic Lie group G with semisimple Lie algebra has very few two-sided ideals. Building upon their recent seminal work on a p-adic analog of Beilinson-Bernstein localization on the flag variety Ardakov and Wadsley make substantial progress towards the classification of primitive ideals (i.e., annihilator ideals of simple modules) in affinoid enveloping algebras. It is not only their results but also the new methods which they develop which open up fruitful new lines of investigation.

T. Schmidt explained his joint work with D. Patel and M. Strauch on localization of p-adic analytic representations. In the classical local Langlands program one considers smooth representations of p-adic reductive groups G over fields of characteristic zero. Here "smooth" means that the underlying vector space of the representation carries the discrete topology. For those representations Schneider/Stuhler constructed a localization functor to G-equivariant constructible sheaves on the Bruhat-Tits building $\mathcal{BT}$ associated with the reductive group G. The rational representation theory of G on the other hand can be localized by Beilinson/Bernstein to $\mathcal{D}$-modules on the flag variety G/B. If one enlarges G/B to its analytic space $(G/B)^{an}$ in the sense of Berkovich then Berkovich has shown that there is a natural equivariant embedding $\mathcal{BT} \longrightarrow (G/B)^{an}$. This combines the targets of the above two localization theories. As far as the sources are concerned, rational representations, being finite dimensional, obviously are admissible locally analytic. Schneider/Teitelbaum have shown that admissible smooth G-representations can be characterized as being those admissible locally analytic G-representations on which the derived Lie algebra action is trivial. This makes the question purposeful whether there is a localization theory for general admissible locally analytic representations leading to some kind of module sheaves on $\mathcal{BT}$ viewed inside $(G/B)^{an}$. The lecture explained a complete positive answer to this question.

J. Kohlhaase addressed the question of admissibility of smooth representations in characteristic p. An admissible unitary p-Banach space representation of a p-adic reductive group G can be reduced modulo p giving rise to an admissible smooth G-representation in a vector space over a finite field of characteristic p. Recent work of Abe and Herzig shows that the basic building blocks of these latter representations are the so called supersingular ones. They are the quotients of a certain universal representation. It is believed that, except in the rank one case, this universal representation is no longer admissible. Kohlhaase explained new results in this direction. This is based on intricate calculations with Koszul complexes and group cohomology with characteristic p coefficients. Using results by Lazard they rely very much on Iwasawa algebra techniques. It becomes more and more clear that this latter methods are of crucial importance in the subject and need a systematic investigation.

5 Outcome of the Meeting

It is clear that the study of Iwasawa algebras now pervades different areas of mathematics, spanning algebra, arithmetic algebraic geometry and representation theory. One of the unique features of this workshop was to bring experts and younger researchers together for a week under a single workshop umbrella and expose them to progress being made in diverse themes and areas related to Iwasawa algebras. In addition, the wide array of areas covered by speakers presenting their results also exposed bouquet of open problems and questions. It is clear to us that this meeting of minds from different aspects of the study of Iwasawa algebras will initiate further collaborations and trigger new research. Typically in pure mathematics, which has a much longer gestation period and shelf life as far as basic research is concerned, the longterm impact of such meetings is also to be measured in the interweaving of these different areas and attacking the plethora of problems in all these emerging areas related to Iwasawa algebras. We attach a sample of the testimonials:

"During the workshop I had a very good opportunity to become familiar with current research questions and developments of two research areas, which even though do not completely coincide with my current research interests, they are closely related to them." –Thanassis Bouganis (Heidelberg University).

"I found the Applications of Iwasawa Algebras workshop at BIRS to be a very timely conference on an area of increasing interest in arithmetic geometry. The choice of speakers and topics was excellent, representing some of the top people in the field (at various stages in their research careers)".—Romyar Sharifi (University of Arizona).

"This well and smoothly organized meeting has been of particular interest to me, and many talks and discussions may have clarified the way how to proceed with my own work". —Jürgen Ritter (University of Augsburg).

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