IMPRIMITIVE IRREDUCIBLE MODULES FOR FINITE QUASISIMPLE GROUPS

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Workshop Permutation Groups
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Contents

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   We call $H := \mathrm{Stab}_G(V_1)$ a block stabilizer.
   We have $V \cong \mathrm{Ind}_H^G(V_1) := KG \otimes_{KH} V_1$ as KG-modules.

Motivation I: Maximal subgroups

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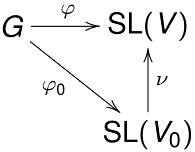

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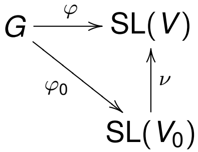

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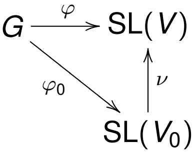

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Is $N_X(G)$ a maximal subgroup of X?

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A table look-up in our lists might help to answer this question.

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   - G a Suzuki or Ree group, $G = G_2(q)$, or G a Steinberg triality group

   (Seitz, H.-Husen-Magaard).

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Proof of 4: For $t \in G$, ${}^tH \cap H = C_G({}^ta, a)$. Hence $t \notin \langle {}^ta, a \rangle$ for $t \in G \setminus H$, since such a t does not centralize ${}^tH \cap H$ by 3.

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Finally, $t \in C_G(s)$ *and* ${}^t H_0 \cap sH_0 = \emptyset$, *thus* $t \in C_G({}^t H \cap H)$.

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This allows to apply Harish-Chandra theory to our classification problem, reducing certain aspects to Weyl groups.

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Analogous results hold for the other classical groups.

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Notice that the minimal polynomial of s is irreducible if and only if $C_G(s) \cong \mathrm{GL}_m(q^d)$ for integers m, d with $md = n$.

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where $W(s)$ is the "Weyl group" of $C_{\mathbf{G}^*}(s)$ (Bonnafé).

Suppose that $\mathcal{E}(G, [s])$ contains Harish-Chandra primitive **and** imprimitive characters.

Then $W(s)^F = S\colon\langle\gamma\rangle$, with $S = S_m \times \cdots \times S_m$, and γ permuting the e factors S_m of S transitively, and $em \mid n$.

Theorem (H.-Magaard)

$\chi_\eta \in \mathcal{E}(G, [s])$ is primitive, if and only if $\mathrm{Res}_S^{S\colon\langle\gamma\rangle}(\eta)$ is irreducible.

Thank you for listening!