REPORT ON BIRS WORKSHOP PERMUTATION GROUPS (13W5033) – 21–26 JULY 2013

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The aim of the workshop was to bring together leading researchers in areas related to permutation groups as well as those whose research centres on permutation groups. This was a follow up to meetings on the subject in Oberwolfach and Banff. In both instances bringing together the leading researchers in different areas related to permutation groups and their applications as well exposing top graduate students and younger researchers has led to significant new results.

The theory of permutation groups is a classical area of algebra. It originates in the middle of the nineteenth century, with very considerable contributions by most of the major figures in algebra over the last two centuries, including Galois, Mathieu, Jordan, Frobenius, Burnside, Schur and Wielandt. In the last twenty years, the direction of the subject has changed substantially. The classification of finite simple groups has had many applications, many of these through thorough investigation of relevant permutation actions. This in turn led to invigoration of the subject of permutation groups, with interesting new questions arising and techniques developed for tackling them. Interestingly, some topics arose in more than one context, forming new connections. The concept of exceptionality was first suggested by work on covers of curves; it then appeared independently in homogeneous factorizations of graphs, and more recently it has found applications in investigations of line-transitive linear spaces. The concept of derangements in groups (that is, fixed-point-free permutations) and their proportions is classical; it has applications to images of rational points for maps between curves over finite fields, in probabilistic group theory and in investigating convergence rates of random walks on groups. Recently a conjecture of Boston and Shalev on the proportion of derangements in simple groups actions has been settled; interestingly, this conjecture fails in the slightly more general case of almost simple groups, through examples of exceptional actions mentioned above. This area continues to be very lively. The exciting results of Helfgott on expansion in groups, extended recently by Pyber and Szab'o, and independently by Breuillard, Green and Tao, have shown not only that Cayley graphs of bounded rank finite simple Lie type groups are expanders, but have also yielded proof of the Weiss Conjecture of 1978 for locally primitive graphs involving only bounded rank composition factors. The topic of fixed point ratios and minimal degrees of elements in permutation groups is classical, going back over 100 years, but there has been significant progress in the last fifteen years both for finite and algebraic groups, and these results have had applications in arithmetic algebraic geometry, besides leading to significant insights in group theory – a striking example is the solution of Wielandt's conjecture on the characterization of subnormal subgroups. The question of base size of permutation actions is of importance

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in computational group theory as well as in the study of the graph isomorphism problem. Recent research has thrown much light on the base sizes of actions of almost simple groups in particular. The concept of quasiprimitive permutation groups is also classical, but there has been revived interest in the subject through investigations of groups of graphs and designs. Algebraic graph theory has developed greatly over the last ten to twenty years; there are interesting connections to association schemes and representation theory. Some of these in turn found an application in the study of derangements mentioned above, as well in the study of random walks on groups.

The theory of permutation groups is essentially the theory of symmetry for mathematical and physical systems. It therefore has major impact in diverse areas of mathematics. Twentieth-century permutation group theory, from the seminal work of Wielandt, focused on the theory of finite primitive permutation groups, and this theory continues to become deeper and more powerful as application of the finite simple group classification, and group representation theory, lead to astonishingly complete classifications and asymptotic results.

Advances in the twenty-first century have most productively involved an interplay between permutation groups and other areas of mathematics, such as group representation theory, algebraic graph theory and other areas of geometry and combinatorics, algebraic geometry, logic and model theory, analysis, and computational and complexity theory. There has been significant theoretical development of permutation group theory contributing to, and benefiting from, these diverse applications.

There have been two workshops in permutation groups since 2007. The first was in Oberwolfach (2007) and the second in Banff (2009). Besides the scientific breakthroughs achieved at these meetings, many young postdocs and graduate students were involved and got to interact with the experts in the field.

We now discuss some of these and related areas and recent developments in more detail.

Moufang sets.

Split BN-pairs of rank at least 2 were classified by Tent and Van Maldeghem using the fundamental classification of Moufang polygons due to Tits and Weiss. At the same time the rank 1 case, the classification of Moufang sets, remains wide open. There are a number of closely related concepts all essentially corresponding to a doubly transitive permuation group whose point stabilizers contain a normal subgroup regular outside the point. There has been recent progress on obtaining properties, mostly due to de Medts, Segev and Weiss, and in particular in the setting where the root groups are abelian. Conjecturally all such groups can be presented as certain Jordan algebras. Very recently Prasad showed that inside a pseudoreductive group the only split BN-pairs are the standard ones. In particular, no anisotropic group contains a split BN-pair of any rank. Various questions remain open for which an answer should be given not using the heavy classification results mentioned above.

Exceptionality of permutation groups. The concept of exceptional permutation groups arose in the context of investigations of exceptional polynomials, which arose originally in the work of Dickson, Schur, Davenport, Fried and others. These are polynomials over finite fields which induce permutations on infinitely many finite extension fields. This can be translated into a question relating orbitals of a permutation group and its automorphism group. An answer led to major progress in our knowledge of exceptional polynomials in the work of Fried, Guralnick and Saxl. As a direct consequence, new families of exceptional polynomials were discovered by Lenstra, Zieve and others. Another application appeared in the recent memoir of Guralnick, Müller and Saxl on the rational function analogue of a question of Schur concerning polynomials with integer coefficients which induce permutations on residue fields for infinitely many primes. The solution involved a substantial amount of permutation group theory and algebraic geometry. At about the same time, exceptional permutation groups arose also in the work of Praeger, Li, and others on homogeneous factorizations of complete graphs. There is a further recent application in the study of line-transitive linear spaces.

Derangements. According to a conjecture attributed to Boston and Shalev, there is an absolute lower bound for the proportion of derangements in any action of any simple permutation group. This has been proved recently in an impressive series of papers (and preprints) by Fulman and Guralnick. An important extension for almost simple permutation groups, still being investigated, is distribution of derangements in the cosets of the simple group in its automorphism group. This is connected to the exceptionality condition above. One wants to classify primitive actions in which most elements in a coset are not derangements. This would yield information about rational maps and maps between curves over finite fields that are close to being bijective over arbitrarily large finite fields.

Algebraic graph theory. Successful modern applications of permutation groups in algebraic graph theory date from the late 1980's with proof of the Sims Conjecture, breakthroughs in the classification of finite distance transitive graphs, and the non-existence proof of finite 8-arc transitive graphs of valency greater than 2. These involved use of the simple group classification and built on the theory of finite primitive permutation groups. More recent applications required the development of the theory of finite quasiprimitive permutation groups. This theory also relies heavily on the finite simple group classification, and has been used successfully to analyse even intransitive finite combinatorial structures such as locally s-arc transitive graphs. It's most recent application, by Praeger, Pyber, Spiga, and Sźabo, enabled results on expansion in groups to be exploited to prove the Weiss Conjecture for a large family of groups. In addition the theory of amalgams and their universal completions forms an important link between infinite graphs and their automorphism groups on the one hand, and classification of finite graphs by their local properties. Much of the geometry associated with the finite simple groups has been elucidated from the study of group amalgams. Combining the amalgam approach and the quasiprimitive graph approach is just beginning to pay significant dividends in our understanding of important classes of graphs and group actions. Returning to the finite distance transitive graphs, a related, slightly more general problem concerns multiplicity free permutation actions. There has been substantial progress towards classification of these actions. Deep character theoretical information on some of these actions has been obtained by Lusztig, Henderson and others. The character tables of the corresponding association schemes have been obtained by Bannai and his coworkers. Some of these actions have been used by Diaconis and others to investigate random walks on groups.

Subgroup structure of finite simple groups. Theory of primitive permutation groups is closely related to the subgroup structure of finite simple groups and their automorphism groups. More precisely, understanding the maximal subgroups of a given group is equivalent to understanding all primitive permutation representations of the group. There has been impressive progress in this area. For sporadic groups, the answer is almost complete. For alternating groups, the question of maximality was settled in the late eighties through the work of Liebeck, Praeger and Saxl on maximal factorizations of almost simple groups. This reduces the question to classification of maximal subgroups of smaller almost simple groups. For classical groups, Aschbacher's theorem focuses attention on modular representations of almost simple groups; more precisely, one must determine triples (X, Y, V), where $X < Y$ are simple groups acting absolutely irreducibly on a kY-module V. The problem naturally splits into several subproblems, each of a different nature. The case where X or Y is an alternating group has been studied by Husen, Saxl, Brundan and Kleshchev. When one of the groups is a sporadic group, the question is approached by the GAP team. When both groups are of Lie type over a field of characteristic equal to the characteristic of the field k, one 'lifts' the question to a question about algebraic groups (which is possible in most cases, by results of Liebeck, Seitz and Testerman). The majority of the remaining configurations which must be handled concern the case where X and Y are both of Lie type and their fields of definition are not equal. Seitz treated the case where the fields of definition of X and Y are of equal characteristic, different from the characteristic of k. Combined with recent work of Magaard, Röhrle and Testerman, this reduces the study to a finite number of configurations, many of which concern symmetric or alternating powers of certain irreducible kX-modules. This then feeds into the recent work of Magaard, Malle, Tiep and others on the irreducibility of tensor powers. In addition to the determination of the triples (X, Y, V), there are various other configurations arising from Aschbacher's theorem, in which maximality must still be determined. One major step has been made in the recent monograph of Hiss, Husen and Magaard on imprimitive modules for quasisimple groups.

Locally compact topological groups Topological groups provide a theory of continuous symmetry for mathematical structures, making them indispensable tools in theoretical physics, astronomy, as well as frameworks for analysing symmetries of differential equations. A general locally compact group has a closed connected normal subgroup such that the quotient is totally disconnected as well as locally compact. Connected groups can be studied using the powerful theory of Lie groups and Lie algebras. Thus the role of totally disconnected groups in the study of locally compact groups is critical, examples being p-adic Lie groups and also all discrete groups. However until the recent pioneering work of George Willis, almost all that was known about a general totally disconnected locally compact group was van Dantzig's 1936 theorem on existence of arbitrarily small compact open subgroups. The two fundamental ingredients in Willis's structure theory are tidy subgroups and the scale function: the latter an analogue of eigenvalues of a linear operator, and the former an analogue of its eigenspaces. They lead to highly symmetrical permutation group actions of totally disconnected locally compact groups on graphs and digraphs (Möller), or on buildings (Baumgartner, Ramagge, Rémy and Willis) that reveal and govern the structure of these groups. Much needs to be done in this emerging area.

Infinite permutation groups and model theory Over the last years there have also been major developments in infinite permutation group theory. One aspect here is the interaction between permutation group theory, combinatorics, model theory, and descriptive set theory, typically in the investigation of first order relational structures with rich automorphism groups. The connections between these fields are seen most clearly for permutation groups on countably infinite sets which are closed (in the topology of pointwise convergence) and oligomorphic (that is, have finitely many orbits on k-tuples for all k); these are exactly the automorphism groups of ω-categorical structures, that is, first order structures determined up to isomorphism (among countable structures) by their first order theory.

Themes of current activity here include the following.

(a) The use of group theoretic means (O'Nan-Scott, Aschbacher's description of maximal subgroups of classical groups, representation theory) to obtain structural results for model-theoretically important classes (totally categorical structures, or much more generally, smoothly approximable structures, finite covers of well-understood structures).

(b) In the classification of groups of finite Morley rank, the case of Tits rank 1, i.e. the (split) 2-transitive groups has been hard to treat. This is partly due to the fact that the study of infinite Moufang sets has only recently picked up momentum, mostly due to the work by de Medts, Segev and Weiss (see above). This has also lead to new progress in the finite Morley rank situation and gives rise to the hope that here a complete classification might be within reach.

(c) Reconstruction of a first order structure (up to isomorphism, up to having the same orbits on finite sequences, up to 'bi-interpretability') from its automorphism group, typically, presented as an abstract group. Partially successful techniques here include the description of subgroups of the automorphism group of countable index (the 'small index property'), and first order interpretation of the structure in its automorphism group.

(d) Properties which the full symmetric group S on a countable set shares with various other closed oligomorphic groups. We have in mind such properties as: complete description of the normal subgroup structure; uncountable cofinality (that is, the group is not the union of a countable chain of proper subgroups); existence of a conjugacy class which is dense in the automorphism group, or, better, comeagre (or better still, the condition of 'ample homogeneous generic automorphisms'); the Bergman property for a group (a recently investigated property of certain groups G, which states that if G is generated by a subset S, then there is a natural number n such that any element of G is expressible as a word of length at most n in $S \cup S^{-1}$); the small index property. Several of these themes have been linked in recent work of Kechris and Rosendahl motivated partly by descriptive set theory. A closely related issue here is the 'extension property' for a class C of finite relational structures, which stated that if $U \in C$ then U embeds in some $V \in C$ such that every partial isomorphism between substructures of U extends to an automorphism of V; this condition, proved for graphs by Hrushovski, has connections to the topology on a free group, and to automata theory and issues on the theoretical computer science/finite model theory border.

Probabilistic generation of finite simple groups The work on generation of finite simple groups in the last decade has led to some beautiful and surprising results. Starting with the classical results on generation of finite simple groups (Dickson, Miller, Steinberg), and the even older work of Hurwitz on $(2, 3, 7)$-generation and its connection with the automorphism groups of compact Riemann surfaces, this has been an active area of research. The most recent work has concentrated on the question of when a randomly chosen element of $\mathrm{Hom}(\Gamma, G)$ for Γ a Fuchsian group and G a finite simple group, is surjective. The combined work of Dixon, Guralnick, Kantor, Luebeck, Liebeck and Shalev shows that if Γ is oriented and of genus at least 2, then the probability that such a randomly chosen element is surjective tends to 1 as $|G|$ tends to infinity. Liebeck and Shalev have applied these results to counting branched coverings of Riemann surfaces and to random walks on symmetric groups. On the other hand, if Γ is of genus 0 or 1, the question remains open.

One family of genus 0 groups consists of the hyperbolic triangle groups, i.e. those having a presentation of the form $T_{p_1,p_2,p_3} := \langle x, y : x^{p_1} = y^{p_2} = (xy)^{p_3} \rangle$, where $p_1 \leq p_2 \leq p_3$ are natural numbers such that $1/p_1 + 1/p_2 + 1/p_3 < 1$. For historical and geometrical reasons, of particular interest is the triangle group $T_{2,3,7}$. Much effort has been given to classifying Hurwitz groups (i.e. finite images of $T_{2,3,7}$), especially the simple ones. The work really began in earnest with Macbeath and was taken up by Conder, Malle, Lucchini, Tamburini, Wilson and others, who endeavored to determine which finite simple groups are indeed Hurwitz groups. While alternating groups of sufficiently large order are Hurwitz, within the finite simple groups of Lie type, there is a dichotomy with respect to their occurrence as quotients of $T_{2,3,7}$ depending on whether the Lie rank is large or not. Indeed many classical groups of large rank are Hurwitz irrespectively of the size of the underlying field (and there is no known example of classical groups of large rank which are not Hurwitz), whereas the behaviour of finite simple groups of Lie type of low rank is rather sporadic. As an illustration if $r \geq 267$ then $\mathrm{PSL}_r(q)$ is Hurwitz for any prime power q, while $\mathrm{PSL}_4(q)$ is never Hurwitz, $G_2(q)$ is Hurwitz if $q \geq 5$, and $\mathrm{PSL}_2(p^e)$ is Hurwitz if and only if $e = 1$ and $p \equiv 0, \pm 1 \mod 7$, or $e = 3$ and $p \equiv \pm 2, \pm 3 \mod 7$.

These results on Hurwitz generation of groups of Lie type illustrate an open conjecture of Liebeck and Shalev which states that for any Fuchsian group Γ, there is an integer $f(\Gamma)$, such that if G is a finite simple classical group of rank greater than $f(\Gamma)$, then the probability that a randomly chosen homomorphism in $\mathrm{Hom}(\Gamma, G)$ is surjective tends to 1 as $|G| \to \infty$.

The recent work of Larsen, Lubotzky and Marion on the space $\mathrm{Hom}(\Gamma, G)$, for $G = G(p^n)$ a group of Lie type of rank 1 or 2 and Γ a general hyperbolic triangle group, has also led to a conjecture relating the asymptotic behavoir of this representation variety and the rigidity of certain triples of conjugacy classes in the associated algebraic group.

One further related active area of research is the work on Beauville surfaces (a compact complex surface arising from a free action of a finite group on curves of genus 2). Very recently, the proof of the conjecture of Bauer, Catanese and Grunewald, stating that all finite simple groups other than the alternating group in degree 5, admit unmixed Beauville structures has been completed (relying upon work of Fuertes, Jones, Garion, Lubotsky, Larsen, Guralnick, Malle and others). A necessary condition for a finite group G to admit an unmixed Beauville structure of a certain type is that G is a quotient of a triangle group satisfying additional hypotheses.

The workshop explored most of these themes with a wide variety of talks by leading experts in the field as well as quite a number of talks by graduate students and postdocs. There has been steady and significant progress on many of these problems including the structure of finite primitive permutation groups and extending some of these results to infinite groups. There were several talks regarding applying results on permutation groups to problems in other areas.

The workshop was structured so that there was lots of time available for people to gather in small groups and continue and start collaborations. The comments we received from the participants were all extremely positive and almost all of them commented on how they found work of others that they were not aware of impacted their own. The fact that there was time scheduled for people to interact (rather than just listen to lectures) was very favorably commented on. The quality of the talks was also commented on. Quite a lot of new collaborations were started.

Attachment: A spreadsheet summarising interactions reported to us by participants.