Simple Cuntz-Pimsner Rings

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Section Cuntz-Pimsner rings

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R-systems

- $R$ is an associative ring.
- $P, Q$ are $R$-bimodules.
- $\psi : P \otimes Q \rightarrow R$ is an $R$-bimodule homomorphism.
- The triple $(P, Q, \psi)$ is called an $R$-system.
- $I, J$ are two-sided ideals of $R$.
- An ideal $I$ of $R$ is a $\Psi$-invariant if

  $$\Psi(PI \otimes Q) = \Psi(P \otimes IQ) \subseteq I.$$
Covariant representations

Let \((P, Q, \psi)\) be an \(R\)-system, then a \textit{covariant representation} is a quadruple \((S', T', \sigma', B)\) satisfying:

1. \(B\) is a ring,
2. \(S' : P \to B\) and \(T' : Q \to B\) are additive maps,
3. \(\sigma' : R \to B\) is a ring homomorphism,
4. Given \(p \in P\), \(q \in Q\) and \(r \in R\),
   \[
   S'(pr) = S'(p)\sigma'(r), \quad S'(rp) = \sigma'(r)S'(p),
   \]
   \[
   T'(qr) = T'(q)\sigma'(r) \quad \text{and} \quad T'(rq) = \sigma'(r)T'(q).
   \]
5. \(\sigma'(\psi(p \otimes q)) = S'(p)T'(q)\) for \(p \in P\) and \(q \in Q\).
Condition (FS)

For \( p \in P \) and \( q \in Q \) let us define \( \theta_{q,p} \in \text{End}_R(Q_R) \) given by

\[
\theta_{q,p}(x) = q\psi(p \otimes x)
\]

for \( x \in Q \), and \( \theta_{p,q} \in \text{End}_R(R_P) \) given by

\[
\theta_{p,q}(y) = \psi(y \otimes q)p
\]

for \( y \in P \).

\[
\mathcal{F}_P(Q) = \text{span}\{\theta_{q,p} : p \in P, q \in Q\} \quad \text{and} \quad \mathcal{F}_Q(P) = \text{span}\{\theta_{p,q} : p \in P, q \in Q\}
\]

Definition 1

\((P, Q, \psi)\) satisfies condition \textbf{(FS)} if for any finite set \( \{q_1, \ldots, q_n\} \subseteq Q \) and any finite set \( \{p_1, \ldots, p_m\} \subseteq P \) exist \( \Theta \in \mathcal{F}_P(Q) \) and \( \Psi \in \mathcal{F}_Q(P) \) such that \( \Theta(q_i) = q_i \) and \( \Psi(p_j) = p_j \) for every \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).
Condition (FS)

For \( p \in P \) and \( q \in Q \) let us define \( \theta_{q,p} \in \text{End}_R(Q_R) \) given by

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**Definition 1**

\((P, Q, \psi)\) satisfies **condition (FS)** if for any finite set \( \{q_1, \ldots, q_n\} \subseteq Q \) and any finite set \( \{p_1, \ldots, p_m\} \subseteq P \) exist \( \Theta \in \mathcal{F}_P(Q) \) and \( \Psi \in \mathcal{F}_Q(P) \) such that \( \Theta(q_i) = q_i \) and \( \Psi(p_j) = p_j \) for every \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).
Cuntz-Pimsner Covariant representations

\[ \Delta : R \to \text{End}_R(Q_R) \text{ given by } \Delta(r)(q) = rq \text{ for } r \in R, q \in Q. \]

**Definition 2**

A two-sided ideal \( I \) of \( R \) is \( \psi \)-compatible if \( I \subseteq \Delta^{-1}(\mathcal{F}_P(Q)) \), and faithful if \( I \cap \ker \Delta = \{0\} \).

\( J \) will denote a fixed faithful and \( \psi \)-compatible ideal in \( R \).

**Definition 3**

A covariant representation \((S', T', \sigma', B)\) is said to be Cuntz-Pimsner invariant relative to \( J \) if

\[ \pi_{T',S'}(\Delta(x)) = \sigma'(x) \text{ for all } x \in J, \]

where \( \pi_{T',S'} : \mathcal{F}_P(Q) \to B \) satisfies \( \pi_{T',S'}(\theta_{q,p}) = T'(q)S'(p) \) for all \( p \in P \) and \( q \in Q \).
Cuntz-Pimsner Covariant representations

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Theorem 4

There is a covariant representation \((S, T, \sigma, \mathcal{O}(P,Q,\psi)(J))\) which is Cuntz-Pimsner invariant relative to \(J\) and \textbf{universal} in the sense that every covariant representation which is Cuntz-Pimsner invariant relative to \(J\) factors through it.
Relative Cuntz-Pimsner rings

Theorem 4

There is a covariant representation \((S, T, \sigma, O_{(P,Q,\psi)}(J))\) which is Cuntz-Pimsner invariant relative to \(J\) and universal in the sense that every covariant representation which is Cuntz-Pimsner invariant relative to \(J\) factors through it.
Given $n \in \mathbb{N}$ exist unique additive maps

$$T^n : Q^\otimes n \to O_{(P,Q,\psi)}(J) \quad \text{and} \quad S^n : P^\otimes n \to O_{(P,Q,\psi)}(J)$$

such that for $q_1, q_2, \ldots, q_n \in Q$ and $p_1, p_2, \ldots, p_n \in P$

$$T^n(q_1 \otimes q_2 \otimes \cdots \otimes q_n) = T(q_1)T(q_2)\cdots T(q_n)$$

$$S^n(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = S(p_1)S(p_2)\cdots S(p_n).$$

Then $O_{(P,Q,\psi)}(J)$ is a $\mathbb{Z}$-graded ring with grading

$$O_{(P,Q,\psi)}(J)^{(n)} = \text{span}(\{ T^{k+n}(q)S^k(p) \mid k \in \mathbb{N}, \ q \in Q^\otimes k, \ p \in P^\otimes k \})$$

$$\cup \{ T^n(q) \mid q \in Q^\otimes n \}).$$

$$O_{(P,Q,\psi)}(J)^{(-n)} = \text{span}(\{ T^k(q)S^{k+n}(p) \mid k \in \mathbb{N}, \ q \in Q^\otimes k, \ p \in P^\otimes k \})$$

$$\cup \{ S^n(p) \mid p \in P^\otimes n \}).$$

$$O_{(P,Q,\psi)}(J)^{(0)} = \text{span}(\{ T^k(q)S^k(p) \mid k \in \mathbb{N}, \ q \in Q^\otimes k, \ p \in P^\otimes k \})$$

$$\cup \{ \sigma(r) \mid r \in R \}).$$
Given $n \in \mathbb{N}$ exist unique additive maps

$$T^n : Q \otimes^n \to \mathcal{O}_{(P, Q, \psi)}(J) \quad \text{and} \quad S^n : P \otimes^n \to \mathcal{O}_{(P, Q, \psi)}(J)$$

such that for $q_1, q_2, \ldots, q_n \in Q$ and $p_1, p_2, \ldots, p_n \in P$

$$T^n(q_1 \otimes q_2 \otimes \cdots \otimes q_n) = T(q_1) T(q_2) \cdots T(q_n)$$

$$S^n(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = S(p_1) S(p_2) \cdots S(p_n).$$

Then $\mathcal{O}_{(P, Q, \psi)}(J)$ is a $\mathbb{Z}$-graded ring with grading

$$\mathcal{O}_{(P, Q, \psi)}(J)^{(n)} = \text{span}(\{ T^k + n(q) S^k (p) \mid k \in \mathbb{N}, q \in Q \otimes^k + n, p \in P \otimes^k \})$$

$$\cup \{ T^n(q) \mid q \in Q \otimes^n \})$$

$$\mathcal{O}_{(P, Q, \psi)}(J)^{(-n)} = \text{span}(\{ T^k (q) S^{k + n} (p) \mid k \in \mathbb{N}, q \in Q \otimes^k, p \in P \otimes^{k + n} \})$$

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$$\cup \{ \sigma(r) \mid r \in R \}).$$
Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph and let $F$ be any field. We define the ring $R_E := \bigoplus_{v \in E^0} R_v$ where each $R_v$ is a copy of $F$. We define the $R_E$-bimodules $Q_E := \bigoplus_{e \in E^1} Q_e$ and $P_E := \bigoplus_{\bar{e} \in E^1} P_{\bar{e}}$ where each $Q_e, P_{\bar{e}}$ is a copy of $F$. The left and the right multiplication are defined by

\[ r_v \cdot q_e \cdot s_w = \delta_{v,s(e)} \delta_{w,r(e)} r_v s_w q_e \]

\[ r_v \cdot p_{\bar{e}} \cdot s_w = \delta_{w,s(e)} \delta_{v,r(e)} r_v s_w p_{\bar{e}}. \]

Finally we define $\psi_E : P_E \otimes_{R_E} Q_E \to R_E$ the $R_E$-bimodule homomorphism given by

\[ \psi_E(p_{\bar{f}} \otimes q_e) = \delta_{s(e),s(f)} p_{\bar{f}} q_e. \]

Then

\[ \ker \Delta = \text{span}_F \{ 1_v \mid v \in E^0 \text{ and } vE^1 = \emptyset \}, \]

\[ \Delta^{-1}(\mathcal{F}_{P_E}(Q_E)) = \text{span}_F \{ 1_v \mid v \in E^0 \text{ and } vE^1 \text{ is finite} \}, \]

\[ J_E = \text{span}_F \{ 1_v \mid v \in E^0 \text{ and } 0 < |vE^1| < \infty \}. \]
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$$J_E = \text{span}_F \{ 1_v \mid v \in E^0 \text{ and } 0 < |vE^1| < \infty \} .$$
Let \((S, T, \sigma, \mathcal{O}(P_E, Q_E, \psi_E)(J_E))\) be the universal covariant representation of \((P_E, Q_E, \psi_E)\). Then if for each \(v \in E^0\) and \(e \in E^1\) define

\[
p_v = \sigma(1_v), \quad x_e = T(1_e) \quad \text{and} \quad y_e = S(1_{\overline{e}}).
\]

\(\mathcal{O}(P_E, Q_E, \psi_E)(J_E)\) is generated by

\[
\{p_v \mid v \in E^0\} \cup \{x_e \mid e \in E^1\} \cup \{y_e \mid e \in E^1\}
\]

and these elements satisfy:

(i) \(p_{s(e)}x_e = x_e = x_ep_{r(e)}\) for \(e \in E^1\),

(ii) \(p_{r(e)}y_e = y_e = y_e p_{s(e)}\) for \(e \in E^1\),

(iii) \(y_ex_f = \delta_{e,f} p_{r(e)}\) for \(e, f \in E^1\),

(iv) \(p_v = \sum_{e \in vE^1} x_e y_e\) for \(v \in E^0\) with \(0 < |vE^1| < \infty\).

In fact, \(\mathcal{O}(P_E, Q_E, \psi_E)(J_E)\) is isomorphic to the Leavitt path algebra \(L_F(E)\) of \(E\).
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\[ p_v = \sigma(1_v), \quad x_e = T(1_e) \quad \text{and} \quad y_e = S(1_e). \]

\(\mathcal{O}_{(P_E, Q_E, \psi_E)}(J_E)\) is generated by 

\[ \{p_v \mid v \in E^0\} \cup \{x_e \mid e \in E^1\} \cup \{y_e \mid e \in E^1\} \]

and these elements satisfy: 

(i) \(p_{s(e)} x_e = x_e = x_e p_{r(e)}\) for \(e \in E^1\), 
(ii) \(p_{r(e)} y_e = y_e = y_e p_{s(e)}\) for \(e \in E^1\), 
(iii) \(y_e x_f = \delta_{e,f} p_{r(e)}\) for \(e, f \in E^1\), 
(iv) \(p_v = \sum_{e \in v E^1} x_e y_e\) for \(v \in E^0\) with \(0 < |v E^1| < \infty\).

In fact, \(\mathcal{O}_{(P_E, Q_E, \psi_E)}(J_E)\) is isomorphic to the Leavitt path algebra \(L_F(E)\) of \(E\).
Section The ideal intersection Property

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Definition 5

For an ideal $I$ in $R$, let $\psi^{-1}(I)$ be the ideal

\[
\{ x \in R \mid \psi(px \otimes q) \in I \text{ for all } q \in Q \text{ and all } p \in P \},
\]

and let $I^{[\infty]}$ be the ideal

\[
\bigcap_{k=1}^{\infty} I^{[k]}
\]

where $I^{[k]}$ is defined recursively by $I^{[1]} = I$ and $I^{[k]} = \psi^{-1}(I^{[k-1]}) \cap I$ for $k > 1$.

Example 6

Let $(P_E, Q_E, \psi_E)$ and let $I$ be an ideal of $R_E$ and let $H = \{v \in E^0 \mid 1_v \in I\}$. Then $I = \text{span}_F \{1_v \mid v \in H\}$ and it follows that

\[
I^{[k]} = \text{span}_F \left\{ 1_v \mid v \in H \text{ and } r(e) \in H \text{ for all } e \in \bigcup_{i=1}^{k-1} vE' \right\}
\]

for $k > 1$. 

Definition 5

For an ideal $I$ in $R$, let $\psi^{-1}(I)$ be the ideal

$$\{ x \in R \mid \psi(px \otimes q) \in I \text{ for all } q \in Q \text{ and all } p \in P \},$$

and let $I^{[\infty]}$ be the ideal

$$\bigcap_{k=1}^{\infty} I^{[k]}$$

where $I^{[k]}$ is defined recursively by $I^{[1]} = I$ and $I^{[k]} = \psi^{-1}(I^{[k-1]}) \cap I$ for $k > 1$.

Example 6

Let $(P_E, Q_E, \psi_E)$ and let $I$ be an ideal of $R_E$ and let $H = \{ v \in E^0 \mid 1_v \in I \}$. Then $I = \text{span}_F \{ 1_v \mid v \in H \}$ and it follows that

$$I^{[k]} = \text{span}_F \left\{ 1_v \mid v \in H \text{ and } r(e) \in H \text{ for all } e \in \bigcup_{i=1}^{k-1} vE^i \right\}$$

for $k > 1$. 
The ideal intersection property

**Definition 7**

A subring $A$ of $\mathcal{O}_{(P,Q,\psi)}(J)$ has the **ideal intersection property** if the implication $K \cap A = \{0\} \implies K = \{0\}$ holds for every ideal $K$ in $\mathcal{O}_{(P,Q,\psi)}(J)$.

**Proposition 8**

The following 3 conditions are equivalent:

- The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ does not have the ideal intersection property.
- There is a non-zero graded ideal $\bigoplus_{k \in \mathbb{Z}} H^{(k)}$ in $\mathcal{O}_{(P,Q,\psi)}(J)$, an $n \in \mathbb{N}$ and a family $(\phi_k)_{k \in \mathbb{Z}}$ of injective $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$-bimodule homomorphisms $\phi_k : H^{(k)} \to \mathcal{O}_{(P,Q,\psi)}(J)^{(k+n)}$ such that $x\phi_k(y) = \phi_{k+j}(xy)$ and $\phi_k(y)x = \phi_{k+j}(yx)$ for $k, j \in \mathbb{Z}$, $x \in \mathcal{O}_{(P,Q,\psi)}(J)^{(j)}$ and $y \in H^{(k)}$.
- There is a non-zero $\psi$-invariant ideal $I_0$ of $R$, an $n \in \mathbb{N}$ and an injective $R$-bimodule homomorphism $\eta : I_0 \to Q^\otimes n$ such that $S_p(T_{\eta(x)}(q)) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I_0$ and $q \in Q$, and such that $I_0 \subseteq J^{[\infty]}$. 
The ideal intersection property

### Definition 7

A subring $A$ of $O_{(P,Q,\psi)}(J)$ has the **ideal intersection property** if the implication $K \cap A = \{0\} \implies K = \{0\}$ holds for every ideal $K$ in $O_{(P,Q,\psi)}(J)$.

### Proposition 8

The following 3 conditions are equivalent:

1. The subring $O_{(P,Q,\psi)}(J)^{(0)}$ does not have the ideal intersection property.

2. There is a non-zero graded ideal $\bigoplus_{k \in \mathbb{Z}} H^{(k)}$ in $O_{(P,Q,\psi)}(J)$, an $n \in \mathbb{N}$ and a family $(\phi_k)_{k \in \mathbb{Z}}$ of injective $O_{(P,Q,\psi)}(J)^{(0)}$-bimodule homomorphisms $\phi_k : H^{(k)} \to O_{(P,Q,\psi)}(J)^{(k+n)}$ such that $x \phi_k(y) = \phi_{k+j}(xy)$ and $\phi_k(y)x = \phi_{k+j}(yx)$ for $k, j \in \mathbb{Z}$, $x \in O_{(P,Q,\psi)}(J)^{(j)}$ and $y \in H^{(k)}$.

3. There is a non-zero $\psi$-invariant ideal $I_0$ of $R$, an $n \in \mathbb{N}$ and an injective $R$-bimodule homomorphism $\eta : I_0 \to Q^{\otimes n}$ such that $S_p(T_{\eta(x)}(q)) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I_0$ and $q \in Q$, and such that $I_0 \subseteq J^{[\infty]}$. 
Section The Cuntz-Krieger uniqueness Property

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The following 4 conditions are equivalent:

1. The ideal $J$ satisfies condition (L).
2. The subring $O_{(P, Q, \psi)}(J)^{(0)}$ has the ideal intersection property.
3. Every non-zero ideal in $O_{(P, Q, \psi)}(J)$ contains a non-zero graded ideal.
4. If $(S', T', \sigma', B)$ is an injective covariant representation of $(P, Q, \psi)$ and $J = J_{(S', T', \sigma', B)}$, then the ring homomorphism $\eta_{(S', T', \sigma', B)} : O_{(P, Q, \psi)}(J) \to B$ is injective.
Condition (L)

Definition 9

We say that a $\psi$-invariant ideal $I$ in $R$ is a $\psi$-invariant cycle if there exist $n \in \mathbb{N}$ and an injective $R$-bimodule homomorphism $\eta : I \to Q \otimes^n$ such that $S_p(T_{\eta(x)}(q)) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I$ and $q \in Q$, and we say that $J$ satisfies condition (L) with respect to the $R$-system $(P, Q, \psi)$ if there are no non-zero $\psi$-invariant cycles $I$ in $R$ such that $I \subseteq J^{[\infty]}$.

Define $J(S', T', \sigma', B) = \{x \in R| \sigma'(x) \in \pi_{T', S'}(\mathcal{F}_P(Q))\}$.

Theorem 10

The following 4 conditions are equivalent:

1. The ideal $J$ satisfies condition (L).
2. The subring $\mathcal{O}_{(P, Q, \psi)}(J)^{(0)}$ has the ideal intersection property.
3. Every non-zero ideal in $\mathcal{O}_{(P, Q, \psi)}(J)$ contains a non-zero graded ideal.
4. If $(S', T', \sigma', B)$ is an injective covariant representation of $(P, Q, \psi)$ and $J = J(S', T', \sigma', B)$, then the ring homomorphism $\eta_{(S', T', \sigma', B)} : \mathcal{O}_{(P, Q, \psi)}(J) \to B$ is injective.
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**Definition 9**

We say that a \( \psi \)-invariant ideal \( I \) in \( R \) is a \( \psi \)-invariant cycle if there exist \( n \in \mathbb{N} \) and an injective \( R \)-bimodule homomorphism \( \eta : I \to Q \otimes R \) such that \( S_p(T_{\eta}(x))(q) = \eta(\psi(px \otimes q)) \) for \( p \in P, x \in I \) and \( q \in Q \), and we say that \( J \) satisfies condition (L) with respect to the \( R \)-system \((P, Q, \psi)\) if there are no non-zero \( \psi \)-invariant cycles \( I \) in \( R \) such that \( I \subseteq J[\infty] \).

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**Theorem 10**

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Define $J_{(S', T', \sigma', B)} = \{x \in R | \sigma'(x) \in \pi_{T', S'}(F_P(Q))\}$.

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Example 11

Let \((P_E, Q_E, \psi_E)\) and let \(J_E\). Then

\[
J_E^{[k]} = \text{span}_F \{1_v \mid v \in E^0 \text{ and } 0 < |vE^i| < \infty \text{ for } i = 1, 2, \ldots, k\}
\]

for each \(k \in \mathbb{N}\) and that

\[
J_E^{[\infty]} = \text{span}_F \{1_v \mid v \in E^0 \text{ and } 0 < |vE^i| < \infty \text{ for all } i \in \mathbb{N}\}.
\]

A non-zero ideal \(I_H\) of \(R_E\) is a \(\psi_E\)-invariant cycle if and only if \(H\) is the union of cycles without exit.

Thus \(J_E\) satisfies condition (L) with respect to the \(R_E\)-system \((P_E, Q_E, \psi_E)\) if and only every closed path in \((E^0, E^1, r, s)\) has an exit.
Cuntz-Krieger uniqueness property

**Definition 12**

We say that the ideal $J$ has the **Cuntz-Krieger uniqueness property** with respect to the $R$-system $(P, Q, \psi)$ if the following holds:

If $(S_1, T_1, \sigma_1, B_1)$ and $(S_2, T_2, \sigma_2, B_2)$ are two injective covariant representations of $(P, Q, \psi)$ and they are both Cuntz-Pimsner invariant relative to $J$, then there is a ring isomorphism $\phi$ between $R\langle S_1, T_1, \sigma_1 \rangle$ and $R\langle S_2, T_2, \sigma_2 \rangle$ such that $\phi \circ \sigma_1 = \sigma_2$, $\phi \circ S_1 = S_2$ and $\phi \circ T_1 = T_2$. 
Cuntz-Krieger uniqueness property

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Cuntz-Krieger uniqueness property

Theorem 13

The following 5 conditions are equivalent:

1. The ideal $J$ has the Cuntz-Krieger uniqueness property.
2. If $(S', T', \sigma', B)$ is an injective covariant representation of $(P, Q, \psi)$ which is Cuntz-Pimsner invariant relative to $J$, then the ring homomorphism
   \[ \eta_{(S', T', \sigma', B)}^J : O_{(P, Q, \psi)}(J) \to B \]
   is injective.
3. The subring $\sigma(R)$ has the ideal intersection property.
4. The subring $O_{(P, Q, \psi)}(J)^(0)$ has the ideal intersection property, and $J$ is a maximal faithful, $\psi$-compatible ideal.
5. The ideal $J$ satisfies condition (L) and is a maximal faithful, $\psi$-compatible ideal.
Graded ideals

If $I$ is a $\psi$-invariant ideal in $R$, then $R_I = R/I$, $Q_I = Q/QI$ and $IP = P/IP$, and $\varphi_I$ denotes the corresponding quotient maps.

There is an $R_I$-bimodule homomorphism $\psi_I : IP \otimes Q_I \rightarrow R_I$ given by $\psi_I(\varphi_I(p) \otimes \varphi_I(q)) = \varphi_I(\psi(p \otimes q))$.

The triple $(IP, Q_I, \psi_I)$ is then an $R_I$-system satisfying condition (FS).

**Definition 14**

A **T-pair** is a pair $(I, J')$ where $I$ and $J'$ are ideals in $R$ such that $I \subseteq J$, $I$ is $\psi$-invariant, and $J' := \varphi_I(J')$ is a faithful, $\psi_I$-compatible ideal in $R_I$.

**Theorem 15**

There is a bijection between the T-pairs $(I, J')$ with $J \subseteq J'$ and the graded ideals of $O_{(P, Q, \psi)}(J)$.
Graded ideals

If $I$ is a $\psi$-invariant ideal in $R$, then $R_I = R/I$, $Q_I = Q/QI$ and $P = P/IP$, and $\varphi_I$ denotes the corresponding quotient maps.

There is an $R_I$-bimodule homomorphism $\psi_I : P \otimes Q_I \to R_I$ given by $\psi_I(\varphi_I(p) \otimes \varphi_I(q)) = \varphi_I(\psi(p \otimes q))$.

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If $I$ is a $\psi$-invariant ideal in $R$, then $R_I = R/I$, $Q_I = Q/QI$ and $P_I = P/IP$, and $\varphi_I$ denotes the corresponding quotient maps.

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Theorem 15

There is a bijection between the T-pairs $(I, J')$ with $J \subseteq J'$ and the graded ideals of $O_{(P,Q,\psi)}(J)$. 
Graded ideals

If $I$ is a $\psi$-invariant ideal in $R$, then $R_I = R/I$, $Q_I = Q/QI$ and $P = P/IP$, and $\varphi_I$ denotes the corresponding quotient maps.

There is an $R_I$-bimodule homomorphism $\psi_I : IP \otimes QI \to R_I$ given by $\psi_I(\varphi_I(p) \otimes \varphi_I(q)) = \varphi_I(\psi(p \otimes q))$.

The triple $(IP, QI, \psi_I)$ is then an $R_I$-system satisfying condition (FS).

**Definition 14**

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**Theorem 15**

There is a bijection between the T-pairs $(I, J')$ with $J \subseteq J'$ and the graded ideals of $O_{(P,Q,\psi)}(J)$.
Condition (K)

Definition 16

We say that the ideal $J$ satisfies **condition (K)** with respect to the $R$-system $(P, Q, \psi)$ if $J'$ satisfies condition (L) with respect to the $R_I$-system $(IP, IQ, \psi_I)$ whenever $(I, J')$ is a $T$-pair of $(P, Q, \psi)$ such that $J \subseteq J'$.

Theorem 17

The following 3 conditions are equivalent:

1. Every ideal of $O_{(P,Q,\psi)}(J)$ is graded.
2. The ideal $J$ satisfies condition (K).
3. If $(S', T', \sigma', B)$ is a covariant representation of $(P, Q, \psi)$ which is Cuntz-Pimsner invariant relative to $J$, and $(I, J') = \omega(S', T', \sigma', B)$, then the ring homomorphism

$$\eta_{(I,J')}^{(S',T',\sigma',B)} : O_{(IP,QI,\psi_I)}(J'_I) \to B$$

is injective.
Definition 16

We say that the ideal $J$ satisfies condition (K) with respect to the $R$-system $(P, Q, \psi)$ if $J'$ satisfies condition (L) with respect to the $R_I$-system $(I_P, Q_I, \psi_I)$ whenever $(I, J')$ is a $T$-pair of $(P, Q, \psi)$ such that $J \subseteq J'$.

Theorem 17

The following 3 conditions are equivalent:

1. Every ideal of $O_{(P, Q, \psi)}(J)$ is graded.
2. The ideal $J$ satisfies condition (K).
3. If $(S', T', \sigma', B)$ is a covariant representation of $(P, Q, \psi)$ which is Cuntz-Pimsner invariant relative to $J$, and $(I, J') = \omega(S', T', \sigma', B)$, then the ring homomorphism

$$\eta_{(I, J')}^{(S', T', \sigma', B)} : O_{(I_P, Q_I, \psi_I)}(J') \to B$$

is injective.
Section Simplicity

1. Cuntz-Pimsner rings

2. The ideal intersection Property

3. The Cuntz-Krieger uniqueness Property

4. Simplicity

5. Examples
**Z-simple Cuntz-Pimsner rings**

**Definition 18**

We say that $J$ is a super maximal $\psi$-compatible ideal if the only $T$-pairs $(I, J')$ of $(P, Q, \psi)$ which satisfies that $J \subseteq J'$, are $(0, J)$ and $(R, R)$.

It follows that $J$ is a super maximal $\psi$-compatible ideal if and only if the only graded ideals in $\mathcal{O}_{(P, Q, \psi)}(J)$ are $\{0\}$ and $\mathcal{O}_{(P, Q, \psi)}(J)$.

**Example 19**

Let $(P_E, Q_E, \psi_E)$ and $J_E$. It follows that $J_E$ is super maximal $\psi_E$-compatible ideal if and only if the only saturated hereditary subsets of $E^0$ are $\emptyset$ and $E^0$. 

\(\mathbb{Z}\)-simple Cuntz-Pimsner rings

**Definition 18**

We say that \(J\) is a **super maximal** \(\psi\)-compatible ideal if the only \(T\)-pairs \((I, J')\) of \((P, Q, \psi)\) which satisfies that \(J \subseteq J'\), are \((0, J)\) and \((R, R)\).

It follows that \(J\) is a super maximal \(\psi\)-compatible ideal if and only if the only graded ideals in \(\mathcal{O}_{(P, Q, \psi)}(J)\) are \(\{0\}\) and \(\mathcal{O}_{(P, Q, \psi)}(J)\).

**Example 19**

Let \((P_E, Q_E, \psi_E)\) and \(J_E\). It follows that \(J_E\) is super maximal \(\psi_E\)-compatible ideal if and only if the only saturated hereditary subsets of \(E^0\) are \(\emptyset\) and \(E^0\).
\[ \mathbb{Z}\text{-simple Cuntz-Pimsner rings} \]

**Definition 18**

We say that \( J \) is a **super maximal** \( \psi \)-compatible ideal if the only \( T \)-pairs \((I, J')\) of \((P, Q, \psi)\) which satisfies that \( J \subseteq J' \), are \((0, J)\) and \((R, R)\).

It follows that \( J \) is a super maximal \( \psi \)-compatible ideal if and only if the only graded ideals in \( \mathcal{O}_{(P, Q, \psi)}(J) \) are \( \{0\} \) and \( \mathcal{O}_{(P, Q, \psi)}(J) \).

**Example 19**

Let \((P_E, Q_E, \psi_E)\) and \( J_E \). It follows that \( J_E \) is super maximal \( \psi_E \)-compatible ideal if and only if the only saturated hereditary subsets of \( E^0 \) are \( \emptyset \) and \( E^0 \).
Simple Cuntz-Pimsner rings

Theorem 20

The following 5 conditions are equivalent:

1. The ring $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple.

2. The subring $\sigma(R)$ has the ideal intersection property and $J$ is a super maximal $\psi$-compatible ideal.

3. The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property and $J$ is a super maximal $\psi$-compatible ideal.

4. The ideal $J$ satisfies condition (L) and is a super maximal $\psi$-compatible ideal.

5. If $(S', T', \sigma', B)$ is a non-zero covariant representation of $(P, Q, \psi)$ which is Cuntz-Pimsner invariant relative to $J$, then the ring homomorphism

$$\eta^J_{(S', T', \sigma', B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$$

is injective.
1. Cuntz-Pimsner rings

2. The ideal intersection Property

3. The Cuntz-Krieger uniqueness Property

4. Simplicity

5. Examples
fractional skew monoid ring

Let $R$ be a ring with local units and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R)R\alpha(R) \subseteq \alpha(R)$.

Let

$$P = \text{span}\{r_1\alpha(r_2) \mid r_1, r_2 \in R\} \quad \text{and} \quad Q = \text{span}\{\alpha(r_1)r_2 \mid r_1, r_2 \in R\}$$

and

$$\psi : P \otimes Q \to R \quad \text{given by} \quad p \otimes q \mapsto pq,$$

then $(P, Q, \psi)$ is an $R$-system.

Then $R$ is a uniquely maximal, faithful, $\psi$-compatible ideal and that if:

1. $\alpha$ is an automorphism then $\mathcal{O}_{(P, Q, \psi)}(R) \cong R \times_\alpha \mathbb{Z}$.
2. $R$ is unital and $\alpha(R) = \alpha(R)R\alpha(R) = \alpha(1)R\alpha(1)$ then $\mathcal{O}_{(P, Q, \psi)}(R) \cong R[t_+, t_-; \alpha]$
fractional skew monoid ring

Let $R$ be a ring with local units and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Let

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We say that an ideal $I$ of $R$ is **strongly $\alpha$-invariant** if $\alpha^{-1}(I) = I$.

### Proposition 21

Let $R$ be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Then there is a bijective correspondence between graded ideals of $R[t_+, t_-; \alpha]$ and strongly $\alpha$-invariant ideals of $R$.

### Corollary 22

Let $R$ be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Then the following three conditions are equivalent:

1. The ring $R$ is a super maximal $\psi$-compatible ideal.
2. The only graded ideals in $R[t_+, t_-; \alpha]$ are $\{0\}$ and $R[t_+, t_-; \alpha]$.
3. The only strongly $\alpha$-invariant ideals in $R$ are $\{0\}$ and $R$. 
fractional skew monoid ring

We say that an ideal $I$ of $R$ is strongly $\alpha$-invariant if $\alpha^{-1}(I) = I$.

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- The only strongly $\alpha$-invariant ideals in $R$ are $\{0\}$ and $R$. 
Definition 23

Let $n \in \mathbb{N}$ and let $R$ be a ring with local units. A ring homomorphism $\alpha : R \to R$ is said to be **inner with periodicity** $n$ if there exist $u, v \in \mathcal{M}(R)$ such that $vu = 1$ (where 1 denotes the unit of $\mathcal{M}(R)$), and $\alpha^n(r) = urv$ and $\alpha(ur) = u\alpha(r)$ for all $r \in R$. If $\alpha$ is not inner of any periodicity, then it is said to be **outer**.

Proposition 24

Let $R$ be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Consider the following three conditions:

1. There exists an $n \in \mathbb{N}$ such that the homomorphism $\alpha$ is inner with periodicity $n$.
2. The ring $R$ is a $\psi$-invariant cycle.
3. The ring $R$ does not satisfy condition (L) with respect to $(P, Q, \psi)$.

Then (1) implies (2), and (2) implies (3). If in addition $R$ is a super maximal $\psi$-compatible ideal, and $\alpha^n$ is strict for every $n \in \mathbb{N}$, then (3) implies (1) and the three conditions are equivalent.
fractional skew monoid ring

**Definition 23**

Let $n \in \mathbb{N}$ and let $R$ be a ring with local units. A ring homomorphism $\alpha : R \to R$ is said to be **inner with periodicity** $n$ if there exist $u, v \in \mathcal{M}(R)$ such that $vu = 1$ (where 1 denotes the unit of $\mathcal{M}(R)$), and $\alpha^n(r) = urv$ and $\alpha(ur) = u\alpha(r)$ for all $r \in R$. If $\alpha$ is not inner of any periodicity, then it is said to be **outer**.

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Corollary 25

Let $R$ be a unital ring and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R) = eRe$ for some idempotent $e \in R$. Then the following two statements are equivalent:

1. The fractional skew monoid ring $R[t_+, t_-; \alpha]$ is simple.
2. The homomorphism $\alpha$ is outer and the only strongly $\alpha$-invariant ideals in $R$ are $\{0\}$ and $R$.

Corollary 26

Let $R$ be a ring with local units and let $\alpha : R \to R$ be a ring automorphism. Then the following two statements are equivalent:

1. The crossed product $R \times_\alpha \mathbb{Z}$ is simple.
2. The automorphism $\alpha$ is outer and the only strongly $\alpha$-invariant ideals in $R$ are $\{0\}$ and $R$. 
Corollary 25

Let $R$ be a unital ring and let $\alpha : R \rightarrow R$ be an injective ring homomorphism such that $\alpha(R) = eRe$ for some idempotent $e \in R$. Then the following two statements are equivalent:

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