The evolution of the orbit distance in the double averaged restricted 3-body problem with crossing singularities

Giovanni Federico Gronchi

*Dipartimento di Matematica, Università di Pisa*

e-mail: gronchi@dm.unipi.it

*New perspectives on the N-body problem*

Banff centre, Canada

January 13-18, 2013
joining works in collaboration with:

A. Milani, C. Tardioli, G. Tommei


3–body problem: Sun, Earth, asteroid

restricted problem: the asteroid does not influence the motion of the two larger bodies.

equations of motion of the asteroid:

\[
\ddot{y} = -G \left[ m_\odot \frac{(y - y_\odot(t))}{|y - y_\odot(t)|^3} + m_\oplus \frac{(y - y_\oplus(t))}{|y - y_\oplus(t)|^3} \right]
\]

- \( y \) is the unknown position of the asteroid;
- \( y_\odot(t), y_\oplus(t) \) are known functions of time, solutions of the two-body problem Sun-Earth.
The restricted 3–body problem

3–body problem: Sun, Earth, asteroid
restricted problem: the asteroid does not influence the motion of the two larger bodies.

Equations of motion of the asteroid:

\[
\ddot{y} = -G \left[ m_\odot \frac{(y - y_\odot(t))}{|y - y_\odot(t)|^3} + m_\oplus \frac{(y - y_\oplus(t))}{|y - y_\oplus(t)|^3} \right]
\]

- \( y \) is the unknown position of the asteroid;
- \( y_\odot(t), y_\oplus(t) \) are known functions of time, solutions of the two-body problem Sun-Earth.
The restricted 3–body problem

In heliocentric coordinates

\[ \ddot{x} = -k^2 \left[ \frac{x}{|x|^3} + \mu \left( \frac{(x - x')}{|x - x'|^3} - \frac{x'}{|x'|^3} \right) \right] \]

- \( x = y - y_\odot, \ x' = y_\oplus - y_\odot; \)
- \( k^2 = Gm_\odot, \ \mu = \frac{m_\oplus}{m_\odot} \) is a small parameter;
- \( -k^2 \mu \frac{(x-x')}{|x-x'|^3} \) is the direct perturbation of the planet on the asteroid;
- \( k^2 \mu \frac{x'}{|x'|^3} \) is the indirect perturbation, due to the interaction Sun-planet.
The restricted 3–body problem

In heliocentric coordinates

\[
\dot{x} = -k^2 \left[ \frac{x}{|x|^3} + \mu \left( \frac{(x - x')}{|x - x'|^3} - \frac{x'}{|x'|^3} \right) \right]
\]

- \( x = y - y_\odot, \ x' = y_\oplus - y_\odot; \)
- \( k^2 = Gm_\odot, \ \mu = \frac{m_\oplus}{m_\odot} \) is a small parameter;
- \(-k^2 \mu \frac{(x-x')}{|x-x'|^3}\) is the direct perturbation of the planet on the asteroid;
- \( k^2 \mu \frac{x'}{|x'|^3} \) is the indirect perturbation, due to the interaction Sun-planet.
The restricted 3–body problem

In heliocentric coordinates

\[
\ddot{x} = -k^2 \left[ \frac{x}{|x|^3} + \mu \left( \frac{(x - x')}{|x - x'|^3} - \frac{x'}{|x'|^3} \right) \right]
\]

- \( x = y - y_\odot, \) \( x' = y_\oplus - y_\odot \);
- \( k^2 = Gm_\odot, \) \( \mu = \frac{m_\oplus}{m_\odot} \) is a small parameter;
- \( -k^2 \mu \frac{(x-x')}{|x-x'|^3} \) is the direct perturbation of the planet on the asteroid;
- \( k^2 \mu \frac{x'}{|x'|^3} \) is the indirect perturbation, due to the interaction Sun-planet.
Canonical formulation of the problem

Use **Delaunay’s variables** $\mathcal{Y} = (L, G, Z, \ell, g, z)$ for the motion of the asteroid:

$$
\begin{align*}
L &= k\sqrt{a} \\
G &= L\sqrt{1 - e^2} \\
Z &= G \cos I
\end{align*}
\quad \begin{align*}
\ell &= n(t - t_0) \\
g &= \omega \\
z &= \Omega
\end{align*}
$$

These are **canonical variables**, representing the **osculating orbit**, solution of the 2-body problem Sun-asteroid.

Denote by $\mathcal{Y}' = (L', G', Z', \ell', g', z')$ Delaunay’s variables for the planet.
Use Delaunay’s variables $\mathcal{Y} = (L, G, Z, \ell, g, z)$ for the motion of the asteroid:

\[
\begin{align*}
L &= k\sqrt{a} \\
G &= L\sqrt{1 - e^2} \\
Z &= G\cos I \\
\ell &= n(t - t_0) \\
g &= \omega \\
z &= \Omega
\end{align*}
\]

These are canonical variables, representing the osculating orbit, solution of the 2-body problem Sun-asteroid.

Denote by $\mathcal{Y}' = (L', G', Z', \ell', g', z')$ Delaunay’s variables for the planet.
Use **Delaunay’s variables** $\mathcal{Y} = (L, G, Z, \ell, g, z)$ for the motion of the asteroid:

$$
\begin{align*}
L &= k \sqrt{a} \\
G &= L \sqrt{1 - e^2} \\
Z &= G \cos I
\end{align*}
$$

$$
\begin{align*}
\ell &= n (t - t_0) \\
g &= \omega \\
z &= \Omega
\end{align*}
$$

These are **canonical variables**, representing the osculating orbit, solution of the 2-body problem Sun-asteroid.

Denote by $\mathcal{Y}' = (L', G', Z', \ell', g', z')$ Delaunay’s variables for the planet.
Hamilton’s equations are

\[ \dot{Y} = J_3 \nabla_Y H, \]

where

\[ J_3 = \begin{bmatrix} O_3 & -I_3 \\ I_3 & O_3 \end{bmatrix}. \]

\[ H = H_0 - R \] is the Hamiltonian, \( H_0 = -\frac{k^2}{2L^2} \) (unperturbed part),

\[ R = k^2 \mu \left( \frac{1}{|\mathcal{X} - \mathcal{X}'|} - \frac{\mathcal{X} \cdot \mathcal{X}'}{|\mathcal{X}'|^3} \right) \] (perturbing function).

Here \( \mathcal{X}, \mathcal{X}' \) denote \( x, x' \) as functions of \( Y, \dot{Y} \).
Let \((E_j, v_j), j = 1, 2\) be the orbital elements of two celestial bodies on confocal Keplerian orbits: 
\(E_j\) represents the trajectory of a body, 
\(v_j\) is a parameter along it. Set \(V = (v_1, v_2)\).

For a given two-orbit configuration \(\mathcal{E} = (E_1, E_2)\), we introduce the Keplerian distance function

\[\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\mathcal{X}_1 - \mathcal{X}_2|\]

We are interested in the local minimum points of \(d\).
The Keplerian distance function $d$

Let $(E_j, v_j), j = 1, 2$ be the orbital elements of two celestial bodies on confocal Keplerian orbits: $E_j$ represents the trajectory of a body, $v_j$ is a parameter along it. Set $V = (v_1, v_2)$.

For a given two-orbit configuration $\mathcal{E} = (E_1, E_2)$, we introduce the Keplerian distance function

$$\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\chi_1 - \chi_2|$$

We are interested in the local minimum points of $d$. 

Giovanni F. Gronchi  
Dynamics, Topology and Computations
Is there still something that we do not know about distance of points on conic sections?

(1) I observed you were quite eager to be kept informed of the work I was doing in conics.

(Apollonius of Perga, *Conics*, Book I)
The local minimum points of $d$ can be found by computing all the critical points of $d^2$.

Apart from the case of two concentric coplanar circles, or two overlapping ellipses, $d^2$ has finitely many critical points.

There exist configurations with 12 critical points, and 4 local minima of $d^2$.

This is thought to be the maximum possible, but a proof is not known yet, see also Albouy, Cabral, Santos (2012).
Some remarks on the critical points of $d^2$

- The local minimum points of $d$ can be found by computing all the critical points of $d^2$.
- Apart from the case of two concentric coplanar circles, or two overlapping ellipses, $d^2$ has finitely many critical points.
- There exist configurations with 12 critical points, and 4 local minima of $d^2$. This is thought to be the maximum possible, but a proof is not known yet, see also Albouy, Cabral, Santos (2012).
Some remarks on the critical points of $d^2$

- The local minimum points of $d$ can be found by computing all the critical points of $d^2$.
- Apart from the case of two concentric coplanar circles, or two overlapping ellipses, $d^2$ has finitely many critical points.
- There exist configurations with 12 critical points, and 4 local minima of $d^2$.

This is thought to be the maximum possible, but a proof is not known yet, see also Albouy, Cabral, Santos (2012).
The local minimum points of $d$ can be found by computing all the critical points of $d^2$.

Apart from the case of two concentric coplanar circles, or two overlapping ellipses, $d^2$ has finitely many critical points.

There exist configurations with 12 critical points, and 4 local minima of $d^2$. This is thought to be the maximum possible, but a proof is not known yet, see also Albouy, Cabral, Santos (2012).
Let \( V_h = V_h(\mathcal{E}) \) be a local minimum point of \( V \mapsto d^2(\mathcal{E}, V) \).

Consider the maps

\[
\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h), \\
\mathcal{E} \mapsto d_{\min}(\mathcal{E}) = \min_h d_h(\mathcal{E}).
\]

The map \( \mathcal{E} \mapsto d_{\min}(\mathcal{E}) \) gives the orbit distance.
(i) \( d_h \) and \( d_{\text{min}} \) are not differentiable where they vanish;

(ii) two local minima can exchange their role as absolute minimum thus \( d_{\text{min}} \) loses its regularity without vanishing;

(iii) when a bifurcation occurs the definition of the maps \( d_h \) may become ambiguous after the bifurcation point.
Singularities of \(d_h\) and \(d_{\text{min}}\)

(i) \(d_h\) and \(d_{\text{min}}\) are not differentiable where they vanish;

(ii) two local minima can exchange their role as absolute minimum thus \(d_{\text{min}}\) loses its regularity without vanishing;

(iii) when a bifurcation occurs the definition of the maps \(d_h\) may become ambiguous after the bifurcation point.
(i) $d_h$ and $d_{\min}$ are not differentiable where they vanish;
(ii) two local minima can exchange their role as absolute minimum thus $d_{\min}$ loses its regularity without vanishing;
(iii) when a bifurcation occurs the definition of the maps $d_h$ may become ambiguous after the bifurcation point.
Smoothing through change of sign

Model problem:

\[ f(x, y) = \sqrt{x^2 + y^2} \quad \tilde{f}(x, y) = \begin{cases} 
-f(x, y) & \text{for } x > 0 \\
 f(x, y) & \text{for } x < 0 
\end{cases} \]

Can we smooth the maps \( d_h(\mathcal{E}), d_{\min}(\mathcal{E}) \) through a change of sign?
Model problem:

\[ f(x, y) = \sqrt{x^2 + y^2} \quad \tilde{f}(x, y) = \begin{cases} 
-f(x, y) & \text{for } x > 0 \\
    f(x, y) & \text{for } x < 0 
\end{cases} \]

Can we smooth the maps \( d_h(\mathcal{E}) \), \( d_{\text{min}}(\mathcal{E}) \) through a change of sign?
Local smoothing of \( d_h \) at a crossing singularity

**Smoothing** \( d_h \), the procedure for \( d_{\text{min}} \) is the same.

Consider the points on the two orbits

\[
\chi_1(h) = \chi_1(E_1, v_1^{(h)}) ; \quad \chi_2(h) = \chi_2(E_2, v_2^{(h)}) .
\]

corresponding to the local minimum point

\[
V_h = (v_1^{(h)}, v_2^{(h)}) \text{ of } d^2 ;
\]
Local smoothing of $d_h$ at a crossing singularity

introduce the tangent vectors to the trajectories $E_1, E_2$ at these points:

\[ \tau_1 = \frac{\partial \vec{X}_1}{\partial v_1} (E_1, v_1(h)), \quad \tau_2 = \frac{\partial \vec{X}_2}{\partial v_2} (E_2, v_2(h)), \]

and their cross product $\tau_3 = \tau_1 \times \tau_2$;
Local smoothing of $d_h$ at a crossing singularity

\[ \Delta = \mathbf{x}_1 - \mathbf{x}_2, \quad \Delta_h = \mathbf{x}_1^{(h)} - \mathbf{x}_2^{(h)}. \]

The vector $\Delta_h$ joins the points attaining a local minimum of $d^2$ and $|\Delta_h| = d_h$.

Note that $\Delta_h \times \tau_3 = 0$
Smoothing the crossing singularity

**smoothing rule:**

\[ \tilde{d}_h = \text{sign}(\tau_3 \cdot \Delta_h) d_h \]

\( \mathcal{E} \mapsto \tilde{d}_h(\mathcal{E}) \) is an analytic map in a neighborhood of most crossing configurations
Smoothing the crossing singularity

**smoothing rule:**

\[ \tilde{d}_h = \text{sign}(\tau_3 \cdot \Delta_h)d_h \]

\[ \mathcal{E} \mapsto \tilde{d}_h(\mathcal{E}) \text{ is an analytic map in a neighborhood of most crossing configurations} \]
The **averaging principle** is used to study the qualitative behavior of solutions of ODEs in perturbation theory, see Arnold, Kozlov, Neishtadt (1997).

**unperturbed**
\[
\begin{align*}
\dot{\phi} &= \omega(I) \\
\dot{I} &= 0
\end{align*}
\]
\(\phi \in \mathbb{T}^n, I \in \mathbb{R}^m\)

**perturbed**
\[
\begin{align*}
\dot{\phi} &= \omega(I) + \epsilon f(\phi, I, \epsilon) \\
\dot{I} &= \epsilon g(\phi, I, \epsilon)
\end{align*}
\]

**averaged**
\[
\dot{J} = \epsilon G(J), \quad G(J) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(\phi, J, 0) \, d\phi_1 \ldots d\phi_n
\]
The averaging principle is used to study the qualitative behavior of solutions of ODEs in perturbation theory, see Arnold, Kozlov, Neishtadt (1997).

unperturbed \[
\begin{align*}
\dot{\phi} &= \omega(I) \\
I &= 0 \\
\phi &\in \mathbb{T}^n, I \in \mathbb{R}^m
\end{align*}
\]

perturbed \[
\begin{align*}
\dot{\phi} &= \omega(I) + \epsilon f(\phi, I, \epsilon) \\
\dot{I} &= \epsilon g(\phi, I, \epsilon)
\end{align*}
\]

averaged \[
\dot{J} = \epsilon G(J), \quad G(J) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(\phi, J, 0) \, d\phi_1 \ldots d\phi_n
\]
The averaging principle is used to study the qualitative behavior of solutions of ODEs in perturbation theory, see Arnold, Kozlov, Neishtadt (1997).

unperturbed \[ \begin{cases} \dot{\phi} = \omega(I) \\ I = 0 \end{cases} \quad \phi \in \mathbb{T}^n, I \in \mathbb{R}^m \]

perturbed \[ \begin{cases} \dot{\phi} = \omega(I) + \epsilon f(\phi, I, \epsilon) \\ \dot{I} = \epsilon g(\phi, I, \epsilon) \end{cases} \]

averaged \[ \dot{J} = \epsilon G(J), \quad G(J) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(\phi, J, 0) \, d\phi_1 \ldots d\phi_n \]
Using the averaged equations corresponds to substituting the time average with the space average.

Case of 2 angles: a problem occurs if there are resonant relations of low order between the motions $\phi_1(t), \phi_2(t)$, i.e. if $k_1\dot{\phi}_1 + k_2\dot{\phi}_2 = 0$, with $k_1, k_2$ small integers.
Averaged Hamilton’s equations:

\[
\dot{\bar{Y}} = -J_2 \nabla_{\bar{Y}} \bar{R},
\]

(1)

with \( Y = (G, Z, g, z) \). We averaged over the fast angles \( \ell, \ell' \).

If no orbit crossing occurs, (1) are equal to

\[
\dot{\bar{Y}} = -J_2 \nabla_{\bar{Y}} \bar{R}
\]

(2)

with

\[
\bar{R} = \frac{1}{(2\pi)^2} \int_{T^2} R d\ell d\ell' = \frac{\mu k^2}{(2\pi)^2} \int_{T^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|} d\ell d\ell'
\]

The average of the indirect term of \( R \) is zero.
If there is an orbit crossing, then averaging on the fast angles $\ell, \ell'$ produces a singularity in the averaged equations: we take into account every possible position on the orbits, thus also the collision configurations.

\[
\overline{R} = \frac{\mu k^2}{(2\pi)^2} \int_{T^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|} d\ell d\ell'
\]

and

\[|\mathcal{X}(E_1, v_1^{(h)}) - \mathcal{X}'(E_2, v_2^{(h)})| = 0.\]
(433) Eros: the first near-Earth asteroid (NEA, with $q = a(1 - e) \leq 1.3$ AU), discovered in 1898; it crosses the trajectory of Mars.

Today (January 15, 2013) we know about 9500 NEAs: several of them cross the orbit of the Earth during their evolution.
Let $\mathcal{E}_c$ be a non–degenerate crossing configuration for $d_h$, with only one crossing point. Given a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$, we set

$$\mathcal{W}^+ = \mathcal{W} \cap \{\tilde{d}_h > 0\},$$

$$\mathcal{W}^- = \mathcal{W} \cap \{\tilde{d}_h < 0\}.$$  

The averaged vector field $\nabla Y R$ is not defined on $\Sigma = \{d_H = 0\}$. 

Giovanni F. Gronchi  
Dynamics, Topology and Computations
Main result

**Theorem:** The averaged vector field $\nabla YR$ can be extended to two Lipschitz–continuous vector fields $(\nabla YR)_h^\pm$ on a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$. These extended vector fields, restricted to $\mathcal{W}^+$, $\mathcal{W}^-$ respectively, correspond to $\nabla YR$. 

\[
\nabla YR = (\nabla YR)_h^-
\]

\[
\nabla YR = (\nabla YR)_h^+
\]
Moreover the following relations hold:

\[
\text{Diff}_h \left( \frac{\partial R}{\partial y_k} \right) \overset{\text{def}}{=} \left( \frac{\partial R}{\partial y_k} \right)_h^+ - \left( \frac{\partial R}{\partial y_k} \right)_h^- = \mu k^2 \left[ \frac{\partial}{\partial y_k} \left( \frac{1}{\sqrt{\det(A_h)}} \right) \tilde{d}_h + \frac{1}{\sqrt{\det(A_h)}} \frac{\partial \tilde{d}_h}{\partial y_k} \right],
\]

where \( y_k \) is a component of Delaunay’s elements \( Y \), and

\[
A_h(\mathcal{E}) = \frac{1}{2} \frac{\partial^2 d^2}{\partial V^2}(\mathcal{E}, V_h(\mathcal{E})).
\]
We write

\[ d^2(\mathcal{E}, V) = d_h^2(\mathcal{E}) + (V - V_h) \cdot A_h(\mathcal{E})(V - V_h) + R_3^{(h)}(\mathcal{E}, V), \]

where

i) \(2A_h(\mathcal{E})\) is the Hessian matrix of \(V \mapsto d^2(\mathcal{E}, V)\) in \(V_h\);

ii) \(R_3^{(h)}\) is Taylor’s remainder in the integral form.

Introduce the \textit{approximated distance}

\[ \delta_h = \sqrt{d_h^2 + (V - V_h) \cdot A_h(V - V_h)}. \]
Consider the following **decomposition**:

\[
\mathcal{W} \setminus \Sigma \ni E \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{d} \, d\ell d\ell' \\
= \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) \, d\ell d\ell' + \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell d\ell'
\]

We prove that:

i) the two maps \( \mathcal{W}^\pm \ni E \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell d\ell' \) admits two different analytic extensions to \( \mathcal{W} \);

ii) the map \( \mathcal{W} \setminus \Sigma \ni E \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) \, d\ell d\ell' \) admits a Lipschitz–continuous extension to \( \mathcal{W} \).
Consider the following decomposition:

\[ \mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{d} \, dl \, dl' \]

\[ = \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) \, dl \, dl' + \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, dl \, dl' \]

We prove that:

i) the two maps \( \mathcal{W}^\pm \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, dl \, dl' \) admits two different analytic extensions to \( \mathcal{W} \);

ii) the map \( \mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) \, dl \, dl' \) admits a Lipschitz–continuous extension to \( \mathcal{W} \).
Consider the following decomposition:

\[ \mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) d\ell d\ell' + \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} d\ell d\ell' \]

We prove that:

i) the two maps $\mathcal{W}^\pm \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} d\ell d\ell'$ admits two different analytic extensions to $\mathcal{W}$;

ii) the map $\mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) d\ell d\ell'$ admits a Lipschitz–continuous extension to $\mathcal{W}$. 
idea of the proof of i)

\[ \mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2} \frac{1}{\delta_h} \, d\ell \, d\ell' \]

Set

\[ \mathcal{D} = \{ V \in \mathbb{T}^2 : (V - V_h) \cdot \mathcal{A}_h (V - V_h) \leq r^2 \} . \]

We have

\[ \int_{\mathcal{D}} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \left( \sqrt{d_h^2 + r^2} - d_h \right) . \]
We obtain

\[
\int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \ d\ell \ d\ell' = \frac{\partial}{\partial y_k} \left( \frac{2\pi}{\sqrt{\det A_h}} \right) \left( \sqrt{d_h^2 + r^2} - d_h \right) + \\
\frac{2\pi}{\sqrt{\det A_h}} \frac{d_h}{\sqrt{d_h^2 + r^2}} \frac{\partial d_h}{\partial y_k} - \frac{2\pi}{\sqrt{\det A_h}} \frac{\partial d_h}{\partial y_k} + \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2 \setminus \mathcal{D}} \frac{1}{\delta_h} \ d\ell \ d\ell'
\]

so that the formula

\[
\left( \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \ d\ell \ d\ell' \right)_{\pm} = \frac{\partial}{\partial y_k} \left( \frac{2\pi}{\sqrt{\det A_h}} \right) \left( \sqrt{d_h^2 + r^2} \pm \tilde{d}_h \right) + \\
\frac{2\pi}{\sqrt{\det A_h}} \frac{\tilde{d}_h}{\sqrt{d_h^2 + r^2}} \frac{\partial \tilde{d}_h}{\partial y_k} \pm \frac{2\pi}{\sqrt{\det A_h}} \frac{\partial \tilde{d}_h}{\partial y_k} + \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2 \setminus \mathcal{D}} \frac{1}{\delta_h} \ d\ell \ d\ell'
\]

defines analytic extensions of \( \mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \ d\ell \ d\ell' \) to \( \mathcal{W} \).
Generalized solutions

\[ Y_{k} - Y_{k-1} + W^+ - \Sigma \]

**Figure:** Runge-Kutta-Gauss method and continuation of the solutions of equations (1) beyond the singularity.

The averaged solutions are piecewise–smooth
Comparison of solutions for (1620) Geographos

Giovanni F. Gronchi
Dynamics, Topology and Computations
Define the **secular evolution of the minimal distances**

\[
\begin{align*}
\overline{d}_h(t) &= \tilde{d}_h(\mathcal{E}(t)), \\
\overline{d}_{\text{min}}(t) &= \tilde{d}_{\text{min}}(\mathcal{E}(t))
\end{align*}
\]

in an open interval containing a crossing time \( t_c \).

**Proposition:** Assume \( t_c \) is a crossing time and \( \mathcal{E}_c = \mathcal{E}(t_c) \) is a non-degenerate crossing configuration with only one crossing point, i.e. \( d_h(\mathcal{E}_c) = 0 \). Then there exists an interval \( (t_a, t_b) \), \( t_a < t_c < t_b \) such that \( \overline{d}_h \in C^1((t_a, t_b); \mathbb{R}) \).
Define the secular evolution of the minimal distances

\[ \overline{d}_h(t) = \tilde{d}_h(\mathcal{E}(t)), \quad \overline{d}_{\text{min}}(t) = \tilde{d}_{\text{min}}(\mathcal{E}(t)) \]

in an open interval containing a crossing time \( t_c \).

**Proposition:** Assume \( t_c \) is a crossing time and \( \mathcal{E}_c = \mathcal{E}(t_c) \) is a non-degenerate crossing configuration with only one crossing point, i.e. \( d_h(\mathcal{E}_c) = 0 \). Then there exists an interval \((t_a, t_b)\), \( t_a < t_c < t_b \) such that \( \overline{d}_h \in C^1((t_a, t_b); \mathbb{R}) \).
Secular evolution of the orbit distance

Proof:

\[
\lim_{t \to t_c^+} \dot{d}_h(t) - \lim_{t \to t_c^-} \dot{d}_h(t) = \text{Diff}_h(\nabla_Y R) \cdot \mathbb{J}_2 \nabla_Y \tilde{d}_h \bigg|_{\varepsilon = \varepsilon_c} \\
= \frac{\mu k^2}{\pi \sqrt{\det A_h}} \{\tilde{d}_h, \tilde{d}_h\}_Y \bigg|_{\varepsilon = \varepsilon_c} = 0,
\]

The secular evolution of \( \tilde{d}_{\text{min}} \) is more regular than that of the orbital elements in a neighborhood of a planet crossing time.
Conclusions and future work

We can compute the secular evolution of planet crossing asteroids, by averaging over the fast angles: the solutions are piecewise–smooth;

the orbit distance along the averaged evolution is more regular than the orbital elements.

Open questions

Can we prove that the averaged solutions are good approximation of the solutions of the full equations?

What can we do in case of mean motion resonances?
Conclusions and future work

- We can compute the secular evolution of planet crossing asteroids, by averaging over the fast angles: the solutions are piecewise-smooth;
- the orbit distance along the averaged evolution is more regular than the orbital elements.

Open questions
- Can we prove that the averaged solutions are good approximation of the solutions of the full equations?
- What can we do in case of mean motion resonances?