Sharp isoperimetric inequalities for Steklov eigenvalues

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The *Steklov spectral problem* on a bounded domain $\Omega \subset \mathbb{R}^d$ is

$$\Delta u = 0 \text{ in } \Omega, \quad \partial_n u = \sigma u \text{ on } \partial \Omega.$$ 

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0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \uparrow \infty
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The Steklov eigenvalues are the eigenvalues of the **Dirichlet–to–Neumann operator** $\Lambda : C^\infty(\partial \Omega) \to C^\infty(\partial \Omega)$, defined by

$$\Lambda f = \partial_n u$$

where $\Delta u = 0$ in $\Omega$ and $u = f$ on $\partial \Omega$. 


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where $\Delta u = 0$ in $\Omega$ and $u = f$ on $\partial \Omega$.

The operator $\Lambda$ is an elliptic self-adjoint $\Psi$do of order 1, with principal symbol $|\xi|$. It follows that

$$\sigma_k \sim C(d) \left( \frac{k}{|\partial \Omega|} \right)^{1/(d-1)} \text{ as } k \nearrow \infty.$$
Isoperimetric problems for Steklov eigenvalues

**Problem**

Maximize $\sigma_k(\Omega)$ among domains $\Omega \subset \mathbb{R}^d$ with $|\partial \Omega| = 1$. 

Given $c > 0$, it is clear that $\sigma_k(c\Omega) = 1/c \sigma_k(\Omega)$. Therefore, the functional $\Omega \mapsto \tilde{\sigma}_k(\Omega) := \sigma_k(\Omega) / |\partial \Omega|^{1/d - 1}$ is scaling invariant.

Equivalent problem

Maximize $\tilde{\sigma}_k(\Omega)$ among all regular domains $\Omega \subset \mathbb{R}^d$. 

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**Equivalent problem**

Maximize $\tilde{\sigma}_k(\Omega)$ among all regular domains $\Omega \subset \mathbb{R}^d$. 
Variational characterization of $\sigma_k$

The starting point of many strategies to obtain isoperimetric results is to use a variational characterization. . .

Let

$$\mathcal{H}_k = \{ V \subset H^1(\Omega) : \dim V = k \}.$$ 

$$\sigma_k = \min_{V \in \mathcal{H}_k} \max_{f \in V \setminus \{0\}} \frac{\int_{\Omega} |\nabla f|^2 \, dx}{\int_{\partial \Omega} f^2 \, dS}$$
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Observation

The *infimum* of $\sigma_k(\Omega)$ among domains with $|\partial \Omega| = 1$ is zero.
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The *infimum* of $\sigma_k(\Omega)$ among domains with $|\partial \Omega| = 1$ is *zero*.

This is related to loss of compactness for the trace map

$$H^1(\Omega) \to L^2(\partial \Omega)$$

Channels, cusps, . . .
Physical interpretation in two dimension

The *non homogeneous Neumann* spectral problem with density $0 < \rho \in C^\infty(\Omega)$ is

$$-\Delta u = \mu \rho u \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial \Omega.$$ 

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One can think of the Steklov problem as a free membrane with its mass uniformly distributed along its boundary.
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$$\int_\Omega |\nabla f|^2 \, dx \quad \text{subject to} \quad \int_\Omega f^2 \rho \, dx$$

If $\rho_n dx \overset{n \to \infty}{\longrightarrow} dS$, then for $f \in H^1(\Omega)$

$$\lim_{n \to \infty} \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_\Omega f^2 \rho_n \, dx} = \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_{\partial \Omega} f^2 \, dS}$$
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Isoperimetric inequalities for planar domains.

Weinstock, 1954
If $\Omega \subset \mathbb{R}^2$ is simply connected,

$$\sigma_1(\Omega)|\partial \Omega| \leq \sigma_1(\mathbb{D})|\partial \mathbb{D}| = 2\pi.$$ 

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Observation
Let $A \in \mathbb{D} \setminus B(0,\epsilon)$. Then for small $\epsilon > 0$ one has

$$\sigma_1(A|\partial A| > 2\pi.$$ 

Simple-connectedness is not merely a technical assumption!

What can we say for multiply connected domains?
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Normalized eigenvalues of $A_\epsilon$
Higher eigenvalues for simply connected domains


If $\Omega \subset \mathbb{R}^2$ is simply connected, then for each $k \in \mathbb{N}$,

$$\sigma_k(\Omega)|\partial \Omega| \leq k\sigma_1(\mathbb{D})|\partial \mathbb{D}| = 2k\pi.$$
Higher eigenvalues for simply connected domains

If \( \Omega \subset \mathbb{R}^2 \) is simply connected, then for each \( k \in \mathbb{N} \),

\[
\sigma_k(\Omega) |\partial \Omega| \leq k \sigma_1(\mathbb{D}) |\partial \mathbb{D}| = 2k\pi.
\]

This inequality is sharp, and attained in the limit by a family of domains \( \Omega_\varepsilon \) degenerating to \( k \) disjoint identical disks.

\( k = 4 \)

This contrasts with Neumann eigenvalues...
Upper bounds for surfaces

Fraser–Schoen, 2011.
If $\Omega$ is a smooth compact surface of genus $\gamma$ with $l$ boundary components, then

$$\sigma_1(\Omega)|\partial \Omega| \leq 2(\gamma + l)\pi.$$
Upper bounds for surfaces

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- **Weinstock**: $\gamma = 0, l = 1, k = 1$.
- **Hersch–Payne–Schiffer**: $\gamma = 0, l = 1$, arbitrary $k \in \mathbb{N}$.
- **Fraser–Schoen, 2011**: $k = 1$, arbitrary $\gamma$ and $l$.  

These inequality are in general not sharp. For instance, Fraser–Schoen, 2011 For $l = 2$ and $\gamma = 0$, the maximum of $\sigma_1(\Omega)|\partial\Omega|$ is attained at the critical catenoid. (max $\approx 4\pi/\sqrt{2}$). Also, not sharp for large $l$. . .
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Open problems/projects

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Problem
Find sharp upper bounds in the general case

Ongoing project with Bruno Colbois
There exists a sequence $\Omega_n$ of surfaces such that $\sigma_1(\Omega_n)|\partial \Omega_n| \to \infty$.
(In this situation, the genus will have to diverge.)
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Thank you for your attention!