Complex spectra of self-adjoint operator pencils

Michael Levitin

University of Reading

based on joint works with

Daniel Elton (Lancaster) and Iosif Polterovich (Montreal)
(http://arxiv.org/abs/1303.2185, now in revision)

and

with E Brian Davies (King’s College London)
(in preparation)
Let
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be a family of operators in a Hilbert space \( \mathcal{H} \), depending on a parameter \( \lambda \in \mathbb{C} \), with self-adjoint operator coefficients
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Self-adjoint pencils

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We say that $\lambda_0 \in \text{spec}(\mathcal{P})$ if $\mathcal{P}(\lambda_0)$ is not invertible, or, equivalently, if $0 \in \text{spec}(\mathcal{P}(\lambda_0))$. 

Let us look in more detail at a linear pencil $\mathcal{P} = A - \lambda B$. Suppose that $B$ is positive. Then for an eigenvalue $\lambda$ of $\mathcal{P}$ we have $Au = \lambda Bu \iff B^{-1/2}AB^{-1/2}v = \lambda v$, with $v = B^{1/2}u$, and the problem is equivalent to a standard one for a self-adjoint operator; the spectrum is real! Thus, the interesting case is when both $A$ and $B$ are not sign-definite — the pencil spectrum can be non-real.
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We say that $\lambda_0$ is an eigenvalue of $\mathcal{P}$ if there exists $u \in \mathcal{H} \setminus \{0\}$ such that $\mathcal{P}(\lambda_0)u = 0$, or, equivalently, if 0 is an eigenvalue of $\mathcal{P}(\lambda_0)$. 

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We consider the following class of problems. Fix an integer $N \in \mathbb{N}$, and define the classes of $N \times N$ matrices $H_{N;c}$ and $D_{m,n;\sigma,\tau}$, where

$$H_{N;c} = \begin{pmatrix} c & 1 & 0 & \ldots & 0 \\ 1 & c & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & c & 1 \\ 0 & \ldots & 0 & 1 & c \end{pmatrix}$$

is tri-diagonal, $c \in \mathbb{R}$ is a parameter, and
Simple matrix pencil (contd.)

\[ D_{m,n;\sigma,\tau} = \begin{pmatrix} \sigma & \cdots & \cdot \cdot \cdot & \cdot \cdot \cdot \\ & & \sigma & \cdot \cdot \cdot \\ & & & & \tau \\ & & & & & \cdot \cdot \cdot \\ & & & & & & \tau \end{pmatrix} \]

is diagonal, where \( m, n \in \mathbb{N} \) and \( \sigma, \tau \in \mathbb{C} \) are parameters, and we assume \( m + n = N \).

We are only going to consider the case \( \sigma = -\tau = 1 \), and denote for brevity

\[ D_{m,n} := D_{m,n;1,-1} \]

We study the eigenvalues of the linear operator pencil

\[ \mathcal{P}_{m,n;c} = \mathcal{P}_{m,n;c}(\lambda) = H_{m+n;c} - \lambda D_{m,n} \]

as \( N = m + n \to \infty \).
We start with the following easy result on the localisation of eigenvalues of the pencil $\mathcal{P}_{m,n;c}$.

**Theorem**

(a) *The spectrum* $\text{spec} \mathcal{P}_{m,n;c}$ *is invariant under the symmetry* $\lambda \rightarrow \bar{\lambda}$. 

(b) All the eigenvalues $\lambda \in \text{spec} \mathcal{P}_{m,n;c}$ satisfy $|\lambda| < 2 + |c|$.

(c) If $|c| \geq 2$, then $\text{spec} \mathcal{P}_{m,n;c} \subset \mathbb{R}$. 

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Rough localisation

Rough asymptotics of eigenvalues as $N \to \infty$ is given by

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\max \{ |\text{Im}(\lambda)| : \lambda \in \text{spec } P_{m,n;c} \}
\leq \max \left\{ \frac{\log(m)}{m}(1 + o(1)), \frac{\log(n)}{n}(1 + o(1)) \right\}
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as \( m, n \to \infty \).

Note that the estimate is sharp in the following sense: it’s attained, and it needs both \( n, m \to \infty \).
Example, $c = 0, n = m = N/2$
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Asymptotics, \( c = 0, \ n = m = N/2 \)

**Theorem**

Let \( c = 0, \ n = m = N/2 \to \infty \). The eigenvalues of \( P_{n,n;0} \) are all non-real, and satisfy

\[
\text{Im} \lambda = \pm 1/N \ast Y(|\text{Re} \lambda|) + o(N^{-1}),
\]

where

\[
Y(u) := \sqrt{4 - u^2} \log \cot(\pi/4 - \arccos(u/2)/2)
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Theorem

Let $c \neq 0$, $n = m = N/2 \to \infty$. The eigenvalues of $\mathcal{P}_{n,n;c}$ satisfy

$$|\text{Im} \, \lambda| \leq 1/N \ast Y_c(|\text{Re} \, \lambda|) + o(N^{-1}),$$

where $Y_c$ is some explicitly described but complicated function.
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Idea of proof

Do not try to analyse directly a characteristic polynomial in \( \lambda \).

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\begin{align*}
\lambda - c &= \frac{z + 1}{z}, \\
\lambda + c &= \frac{w + 1}{w}
\end{align*}
\]

Then for non-real eigenvalues

\[
F_m(z) F_n(w) = -1
\]

where

\[
F_m(z) = z^n + 1 - z^{-n} - 1
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\[
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$$F_m(z) = \frac{z^{n+1} - z^{-n-1}}{z^n - z^{-n}} = \frac{\sinh((n + 1) \log z)}{\sinh(n \log z)}.$$
Define a self-adjoint operator

\[ T_V = \begin{pmatrix} V + k & -\nabla \\ \nabla & V - k \end{pmatrix} = -i\sigma_2 \nabla + k\sigma_3 + V, \]

where \( \nabla = \frac{d}{dx}, \) \( \sigma_2, \sigma_3 \) are Pauli matrices, \( k \) is the mass, and \( V(x) \) is a potential.
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For a given potential \( V \), we denote by \( \Sigma_V \) the spectrum of the linear operator pencil

\[ \gamma \mapsto T_0 + \gamma V = \begin{pmatrix} k & -\nabla \\ \nabla & -k \end{pmatrix} + \gamma \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}. \]

(The spectral parameter is denoted \( \gamma \) in this problem for historical reasons.)
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Equivalently,

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Similar problems, as well as some other related questions, have been studied in a variety of situations in mathematical literature, e.g. [Birman Solomyak 1977], [Klaus 1980], [Gesztesy et al. 1988], [Birman Laptev 1994], [Safronov 2001], [Schmidt 2010].

In physical literature, our problem appears in the study of electron waveguides in graphene (see [Hartmann Robinson Portnoi 2010], [Stone Downing Portnoi 2012] and many references there).

It was shown in [Hartmann Robinson Portnoi 2010] that for the potential \( V_{\text{HRP}}(x) = -1/\cosh(x) \) the solutions can be found explicitly in terms of special functions. Moreover, there exists an infinite sequence of coupling constants \( \gamma \) such that 0 is an eigenvalue of the operator \( T_\gamma V_{\text{HRP}} \).
Function classes

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Let $V_1$ denote the class of real valued locally $L^2$ potentials which satisfy

$$\int_{\mathbb{R}} |V(x)| \, dx < +\infty;$$

that is, we require $V$ to be integrable. Equivalently, we can define $V_1 = V_0 \cap L^1$. The class $V_1$ is sometimes denoted as $\ell^1(L^2)$. 

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Firstly we consider the number of points of $\Sigma_V$ lying inside the disc $\{z \in \mathbb{C} : |z| \leq R\}$ of radius $R \geq 0$. 

Theorem: Suppose $V \in V_1$. Then $\#(\Sigma_V \cap \{z \in \mathbb{C} : |z| \leq R\}) \leq C \|V\|_{L^1} R$ for any $R \geq 0$, where $C$ is a universal constant (we can take $C = 4e/\pi$).
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Restricting our attention to real points we have the following complementary lower bound

**Theorem**

Suppose $V \in V_1$. Then

$$\#(\Sigma_V \cap [0, R]) \geq \frac{R}{\pi} \left| \int_{\mathbb{R}} V(x) \, dx \right| + o(R)$$

as $R \to \infty$, while the same estimate holds for $\#(\Sigma_V \cap [-R, 0])$ (by symmetry).
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as $R \to \infty$, while the same estimate holds for $\#(\Sigma_V \cap [-R, 0])$ (by symmetry). In particular, $\Sigma_V \cap \mathbb{R}$ contains infinitely many points if $\int_{\mathbb{R}} V(x) \, dx \neq 0$. 

Single-signed potentials

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**Theorem**

Suppose $V \in \mathbb{V}_1$ is single-signed. Then

$$
\#(\Sigma_V \cap [0, R]) = \frac{R}{\pi} \left| \int_{\mathbb{R}} V(x) \, dx \right| + o(R) = \frac{\|V\|_{L^1}}{\pi} R + o(R)
$$

as $R \to \infty$. 
Anti-symmetric potentials

For potentials of variable sign the behaviour of the $\gamma$-spectrum may be different, in some cases quite drastically so. For anti-symmetric potentials we have the following

Theorem

If $V \in V_0$ is anti-symmetric then $\Sigma V \cap \mathbb{R} = \emptyset$.

Note that, the $\gamma$-spectrum may still contain an infinite number of complex eigenvalues. The absence of real points in the $\gamma$-spectrum shows that the general lower bound obtained is quite sharp. Theorem also applies to potentials $V$ satisfying the condition $V(a+x) = -V(a-x)$ for some $a \in \mathbb{R}$ and all $x \in \mathbb{R}$.
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Theorem also applies to potentials $V$ satisfying the condition $V(a + x) = -V(a - x)$ for some $a \in \mathbb{R}$ and all $x \in \mathbb{R}$.
Discussion of the results

Our results give information about the asymptotics of the counting function $\#(\Sigma \cap [0, R])$ as $R \to \infty$. We’ve already seen two cases when the results give leading term asymptotic behaviour of

$$\frac{R}{\pi} \int_{\mathbb{R}} |V(x)| \, dx \quad \text{and} \quad \frac{R}{\pi} \left| \int_{\mathbb{R}} V(x) \, dx \right|$$

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\]

respectively. (Though they coincide if \( V \) is sign-definite).
The above results may lead to a hypothesis that, in fact, the lower bound always gives the leading order term in the asymptotics of the counting function of the spectrum. However, this is not the case; for general (variable-signed) potentials the precise asymptotic behaviour of \( \#(\Sigma_V \cap [0, R]) \) as \( R \to \infty \) appears to depend on \( V \) in a rather subtle way. In particular, this behaviour appears to be sensitive to ‘gaps’ in the potential, namely intervals where \( V \equiv 0 \) appearing between components of \( \text{supp}(V) \).
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**Surprise**

We can construct potentials for which the actual asymptotic coefficient lies anywhere between the modulus of the integral of the potential and the \( L^1 \) norm, modulo multiplication by \( R/\pi \).
Examples — general setup

We restrict our attention mostly to piecewise constant potentials with compact support; these allow the easiest analysis and already demonstrate the full range of effects. Consider points $a_0 < a_1 < \cdots < a_m$ which partition the real line into $m$ finite intervals $l_j = (a_{j-1}, a_j)$, $j = 1, \ldots, m$, and two semi-infinite intervals $l_- = (-\infty, a_0)$ and $l_+ = (a_m, +\infty)$. Consider a potential
We restrict our attention mostly to piecewise constant potentials with compact support; these allow the easiest analysis and already demonstrate the full range of effects. Consider points $a_0 < a_1 < \cdots < a_m$ which partition the real line into $m$ finite intervals $I_j = (a_{j-1}, a_j), j = 1, \ldots, m,$ and two semi-infinite intervals $I_- = (-\infty, a_0)$ and $I_+ = (a_m, +\infty)$.

Consider a potential
\[
V(x) = W(x; [a_0, \ldots, a_m]; \{v_1, \ldots, v_m\}) := \begin{cases} 
  v_j, & x \in I_j, \ j = 1, \ldots, m, \\
  0, & x \in I_- \cup I_+, 
\end{cases}
\]
with some given real constants $v_j$. 

(3)
On each interval, we can solve the equations explicitly in trigonometric functions; matching conditions lead to an explicit characteristic equation for eigenvalues: \( \gamma \in \Sigma_V \) if and only if \( D_V(\gamma) = 0 \).
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Thus, in each particular case our problem is reduced to constructing $D_V(\gamma)$ and finding its real or complex roots. We visualise the real roots of $D_V(\gamma)$ by simply plotting its graph for real arguments.
Consider the one-gap potentials $V_{3,g}(x) := W(x; [-g - 1, -g, 0, 2]; \{-1, 0, 1\})$ parametrised by the gap length $g$. For each of these potentials, $\int_{\mathbb{R}} V_{3,g} = 1$ and $\|V_{3,g}\|_{L^1} = 3$. The graphs of $D_{V_{3,g}}(\gamma)$ for real $\gamma$ and $g = 0$ or $g = 1$: 
We can expect asymptotics of the form

$$\#(\Sigma V_{3,g} \cap [0, R]) = C_g \frac{R}{\pi} + O(1),$$

as $R \to \infty$. For the no-gap potential $V_{3,0}$ one of our Theorems gives such an asymptotics with $C_0 = 1 = \int_{\mathbb{R}} V_{3,1}$. On the hand, $D_{V_{3,1}}(\gamma)$ has three times as many real roots as $D_{V_{3,0}}(\gamma)$ (for sufficiently large $\gamma$). This leads to a constant $C_1 = 3 = \|V_{3,0}\|_{L^1}$ in the asymptotics for the gap potential $V_{3,1}$. 

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This example is just a partial case of a more complicated phenomenon. Consider a general (not necessarily piecewise constant) one gap compact potential $V(x)$ such that $\text{supp}(V) = I_1 \cup I_2$, where $I_1$ and $I_2$ are compact intervals separated by a gap of length $g > 0$, and assume additionally that $V(x)$ does not change sign on either $I_j$. If the signs of $V|_{I_1}$ and $V|_{I_2}$ coincide, then the asymptotic counting function involves

$$C = \| V \|_{L^1} = \left| \int_{\mathbb{R}} V \right|.$$

If, however, the signs of $V|_{I_1}$ and $V|_{I_2}$ are different, then the asymptotic behaviour is given by a complicated formula which depends not only upon the gap length $g$ and the values of $\left| \int_{I_j} V \right|$ but also upon the rationality or irrationality of the ratio of these two integrals! The rigorous approach to this involves an intricate analysis based on the following version of a classical problem.
Counting zeros

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \cos(x) + a \cos(bx),$$

where $a$ and $b$ are real parameters satisfying $0 \leq a < 1$ and $b > 0$. For any function $\phi : \mathbb{R} \to \mathbb{R}$ we also set $f_\phi = f + \phi$. We want to consider $f_\phi$ as a perturbation of $f = f_0$ for large $x$. To this end introduce the family of conditions

$$\phi \in C^k(\mathbb{R}), \quad \phi^{(k)}(x) = o(1) \text{ as } x \to \infty \quad \text{(Ak)}$$

where $k \in \mathbb{N}_0$ (we’ll only need to consider $k = 0, 1, 2$).

Fix a perturbation $\phi$. Introduce the counting function

$$N_\phi(R) = \# \{ x \in [0, R) : f_\phi(x) = 0 \} \in \mathbb{N} \cup \{0, \infty\}$$

We are interested in the asymptotic behaviour of $N_\phi(R)$ as $R \to \infty$, and how this behaviour depends on the parameters $a$ and $b$. 
Proposition

Suppose $ab < 1$ and $\phi$ satisfies (A0), (A1). Then

$$N_\phi(R) = \frac{1}{\pi} R + O(1) \text{ as } R \to \infty.$$ 

Remark

When $ab < 1$ we get the same asymptotic behaviour for $N_\phi(R)$ as in the case $a = 0$ (that is, when $f = \cos$).
Counting zeros — large $ab$, irrational case

When $ab > 1$ we can define $\alpha, \beta \in (0, \pi/2)$ by

\[
\alpha = \arcsin \frac{\sqrt{a^2 b^2 - 1}}{\sqrt{b^2 - 1}} \quad \text{and} \quad \beta = \arcsin \frac{\sqrt{1 - a^2}}{a\sqrt{b^2 - 1}}.
\]

Also set $u = \frac{2}{\pi} (b\alpha + \beta)$. If we fix $b > 1$ and vary $a$ from $1/b$ to $1$ it is easy to check that $\alpha$ increases from $0$ to $\pi/2$ and $\beta$ decreases from $\pi/2$ to $0$; it follows that $u$ varies from $1$ to $b$.

**Proposition**

Suppose $ab > 1$, $b$ is irrational and $\phi$ satisfies (A0), (A1), (A2). Then

\[
\lim_{R \to \infty} \frac{N_\phi(R)}{R} = \frac{1}{\pi} u.
\]
Counting zeros — large $ab$, rational case

**Proposition**

Suppose $ab > 1$, $b$ is rational and $\phi$ satisfies (A0), (A1). Write $b = p/q$ where $p, q \in \mathbb{N}$ are coprime. If $p$ and $q$ are odd set $P = p$ and $Q = q$; if $p$ and $q$ have opposite parity set $P = 2p$ and $Q = 2q$. If $P + Qu \notin 4\mathbb{Z}$ then

$$
\lim_{R \to \infty} \frac{N_\phi(R)}{R} = \frac{1}{\pi} \left( \frac{4}{Q} \left\lfloor \frac{1}{4}(P + Qu) \right\rfloor - \frac{P}{Q} + \frac{2}{Q} \right).
$$

(5)

We are using $\lfloor x \rfloor$ to denote the largest integer which does not exceed $x$. 

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Remark

From (5) and the bounds $x - 1 \leq \lfloor x \rfloor \leq x$ we get

$$\frac{1}{\pi} u - \frac{2}{Q \pi} \leq \lim_{R \to \infty} \frac{N_\phi(R)}{R} \leq \frac{1}{\pi} u + \frac{2}{Q \pi}.$$

Using the size of $Q$ as a measure of ‘how irrational’ $b$ is it follows that the result for irrational $b$ can be viewed as a limit of the rational case.