On the spectrum of the Hodge Laplacian and the John ellipsoid

Alessandro Savo, Sapienza Università di Roma

We give upper and lower bounds for the first eigenvalue of the Hodge Laplacian acting on $p$-forms of a compact, convex Euclidean domain $\Omega$ (smooth boundary, absolute boundary conditions). We denote this eigenvalue by the symbol

$$\lambda_1^{[p]}(\Omega).$$

Perhaps the main scope is to stress the geometric meaning of this eigenvalue, and to relate it with a classical object in convex geometry: the John ellipsoid of the domain.
Known estimates on functions: the Dirichlet problem.

\( \Omega = \text{compact, convex domain in } \mathbb{R}^n \) with smooth boundary.

- Classical Dirichlet eigenvalue problem:
  \[
  \begin{aligned}
  \Delta f &= \lambda f \quad \text{on } \Omega, \\
  f &= 0 \quad \text{on } \partial \Omega.
  \end{aligned}
  \]

Let \( \lambda_1^D(\Omega) \) be its first eigenvalue.

Classical bounds:

\[
\frac{\pi^2}{4R(\Omega)^2} \leq \lambda_1^D(\Omega) \leq \frac{c_n}{R(\Omega)^2}
\]

where

\( R(\Omega) = \text{inner radius of } \Omega. \)

Lower bound: Hersch, Li and Yau.
Upper bound: domain monotonicity.
Given geometric functionals $\Gamma_1(\Omega), \Gamma_2(\Omega)$ we say that $\Gamma_1(\Omega)$ is comparable to $\Gamma_2(\Omega)$ if there exist constants $a, b$ not depending on $\Omega$ such that

$$a\Gamma_1(\Omega) \leq \Gamma_2(\Omega) \leq b\Gamma_1(\Omega).$$

and we will write:

$$\Gamma_1(\Omega) \sim \Gamma_2(\Omega).$$

**Theorem.** For any convex domain $\Omega$ one has

$$\lambda_1^D(\Omega) \sim \frac{1}{R(\Omega)^2}.$$  In other words:

$$\frac{1}{\sqrt{\lambda_1^D(\Omega)}} \sim R(\Omega).$$

That is, the fundamental wavelength for the Dirichlet problem is comparable to the inner radius (large drums produce a low tone).

**Remark.** The above fact does not hold in other spaces: for example, in hyperbolic space $H^n$ one has the Mc Kean inequality:

$$\lambda_1^D(\Omega) \geq \frac{(n-1)^2}{4}$$

for any compact domain $\Omega$ (not necessarily convex).
Known estimates on functions: the Neumann problem.

- The Neumann eigenvalue problem:

\[
\begin{cases}
\Delta f = \lambda f & \text{on } \Omega, \\
\frac{\partial f}{\partial N} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( N \) is the unit normal vector. Let \( \lambda_1^N(\Omega) \) be its first positive eigenvalue. One knows (Polya):

\[\lambda_1^N(\Omega) < \lambda_1^D(\Omega)\]

and that:

\[
\frac{\pi^2}{\text{diam}(\Omega)^2} \leq \lambda_1^N(\Omega) \leq \frac{n\pi^2}{\text{diam}(\Omega)^2}.
\]

Lower bound: Payne and Weinberger.
Upper bound: particular case of an estimate of Cheng.
**Theorem.** For any convex domain $\Omega$ one has
\[
\lambda_1^N(\Omega) \sim \frac{1}{\text{diam}(\Omega)^2}.
\]
In other words:
\[
\frac{1}{\sqrt{\lambda_1^N(\Omega)}} \sim \text{diam}(\Omega).
\]

*Hence: the fundamental wavelength for the Neumann problem is comparable to the diameter.*

- Conclusion: given the fundamental tones for the Dirichlet and Neumann problems:
  \[
  \lambda_1^D(\Omega), \lambda_1^N(\Omega)
  \]
one can *roughly hear* both the inner radius and the diameter of the domain.
The John ellipsoid. The shape of a convex domain $\Omega$ can be roughly described by a suitable ellipsoid.

**Theorem** (F. John, 1948) Given any convex domain $\Omega$ in $\mathbb{R}^n$ there exists a unique ellipsoid of maximal volume included in $\Omega$, denoted by $\mathcal{E}_\Omega$. Moreover one has:

$$\mathcal{E}_\Omega \subseteq \Omega \subseteq n \cdot \mathcal{E}_\Omega,$$

(origin in the center of $\mathcal{E}_\Omega$).

Uniqueness apparently due to Löwner.

- $\mathcal{E}_\Omega$ is called the **John ellipsoid** of $\Omega$. Set:

$$D_p(\mathcal{E}_\Omega) = p\text{—th longest principal axis of }\mathcal{E}_\Omega.$$

Ordering:

$$D_1(\mathcal{E}_\Omega) \geq D_2(\mathcal{E}_\Omega) \geq \cdots \geq D_n(\mathcal{E}_\Omega).$$
Observe that
\[ D_1(\mathcal{E}_\Omega) \sim \text{diam}(\Omega), \quad D_n(\mathcal{E}_\Omega) \sim 2R(\Omega) \sim R(\Omega). \]
(clear for true ellipsoids; in general apply the inclusions in John theorem). The classical estimates give:

**Theorem.** Let \( \Omega \) be a convex domain in \( \mathbb{R}^n \) and \( \mathcal{E}_\Omega \) its John ellipsoid. Then:

\[ \frac{1}{\sqrt{\lambda_1^N(\Omega)}} \sim D_1(\mathcal{E}_\Omega) \quad \text{and} \quad \frac{1}{\sqrt{\lambda_1^D(\Omega)}} \sim D_n(\mathcal{E}_\Omega). \]

In particular, if \( \Omega \) is a convex plane domain then the two fundamental tones determine (up to constants) the two principal axis of the John ellipse of \( \Omega \).

- Satisfactory in dimension 2, but incomplete in dimensions \( n \geq 3 \).

- Do the other principal axes of the John ellipsoid have a similar spectral interpretation?
**The Hodge Laplacian.** Laplacian acting on differential $p$-forms:

$$
\Delta = d\delta + \delta d
$$

where $\delta = d^*$.  

- Eigenvalue problem for the *absolute boundary conditions*:

$$
\begin{aligned}
\Delta \omega &= \lambda \omega \quad \text{on } \Omega, \\
i_N \omega &= 0 \quad \text{on } \partial \Omega, \\
i_N d\omega &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
$$

$N$ = inner unit normal vector  
$i_N$ = interior multiplication by $N$.  

The spectrum is discrete:

$$
\lambda_1^{[p]} \leq \lambda_2^{[p]} \leq \cdots \leq \lambda_k^{[p]} \leq \cdots
$$

(the degree is in the superscript).

- Variational characterization.

$$
\lambda_1^{[p]}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |d\omega|^2 + |\delta \omega|^2}{\int_{\Omega} |\omega|^2} : \omega \in \Lambda^p(\Omega), i_N \omega = 0 \text{ on } \partial \Omega \right\}
$$
• Identify 1-forms and vector fields via the metric. In 3-space:

\[
\lambda_1^1(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\text{div} X|^2 + |\text{curl} X|^2}{\int_{\Omega} |X|^2} : \langle X, N \rangle = 0 \right\},
\]

that is, the infimum is taken over all vector fields which are tangent to the boundary.

Motivation for the boundary conditions:

The space of harmonic \( p \)-forms satisfying the absolute conditions is isomorphic with the absolute de Rham cohomology of \( \Omega \) in degree \( p \).

If \( \Omega \) is convex one has \( \lambda_1^p > 0 \) for all \( p \geq 1 \).

• 0—forms are functions: absolute boundary conditions .... Neumann conditions. Then:

\[
\lambda_1^0 = \lambda_1^N.
\]

• \( n \)—forms are identified with functions (through the \(*\) operator). Dual conditions ... Dirichlet. Then

\[
\lambda_1^n = \lambda_1^D.
\]
The Hodge $\star$ operator transforms absolute boundary conditions into relative boundary conditions ... corresponding dual eigenvalue problem:

$$
\begin{cases}
\Delta \omega = \mu \omega \quad \text{on } \Omega, \\
J^\star \omega = 0 \quad \text{on } \partial \Omega, \\
J^\star \delta \omega = 0 \quad \text{on } \partial \Omega
\end{cases}
$$

where $J^\star$ denotes restriction of forms to the boundary.

We will call, for $p = 0, \ldots, n$:

- $\lambda_1^{[p]}$: fundamental tone in degree $p$
- $\frac{1}{\sqrt[\lambda_1^{[p]}}$: fundamental wavelength in degree $p$

**Problem**: estimate $\lambda_1^{[p]}$ for all degrees $p$. 

Eigenvalue estimates for the Hodge Laplacian.

- Estimating the first eigenvalue for \( p \)-forms is, generally speaking, more difficult than for functions.

- Main tool: Bochner formula, giving estimates involving pointwise lower bounds of the principal curvatures of the boundary (joint works with P. Guerini and S. Raulot).

- For a convex domain it is desirable to have lower bounds depending on global invariants, rather than local ones.

**Theorem** (Guerini-S. 2004) *For any convex domain in \( \mathbb{R}^n \) one has:

\[
\lambda_1^{[0]} = \lambda_1^{[1]} \leq \lambda_1^{[2]} \leq \cdots \leq \lambda_1^{[n]}. 
\]

That is, the fundamental tones of the Hodge Laplacian form an increasing sequence (with respect to the degree).

- All fundamental tones of the Hodge Laplacian belong to the interval \([\lambda_1^N, \lambda_1^D]\).
Sketch of proof.

1. Let \( \omega \) be an eigenform associated to \( \lambda_1^{[p]} \) and \( V \) a parallel vector field in \( \mathbb{R}^n \) (of unit length). Then

\[
i_V \omega
\]

is a test-form for the eigenvalue \( \lambda_1^{[p-1]} \) (because \( i_N i_V = -i_N i_V \)).

2. min-max principle:

\[
\lambda_1^{[p-1]} \int_\Omega |i_V \omega|^2 \leq \int_\Omega |d i_V \omega|^2 + |\delta i_V \omega|^2.
\]

3. Identify the set of parallel vector fields of unit length with \( S^{n-1} \) and integrate both sides with respect to \( V \in S^{n-1} \). After some work and the Bochner formula, get:

\[
\lambda_1^{[p-1]} \leq \lambda_1^{[p]}.
\]

In particular, \( \lambda_1^{[0]} \leq \lambda_1^{[1]} \).

Note: convexity is needed!

4. \( \lambda_1^{[0]} \geq \lambda_1^{[1]} \) is always true (by differentiating Neumann eigenfunctions).
Hence:

$$\lambda_1^{[0]} = \lambda_1^{[1]}$$

and equality holds at the first step. ●

Monotonicity property: $\lambda_1^N \leq \lambda_1^{[p]} \leq \lambda_1^D$. Hence, for all $p$:

$$\frac{\pi^2}{\text{diam}(\Omega)^2} \leq \lambda_1^{[p]} \leq \frac{c_n}{R(\Omega)^2}.$$  

But we can do much better than that.
The main estimate. From the previous results we have only $n$ significant fundamental tones: these can be estimated in terms of the John ellipsoid of the domain.

**Theorem** (S. 2011) Let $\Omega$ be a convex body and $E_\Omega$ its John ellipsoid. Order the principal axes of $E_\Omega$ from longest to shortest:

$$D_1(E_\Omega) \geq D_2(E_\Omega) \geq \cdots \geq D_n(E_\Omega).$$

Then, for all $p = 1, \ldots, n$ one has:

$$\frac{a_{n,p}}{D_p(E_\Omega)^2} \leq \lambda_1^{[p]}(\Omega) \leq \frac{a'_{n,p}}{D_p(E_\Omega)^2},$$

where $a_{n,p}$ and $a'_{n,p}$ are explicit constants. Precisely:

$$a_{n,p} = \frac{4}{n^2 \cdot \binom{n}{p-1}}, \quad a'_{n,p} = 4p(n + 2)n^n.$$ 

**Remark.** The constants are not sharp.
Main result is that the fundamental wavelength in degree $p$ is comparable with the $p$-th longest principal axis of its John ellipsoid:

$$\frac{1}{\sqrt[1]{\lambda^{[p]}_1(\Omega)}} \sim D_p(\mathcal{E}_\Omega)$$

for all $p = 1, \ldots, n$.

Philosophy: knowing all fundamental tones:

$$\lambda^{[1]}_1, \lambda^{[2]}_1, \ldots, \lambda^{[n]}_1$$

one can roughly hear the John ellipsoid (hence the shape) of the domain.

What is the physical interpretation of $\lambda^{[p]}_1(\Omega)$?
Spectrum and volume of cross-sections. Set, for \( p = 1, \ldots, n \):

\[
\text{vol}^{[p]}(\Omega) = \sup\{\text{vol}(\Sigma) : \Sigma = \Omega \cap \pi_p, \\
\pi_p \text{ is a } p\text{-dimensional plane}\}.
\]

\( \text{vol}^{[p]}(\Omega) \) is the maximal volume of a \( p \)-dimensional cross-section of \( \Omega \).

- **Note:**

\[
\begin{aligned}
\text{vol}^{[1]}(\Omega) &= \text{diam}(\Omega) \\
\text{vol}^{[n]}(\Omega) &= \text{vol}(\Omega)
\end{aligned}
\]

- The functional \( \text{vol}^{[p]} \) is monotone increasing with respect to inclusion. Recall John’s theorem:

\[
\mathcal{E}_\Omega \subseteq \Omega \subseteq n \cdot \mathcal{E}_\Omega.
\]

Hence:

\[
\text{vol}^{[p]}(\mathcal{E}_\Omega) \leq \text{vol}^{[p]}(\Omega) \leq n^p \text{vol}^{[p]}(\mathcal{E}_\Omega),
\]
and

\[
\text{vol}^{[p]}(\Omega) \sim \text{vol}^{[p]}(\mathcal{E}_\Omega) \\
\sim D_1(\mathcal{E}_\Omega) \cdots D_p(\mathcal{E}_\Omega) \\
\sim \frac{1}{\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[p]}}}
\]

Therefore we get a spectral estimate involving cross-sections:

**Corollary.** For every \( p = 1, \ldots, n \) one has:

\[
\text{vol}^{[p]}(\Omega) \sim \frac{1}{\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[p]}}}
\]

This means of course:

\[
\frac{c_{n,p}}{\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[p]}}} \leq \text{vol}^{[p]}(\Omega) \leq \frac{c'_{n,p}}{\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[p]}}}.
\]

for explicit (but not sharp) constants.
An inequality for the volume. Taking $p = n$:

$$\text{vol}(\Omega) \sim \frac{1}{\sqrt[1]{\lambda_1^{[1]} \cdots \lambda_1^{[n]}}}.$$ 

That is, the volume is comparable to the product of all fundamental wavelengths.

**Remark.** Is something like this true in a more general situation? (for example, closed manifolds with some curvature assumptions?)

- Consequence: (weak) Faber-Krahn inequality.

In fact, from monotonicity: $\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[n]}} \leq (\lambda_1^{[n]})^{n/2}$

hence:

$$\lambda_1^{[n]} \geq \frac{c_n}{\text{vol}(\Omega)^{2/n}},$$

and we know that $\lambda_1^{[n]} = \lambda_1^D$. That is:

$$\lambda_1^D \geq \frac{c_n}{\text{vol}(\Omega)^{2/n}}.$$ 

(of course, $c_n$ can’t be sharp).
A Faber-Krahn type inequality for $\lambda_1^{[p]}$.

Again from monotonicity: $\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[p]}} \leq \left(\lambda_1^{[p]}\right)^{p/2}$.

**Corollary.** For all $p = 1 \ldots, n$:

$$\lambda_1^{[p]}(\Omega) \geq \frac{c_{n,p}}{\left(\text{vol}^{[p]}(\Omega)\right)^{2/p}}.$$  

- **Case $p = 1$.** We have
  $$\text{vol}^{[1]}(\Omega) = \text{diam}(\Omega) \quad \text{and} \quad \lambda_1^{[1]} = \lambda_1^N$$
  hence
  $$\lambda_1^N(\Omega) \geq \frac{c_n}{\text{diam}(\Omega)^2},$$
  ... Payne-Weinberger inequality for the first Neumann eigenvalue.

- **Case $p = n$:** Faber-Krahn inequality for the first Dirichlet eigenvalue. Then:

  - the bound in the Corollary is an isoperimetric inequality for forms connecting these two classical inequalities on functions.

- **Problem:** find the optimal constant for all $p$.  

19
**Conjecture.** Let $\Omega$ be convex in $\mathbb{R}^n$ and $p = 1, \ldots, n$. Let $\bar{\Omega}_p$ be the $p$-th dimensional ball such that
\[ \text{vol}(\bar{\Omega}_p) = \text{vol}^{[p]}(\Omega). \]
Then
\[ \lambda_1^{[p]}(\Omega) \geq \lambda_1^{[p]}(\bar{\Omega}_p) = \lambda_1^D(\bar{\Omega}_p). \]

For $p = 1$ this is the Payne-Weinberger inequality (with the optimal constant). In fact:
\[ \text{vol}^{[1]}(\Omega) = \text{diam}(\Omega), \quad \bar{\Omega}_1 = [0, \text{diam}(\Omega)] \]
hence
\[ \lambda_1^{[1]}(\bar{\Omega}_1) = \lambda_1^D(\bar{\Omega}_1) = \frac{\pi^2}{\text{diam}(\Omega)^2}. \]
As $\lambda_1^{[1]} = \lambda_1^N$ the above reads: $\lambda_1^N \geq \frac{\pi^2}{\text{diam}(\Omega)^2}$.

- Equivalent form of the conjecture:
\[ \lambda_1^{[p]}(\Omega) \geq \frac{c_p}{\left(\text{vol}^{[p]}(\Omega)\right)^{2/p}} \]
where
\[ c_p = \lambda_1^D(B_p) \cdot \text{vol}(B_p)^{2/p} \]
Scheme of the proof.

- Recall the statement: $\lambda_1^{[p]}(\Omega) \sim 1/D_p(\mathcal{E}_\Omega)^2$.
- The upper bound is given in terms of any ellipsoid $\mathcal{E}_-$ contained in $\Omega$ (no convexity needed).

**Theorem 1.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$ and let $\mathcal{E}_-$ be an ellipsoid contained in $\Omega$, with principal axes $D_1(\mathcal{E}_-) \geq D_2(\mathcal{E}_-) \geq \cdots \geq D_n(\mathcal{E}_-)$. Then:

$$\lambda_1^{[p]}(\Omega) \leq 4p(n + 2) \cdot \frac{\text{vol}(\Omega)}{\text{vol}(\mathcal{E}_-)} \cdot \frac{1}{D_p(\mathcal{E}_-)^2}$$

If $\Omega$ is convex ... take $\mathcal{E}_- = \mathcal{E}_\Omega$, then

$$\frac{\text{vol}(\Omega)}{\text{vol}(\mathcal{E}_\Omega)} \leq n^n$$

because $\Omega \subseteq n\mathcal{E}_\Omega$. Get

$$\lambda_1^{[p]}(\Omega) \leq \frac{c_{n,p}}{D_p(\mathcal{E}_\Omega)^2}$$

where $c_{n,p} = 4p(n + 2)n^n$. 

• Main tool: Hodge decomposition for manifolds with boundary.

• Test-form. Fix coordinates so that $\mathcal{E}_-$ has equation:
  \[ \frac{x_1^2}{D_1^2} + \cdots + \frac{x_n^2}{D_n^2} \leq 4, \]
  where $D_k = D_k(\mathcal{E}_-)$. Let $\omega = dx_1 \wedge \cdots \wedge dx_{p+1}$.
  The test form will be the canonical primitive of $\omega$ restricted to $\mathcal{E}_-$ (this is explicitly computable).

• The **canonical primitive** of $\omega$ is the unique co-exact $(p - 1)$-form $\theta$ such that $d\theta = \omega$ and $i_N \theta = 0$
  on $\partial \Omega$.
  It minimizes the $L^2$-norm among all primitives of $\omega$. 
Lower bound.

Lower bound is given in terms of any ellipsoid $E_+$ containing $\Omega$ (convexity is needed!).

**Theorem 2.** Let $\Omega$ be a convex body in $\mathbb{R}^n$ and $E_+$ an ellipsoid containing $\Omega$, with principal axes $D_1(E_+) \geq D_2(E_+) \geq \cdots \geq D_n(E_+)$. Then, for all $p \geq 2$:

$$\lambda_1^{[p]}(\Omega) \geq 4 \left( \frac{n}{p-1} \right)^{-1} \cdot \frac{1}{D_p(E_+)^2}.$$ 

Now take $E_+ = nE_\Omega$. Get

$$\lambda_1^{[p]}(\Omega) \geq 4 \frac{1}{n^2 \binom{n}{p-1}} \cdot \frac{1}{D_p(E_\Omega)^2}.$$

Thus, John’s theorem is used to relate the upper and lower bounds.
Main steps.

- First step: reduce the problem to a lower bound of the energy.

Let $\omega$ be a $p$-eigenform. Bochner formula:

$$\langle \Delta \omega, \omega \rangle = |\nabla \omega|^2 + \frac{1}{2} \Delta |\omega|^2.$$

Integrating on $\Omega$:

$$\lambda_1^{[p]} \int_{\Omega} |\omega|^2 = \int_{\Omega} |\nabla \omega|^2 + \frac{1}{2} \int_{\Omega} \Delta |\omega|^2.$$

Now:

$$\frac{1}{2} \int_{\Omega} \Delta |\omega|^2 = \frac{1}{2} \int_{\partial \Omega} \frac{\partial}{\partial N} |\omega|^2$$

$$= \int_{\partial \Omega} \langle \nabla_N \omega, \omega \rangle$$

$$= \int_{\partial \Omega} \langle S^{[p]} \omega, \omega \rangle$$

$$\geq 0$$

where $S^{[p]} = \text{self-adjoint extension of the shape operator } S$ to $\Lambda^{[p]}(\partial \Omega)$ (by convexity, one has $S \geq 0$ hence also $S^{[p]} \geq 0$).
• Hence for any $p$-eigenform:

$$\lambda_1^{[p]} \geq \frac{\int_{\Omega} |\nabla \omega|^2}{\int_{\Omega} |\omega|^2}.$$ 

• Second step: estimate from below the energy of co-closed, tangential forms.

**Theorem 3.** Let $\omega$ be a co-closed $(p-1)$-form on $\Omega$ such that $i_N \omega = 0$ on $\partial \Omega$. Let $\mathcal{E}_+$ be an ellipsoid containing $\Omega$, with principal axes $D_1(\mathcal{E}_+) \geq D_2(\mathcal{E}_+) \geq \cdots \geq D_n(\mathcal{E}_+)$. Then:

$$\frac{\int_{\Omega} |\nabla \omega|^2}{\int_{\Omega} |\omega|^2} \geq 4 \left( \frac{n}{p - 1} \right)^{-1} \cdot \frac{1}{D_p(\mathcal{E}_+)^2}.$$ 

• Use the Payne-Weinberger lower bound on suitable cross-sections of $\Omega$ to obtain a lower bound for the energy of the components of $\omega$. 
Proof of the upper bound. As usual, to produce upper bounds we need suitable test-forms. Recall the variational property of the first Hodge-eigenvalue:

$$\lambda_1^{[p]}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |d\omega|^2 + |\delta \omega|^2}{\int_{\Omega} |\omega|^2} : \omega \in \Lambda^p(\Omega), i_N \omega = 0 \text{ on } \partial \Omega \right\}$$

As $\Delta$ commutes with both $d$ and $\delta$, it preserves the space of exact (resp. co-exact) forms. Hence:

$$\lambda_1^{[p]} = \min \{ \lambda_1^{[p]'}, \lambda_1^{[p]''} \}$$

where $\lambda_1^{[p]'}$ (resp. $\lambda_1^{[p]'''}$) is the first eigenvalue of $\Delta$ when restricted to exact (resp. co-exact) $p$-forms. By differentiating eigenforms one sees that:

$$\lambda_1^{[p]'} = \lambda_1^{[p-1]'''}.$$  

From the Hodge decomposition theorem for manifolds with boundary (Hodge-Morrey decomposition), one sees that, if $\omega$ is an exact $p$-form on $\Omega$, then there exists a unique $(p - 1)$-form $\theta = \theta_{\omega, \Omega}$ such that:

$$\begin{cases}
\omega = d\theta, \\
\theta \text{ is co-exact and } i_N \theta = 0 \text{ on } \partial \Omega.
\end{cases}$$

The form $\theta$ above is called the canonical primitive of $\omega$. It has the following important property:
the canonical primitive is the primitive with the least $L^2$-norm.

Now $\theta = \theta_{\omega,\Omega}$ is a test-form for the eigenvalue $\lambda_1^{[p-1]''}$. Hence:

$$\lambda_1^{[p]} \leq \lambda_1^{[p]'}$$

$$= \lambda_1^{[p-1]''}$$

$$\leq \frac{\int_{\Omega} |d\theta|^2}{\int_{\Omega} |\theta|^2} = \frac{\int_{\Omega} |\omega|^2}{\int_{\Omega} |\theta_{\omega,\Omega}|^2}$$

Now if $E_- \subseteq \Omega$ we see that, for any exact $p$-form $\omega$ one has:

$$\lambda_1^{[p]}(\Omega) \leq \frac{\int_{\Omega} |\omega|^2}{\int_{E_-} |\theta_{\omega,E_-}|^2}$$

where $\theta_{\omega,E_-}$ is the canonical primitive of $\omega$ on $E_-$. Let us choose $\omega$ so that everything will be computable. Fix coordinates so that $E_-$ has equation:

$$\frac{x_1^2}{D_1^2} + \cdots + \frac{x_n^2}{D_n^2} \leq 4,$$

and take

$$\omega = dx_1 \wedge \cdots \wedge dx_p.$$ 

Then $|\omega|^2 = 1$ and its canonical primitive on the ellipsoid $E_-$ is explicitly computable. One ends-up
with the desired upper bound:

\[ \lambda_1^{[p]}(\Omega) \leq 4p(n + 2) \cdot \frac{\text{vol}(\Omega)}{\text{vol}(\mathcal{E}_-)} \cdot \frac{1}{D_p(\mathcal{E}_-)^2}. \]