Geometric Variational Problems

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1 Overview of the Field and Recent Developments

In recent years there has been striking progress on geometric variational problems and geometric flows with important applications to geometry and topology. This includes the resolution of some of the old and famous problems. Notably, Perelman's resolution of the Poincare and geometrization conjecture via the Ricci flow method; the proof of the differentiable sphere theorem by Brendle-Schoen, the proof of the Willmore Conjecture by Marques-Neves, the proof of Lawson's conjecture on minimal surfaces by Brendle and Andrews's work on mean curvature flow. The success of the min-max variational approach to the Willmore conjecture has stimulated a lot of research activities in minimal surface theory and on the geometry of Willmore surfaces.

Other highlights include understanding the blow-up sets and limit laminations of sequences of embedded minimal surfaces in 3-manifolds. In particular, this has led to new developments in the classical theory of minimal surfaces in $\mathbb{R}^3$, such as the classification of topologically simple minimal surfaces. Important applications of minimal surfaces to other fields of mathematics have been discovered. In fact minimal surfaces play a crucial role in the proof of the Poincare conjecture. In relativity there is an immense interest in dynamical horizons which are 3-dimensional spacelike hypersurfaces foliated by apparent horizons in a slicing of a spacetime. Apparent horizons are minimal 2-spheres in some cases, but usually solutions of a prescribed mean curvature equation of a particular type. There is also interest in higher dimensional black holes related to string theory. One focus of attention has been on existence theory. Constructing special submanifolds such as minimal surfaces and constant mean curvature surfaces is an important and classical topic in differential geometry with applications in topology and physics (e.g. the theory of general relativity). A further important topic is calibrated submanifolds , i.e. submanifolds which minimize the volume functional in their homology classes. Typical examples are complex submanifolds in Kähler manifolds and special Lagrangian submanifolds in Calabi-Yau manifolds, and they are the building blocks in various duality phenomena in mirror symmetry. Minimal surfaces/submanifolds are characterized by nonlinear partial differential equations which are difficult to solve in general. Douglas-Morrey's solution to the Plateau problem is perhaps the first major advance. One may use the gluing method (a delicate application of the implicit function theorem), variational method (such as min-max principle or minimizing certain functionals, e.g. Sacks-Uhlenbeck's perturbed energy functional) and the heat flow method (e.g. the mean curvature flow).

One focus of the workshop was on Willmore surfaces. These are solutions of a variational problem for which minimal surfaces are minimizers. Willmore surfaces arise naturally in a large number of different fields such as elasticity theory, general relativity and conformal geometry. Moreover, since the system of PDE's solved by Willmore surfaces is of fourth order, they can also be seen as an interesting model for nonlinear higher order systems of PDE's. Recently there have been a number of new analytic and geometric results on the existence, regularity and energy quantization for Willmore surfaces in Euclidean space and in Riemannian manifolds. One of the difficulties in the study of Willmore surfaces is the fact that the system of PDE's is not elliptic since it is invariant under diffeomorphisms. This problem can be overcome by choosing a good gauge, e.g. by considering graphical solutions or by working with a conformal parametrization of the surface. This last approach recently led to the introduction of $W^{2,2}$-conformal immersions and some interesting applications on the existence of minimizing Willmore surfaces in a given conformal class have been obtained. The recent work of Riviére led to a major breakthrough in the regularity theory, of two-dimensional conformally invariant variational problems such as Willmore surfaces, harmonic maps, CMC surfaces or more generally surfaces of prescribed mean curvature in Riemannian manifolds. The workshop brought together people working on applications of these new methods to related problems including geometric flows.

2 Presentation Highlights

In this section, we give a brief summary of the 23 talks delivered at the workshop.

Spyros Alexakis from the University of Toronto spoke about *Boundary regularity and bubbling for Willmore surfaces with bounded weighted energy* which is joint work with Rafe Mazzeo. He considered the space of Willmore surfaces in H^3 and in the unit ball in E^3, with boundaries lying on the sphere. As main results he obtained boundary regularity and a bubbling criteria for such surfaces in terms of a certain weighting of the traceless total curvature. The weight function here blows up logarithmicaly at the boundary. Analogous questions were also posed in the context of asymptotically hyperbolic Einstein metrics and for harmonic maps.

Costante Bellettini from the IAS and Princeton University lectured on *Calibrations of degree 2 and blow up analysis*. These calibrated currents naturally appear when dealing with several geometric questions, some aspects of which require a deep understanding of regularity properties of calibrated currents. He reviewed some of these issues and then he focused on the two-dimensional case where he showed a surprising connection with pseudo-holomorphic curves and performed a blow up analysis, where he was especially interested in the uniqueness of tangent cones and in the rate of decay for the mass ratio.

Jacob Bernstein from the Johns Hopkins University gave a talk with the title *The topology of the limits of a sequence of embedded minimal disks*. The work of Colding-Minicozzi says that a sequence of embedded minimal disks in a three-manifold subconverges to a minimal lamination away from a closed set of singular points. Motivated by this, a great deal of recent work has been done constructing examples of possible limits. Notably, Hoffman-White have produced examples where some of the leaves are annuli – leading them to ask whether leaves of more complicated topology were possible. He showed that under natural geometric conditions on the ambient three-manifold the answer is no. This is joint work with G. Tinaglia.

Christine Breiner from the Fordham University spoke about *Compactness theory for biharmonic maps into spheres*. She studied critical points for the functional

$$E(u) = \int |\Delta u|^2,$$

which are called biharmonic maps and are natural fourth order analogues of harmonic maps. Compactness theory for harmonic maps in two dimensions is well understood. She discussed a recent work with T. Lamm in which they determine the energy quantization and the C^0 limit picture for sequence of approximate biharmonic maps from four dimensional manifolds into spheres. The analogous results for harmonic maps is true even for general targets and relies on results of Sacks-Uhlenbeck, Jost, and Parker. She reviewed the main ideas behind the proof of energy quantization and the C^0 limit picture for harmonic maps and explained some of the pitfalls that arise when trying to extend this proof to the biharmonic setting. She then explained what can be gained by mapping into spheres and sketched a few aspects of the proof.

Philippe Castillon from the Université de Montpellier II and UBC lectured on *Submanifolds, isoperimetric inequalities and optimal transport* He showed how one can use optimal transport methods in order to prove isoperimetric inequalities on submanifolds of the Euclidean space. In particular, he gave a new proof of the Michael-Simon inequality with a constant better than those previously known. This proof relies on the description of a solution to the Monge problem when the initial measure is supported in a submanifold and the target one in a linear subspace.

Albert Chau from the University of British Columbia discussed he recent joint work with L.F. Tam and K.F Li with the title *The Kähler Ricci flow on complete Kähler manifolds with unbounded curvature*. He considered the Kähler Ricci flow on a complete non-compact Kähler manifold (M, g). A classical result of W.-X. Shi says that the flow has a short time solution if g has bounded curvature. This talk addressed the problem of finding a solution when g has unbounded curvature. A. Chau began by discussing some a priori estimates and general existence results for the flow and then he discussed applications in the case g is a $U(n)$ invariant Kähler metric on $\mathbb{C}^n$.

Jaigyoung Choe from the Korea Institute for Advanced Study reported on *Stable capillary hypersurfaces in a wedge*. For an immersed stable constant mean curvature hypersurface Σ in a wedge bounded by two hyperplanes in $\mathbb{R}^n$ which meets those two hyperplanes in constant contact angles and is disjoint from the edge of the wedge he showed that if $\partial\Sigma$ is embedded for $n = 3$, or if $\partial\Sigma$ is convex for $n = 4$, then Σ is part of the sphere.

Robert Gulliver from the University of Minnesota discussed his recent work on *Branch points of minimizing nonorientable surfaces*. He started by recalling that the construction of minimal surfaces in a Riemannian manifold is accomplished by minimizing energy under Plateau boundary conditions, which allow reparameterization of the given boundary curves, if any. The minimizer is a conformally parameterized minimal surface with possible isolated singularities, called *branch points*. True branch points are those which are locally not a branched covering of an immersed surface. They are only possible on an area-minimizing minimal surface when the codimension is at least 2. The absence of false branch points, or more generally of ramified branch points, requires a global topological hypothesis, the Douglas hypothesis; but that has so far only been shown to suffice for orientable surfaces. A branch point is *ramified* if in every neighborhood of the point there are two open sets which define the same piece of surface. In the talk, R. Gulliver outlined these arguments for oriented surfaces, extended a basic theorem to non-orientable surfaces, and then discussed ramified branch points of non-orientable minimal surfaces, focusing on surfaces of small Euler characteristic, using the Riemann-Hurwitz formula. For example, he showed that in codimension one, a mapping from the projective plane, which minimizes area among homotopically non-trivial mappings, is an immersion.

Robert Haslhofer from the Courant Institute spoke on a joint work with Bruce Kleiner with the title *Mean curvature flow of mean convex hypersurfaces*. In the last 15 years, White and Huisken-Sinestrari developed a far-reaching structure theory for the mean curvature flow of mean convex hypersurfaces. Their papers provide a package of estimates and structural results that yield a precise description of singularities and of high curvature regions in a mean convex flow. In the present talk, R. Haslhofer explained a new treatment of the theory of mean convex (and k-convex) flows. This included: (1) an estimate for derivatives of curvatures, (2) a convexity estimate, (3) a cylindrical estimate, (4) a global convergence theorem, (5) a structure theorem for ancient solutions, and (6) a partial regularity theorem. The new proofs are both more elementary and substantially shorter than the original arguments. The estimates are local and universal. A key ingredient in the new approach is the new non-collapsing result of Andrews. Some parts were also inspired by the work of Perelman.

Tom Ilmanen from the ETH Zürich spoke on the *Mean curvature flow in* $\mathbb{R}^3$. The mean curvature flow typically develops finite time singularities, and it is important to understand self-similar solutions at these singularities. Shrinkers are special solutions of mean curvature flow that evolve by rescaling and model the singularities. He proved here that round cylinders are rigid in a very strong sense. Namely, any other shrinker that is sufficiently close to one of them on a large, but compact, set must itself be a round cylinder. This is a joint work with T. Colding and W. Minicozzi.

Brett Kotschwar from the Arizona State University reported on his work about *A frequency approach to unique continuation for the Ricci flow*. He discussed an alternative approach, based on the consideration of an appropriate frequency-type quantity, to certain problems of unique continuation arising in the study of the Ricci flow. This technique allows for short – and transparently quantitative – proofs of a class of global backwards-uniqueness results for the flow, and has further applications to some questions of asymptotic rigidity for self-similar solutions.

Brian Krummel from the University of Cambridge spoke about joint work with Neshan Wickramasekra with the title *Structure of branch sets of harmonic functions and minimal submanifolds*. He discussed some recent results on the structure of the branch set of multiple-valued solutions to the Laplace equation and minimal surface system. It is known that the branch set of a multiple-valued solution on a domain in $\mathbb{R}^n$ has Hausdorff dimension at most $n-2$. The two authors investigated the fine structure of the branch set, showing that the branch set is countably $(n-2)$-rectifiable. The result follows from the asymptotic behavior of solutions near branch points, which is established using a modification of the frequency function monotonicity formula due to F. J. Almgren and an adaptation to higher-multiplicity of a "blow-up" method due to L. Simon that was originally applied to "multiplicity one" classes of minimal submanifolds satisfying an integrability hypothesis.

Chikako Mese from the Johns Hopkins Univerisity spoke about *Harmonic maps in rigidity problems*. She discussed harmonic maps into non-positively curved metric spaces (NPC spaces). Of particular interest is the regularity issue for these maps into special classes of NPC spaces that include the Weil-Petersson completion of Teichmüller space. As an application of the regularity theory, she studied some rigidity questions as well.

Andrea Mondino from ETH Zürich talked about his joint work with Tristan Rivière with the title *A frame energy for tori immersed in $\mathbb{R}^m$: sharp Willmore-conjecture type lower bound, regularity of critical points and topological applications*. He spoke about recent results on the Dirichlet energy of moving frames on 2-dimensional tori immersed in the euclidean $3 \leq m$-dimensional space. This functional, called Frame energy, is naturally linked to the Willmore energy of the immersion and on the conformal structure of the abstract underlying surface. As first result, a sharp Willmore-conjecture type lower bound is established in arbitrary codimension. Smootheness of the critical points of the frame energy is proved after the discovery of hidden conservation laws and, as application, the minimization of the Frame energy in regular homotopy classes of immersed tori in $\mathbb{R}^3$ is performed.

Huy The Nguyen from the University of Queensland reported on his work about *Singular Willmore Spheres in $\mathbb{R}^4$*. In the talk he extended a recent classification result by Lamm-Nguyen of singular Willmore spheres in $\mathbb{R}^3$ to the setting of $\mathbb{R}^4$. He showed that if the order of singularities is bounded then a singular Willmore sphere must be either an umbilic sphere, the Penrose twistor projection of an (anti)-holomorphic curve in CP^3 or the Möbius transformation of a complete non-compact minimal surface in $\mathbb{R}^4$. He then also gave an application to the Willmore flow of spheres in $\mathbb{R}^4$.

Frédéric Robert from the Université Henri Poincaré at Nancy/ UBC spoke about his joint work with Jérôme Vétois on *Glueing of a Bubble to a degenerate metric*. The authors investigate the existence of blowing-up solutions to scalar-curvature equations by glueing a Bubble to a given ground state. The main difficulty occurs when the ground-state is degenerate. Using analytic expansions, they prove that the construction can be carried out for isolated ground states.

Felix Schulze from the University College London reported on his joint work with Tom Ilmanen and Andre Neves with the title *A local regularity theorem for the network flow*. The network flow is the evolution of a network of curves under curve shortening flow in the plane, where it is allowed that at triple points three curves meet under a 120 degree condition. He presented a local regularity theorem for the network flow, which is similar to the result of B. White for smooth mean curvature flow.

Benjamin Sharp from the Imperial College London discussed some aspects of the *Interior and free boundary regularity for Dirac-harmonic maps, harmonic maps and related PDE*. Since Hélein's celebrated proof of the regularity of weakly harmonic maps from surfaces to Riemannian manifolds there have been huge improvements and generalisations to the theory, with applications in many areas of analysis and geometry. Notably the work of Tristan Rivière has provided analytical insight to these problems leading to suitable generalisations. In his talk he gave an overview of some of these ideas and presented new theorems leading to proofs (and hopefully some insight) for both new and classical results. Some of the work presented was joint with Peter Topping (Warwick), Miaomiao Zhu (MPI Leipzig) and Tobias Lamm (KIT).

Miles Simon from the University of Magdeburg spoke about*Some local results for the Ricci flow*. In dimension two he generalized G.Perelman's "Pseudolocality" result to allow cone like regions (which are by definition not locally euclidean). He showed that under certain assumptions one also obtains a local result (different from that of Perelman) in three dimensions.

Jeff Streets from the University of California at Irvine talked about his recent results about *Singularities of the L^2 curvature flow*. The L^2 norm of the Riemannian curvature tensor is a natural energy to associate to a Riemannian manifold, especially in dimension 4. A natural path for understanding the structure of this functional and its minimizers is via its gradient flow, the "L^2 flow." This is a quasi-linear fourth order parabolic equation for a Riemannian metric, which one might hope shares behavior in common with the Yang-Mills flow. He verified this idea by exhibiting structural results for finite time singularities of this flow resembling results on Yang-Mills flow. He also exhibited a new short-time existence statement for the flow exhibiting a lower bound for the existence time purely in terms of a measure of the volume growth of the initial data. As corollaries he obtained new compactness and diffeomorphism finiteness theorems for four-manifolds generalizing known results to ones with minimal dependencies. These results all rely on a new technique for controlling the growth of distances along a geometric flow, which is especially well-suited to the L^2 flow.

Glen Wheeler from the University of Wollongong discussed his work *Curvature contraction of convex hypersurfaces by non-smooth speeds*. In fully nonlinear curvature flow we typically aim to prove that suitably pinched initial data shrinks to a point in finite time, becoming asymptotically close to a self-similar solution as it does so. Theorems of this type trace back to Huisken's seminal contribution for the mean curvature flow of convex hypersurfaces. Since then, the efforts of a number of researchers have established similar results for quite broad classes of (typically fully nonlinear) curvature flow with smooth velocity. The new contribution is an analysis of the case where the speed of the flow, as a function of the eigenvalues of the Weingarten map, is not differentiable. Other requirements on the flow speed are reminiscent of Andrews early work from 1993 and 1994. The main result is that up to necessary modifications, a Huisken-esque result holds. Glen Wheller described the proof of this result, highlighting the essential new contributions, and mentioned possible future directions. This is joint work with Ben Andrews, Andrew Holder, James McCoy, Valentina-Mira Wheeler, and Graham Williams.

Eric Woolgar from the University of Alberta spoke about *APEs, their close relatives, and their evolution*. APEs are Asymptotically Poincaré-Einstein manifolds. He first reviewed the zoology of APEs and their relatives, which are various classes of Conformally Compactifiable manifolds. He showed that APEs have nice properties under the Ricci flow. Namely, if a manifold is initially APE, it remains APE under the flow, and if the Ricci curvature obeys the natural lower bound

$$Ric \geq -(n-1)$$

initially, then this is preserved. The mass, when defined, is monotonic, as is the renormalized volume when the aforementioned Ricci curvature bound holds initially. This has a nice interpretation for the Hawking-Page phase transition in black hole physics. He also discussed the situation for more general classes of asymptotically hyperbolic manifolds, and some open problems. The talk was based in part on joint work Eric Bahuaud and Rafe Mazzeo and with Tracey Balehowsky.

Yu Yuan from the University of Washington reported on *Self similar solutions for curvature flows*. He discussed some rigidity results of self similar solutions for mean curvature flow and Kähler-Ricci flow. One of the results discussed in the talk was: Suppose u is an entire smooth pluri-subharmonic solution on $\mathbb{C}^m$ to the complex Monge-Ampére equation

$$\ln \det(u_{\alpha\beta}) = \frac{1}{2} x \cdot Du - u.$$

Assuming that the corresponding Kähler metric $g = (u_{\alpha\bar{\beta}})$ is complete, implies that u is a quadratic function.

3 Outcome of the Meeting

The 5-day workshop provided excellent opportunities for people to get together to discuss recent developments in the field and talk about their recent and on-going work. It brought experts in minimal surfaces, harmonic mappings, Willmore type surfaces, geometric measure theory, geometric curvature flows (mean curvature flow, Ricci flow, Kähler Ricci flow) in a stimulating environment to share their knowledge. One exciting aspect of the meeting is that there were a lot of young speakers presenting their very interesting work, actively involved in discussions, and initiating future collaborations.