Refined invariants in geometry, topology and string theory

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1 Overview of the Field

Since the spectacular advances in quantum field theory and string theory of the early 90s, BPS states have been a central object of study in theoretical physics with deep mathematical implications. In a nutshell, BPS states are quantum states classified by short supersymmetry multiplets in supersymmetric theories. Such multiplets exhibit remarkable stability properties under quantum fluctuations, providing a reliable interpolation tool between strong and weak coupling in many theories of physical interest. This special property leads to striking results in string theory compactifications. A prominent example is the relation between the low energy spectrum of Seiberg-Witten theories and supersymmetric D-brane states in geometric engineering constructions, as pointed out by Kachru and Vafa [28], and also Katz, Klemm and Vafa [29]. This relation has led in particular to the connection between D-brane bound states and instanton sums discovered by Nekrasov [43].

From a mathematical point of view, supersymmetric D-brane states in string theory compactifications are loosely speaking cohomology classes on moduli spaces of stable bundles on Calabi-Yau threefolds. More generally, it has become increasingly clear in the past years due to work of Douglas and Aspinwall [3] that a complete mathematical construction must involve moduli spaces of stable objects in Calabi-Yau derived categories. Virtual numbers of such objects, Donaldson–Thomas (DT) invariants [17, 49], are identified with BPS indices, which count BPS states with a sign determined by their intrinsic spin quantum number. The special properties of the BPS spectrum of the underlying physical theories lead to deep predictions concerning such invariants, which very often crystalize into beautiful mathematical results.

A particular long range mathematical program prompted by the physics connection is concerned with motivic and categorical generalizations of DT invariants. A first step in this direction is focused on refined DT invariants, a construction loosely analogous to promoting the Euler characteristic of a topological space to the Poincaré polynomial, or refining the Jones polynomial of a knot to a multi-variable polynomial coming from Khovanov–Rozansky theory. As pointed out by Gukov, Schwartz and Vafa [23] , and also more recently by Gaiotto, Moore and Nietzke [19], such constructions are motivated by a refinement of the BPS index in physics theories, which is promoted to a generating function which counts BPS states keeping track of their space-time spin quantum number. In the toric context, a refinement of topological string amplitudes was provided by Iqbal, Kozcaz and Vafa [27], employing the topological vertex formalism of Aganagic-Klemm-Marino-Vafa [1] for topological string amplitudes. The resulting refined vertex was related to refined DT invariants by Dimofte and Gukov [16].

From a more technical point of view, the purpose of our workshop was to study a picture that has been emerging recently from these considerations, relating seemingly different mathematical objects ranging from Macdonald polynomials, knot invariants, invariants coming from enumerative algebraic geometry of local Calabi-Yau threefolds, to equivariant integrals, cohomological invariants of the Hitchin system, instanton moduli spaces, Seifert fibered 3-manifolds and finally to double affine Hecke algebras. While the connections between these areas involve a variety of conjectural dualities in string theory, the refined invariants themselves are mostly (though not in all cases) well-defined mathematical objects.

We illustrate the objects involved and their connections by introducing four generating series F_{GV}, Z_{DT}, Z_{CS} and Z_{gauge}, all functions of a certain number of variables. In our discussion these are all attached to a smooth projective curve (compact Riemann surface) of genus g, though other starting points (such as quiver theories) are also possible.

The series Z_{DT} involves DT invariants of a local curve, a Calabi-Yau 3-fold which is the total space of a certain rank 2 line bundle over our curve. In the unrefined case, this series was studied by Bryan–Pandharipande [5] and Okounkov–Pandharipande [42]. A recent paper by Chuang–Diaconescu–Pan [9] understands these unrefined invariants in terms of Euler characteristics of certain moduli spaces of semi-stable Higgs bundles, and in turn they paint a conjectural picture for the (not yet rigorously defined) refined invariants in terms of the full cohomology of these Higgs moduli spaces. Refined DT invariants have recently been studied in several papers, notably by Kontsevich–Soibelman [30, 31] and Behrend–Bryan–Szendroi [4]; the local curve case however is only understood in some cases in genus 0 [39].

Let us turn to the series F_{GV}, the generating function of Gopakumar–Vafa's BPS-invariants attached to our local curve; they arise in M-theory and mathematically are still mysterious. Type IIA / M-theory duality leads to a conjectural relationship of the form $\exp(F_{GV}) = Z_{DT}$. This formula is in fact the way mathematicians are defining GV invariants, and it has been checked that the so-obtained unrefined BPS-invariants are integers in the local curve case. In work of Diaconescu et al [15], refined BPS invariants are defined using the perverse filtration on the cohomology of Higgs moduli spaces. For refined invariants, the conjectural equation $\exp(F_{GV}) = Z_{DT}$ turns out to be equivalent to the main conjecture of Hausel–Villegas [25] about the mixed Hodge polynomials of character varieties, provided one also assumes the perverse/weight filtration conjecture $P = W$ of de Cataldo–Hausel–Migliorini [11].

The reason one can formulate conjectures for the (partially undefined) refined Z_{DT} is the above mentioned geometric engineering principle, connecting DT invariants to gauge theoretic invariants on $\mathbb{R}^4$. In this story, one considers a $U(1)$ gauge theory on $\mathbb{R}^4$ with g adjoint matter fields, and defines the refined gauge theory partition function Z_{gauge} as a certain mathematically rigorous equivariant integral of a certain K-theory class over the Hilbert scheme of n points on $\mathbb{C}^2$. Geometric engineering then predicts that up to a change of variables, $Z_{DT} = Z_{gauge}$, where the latter is one variant of the famous Nekrasov partition function [43].

The final series Z_{CS} is the generating function of Chern–Simons invariants of three-manifolds in a certain large N limit. Another famous conjecture of Gopakumar–Vafa [20] together with Witten's open/closed string duality predicts that in appropriate variables, $Z_{CS} = Z_{DT}$. In recent work of Aganagic–Shakirov [2], this relation was studied in the refined case in the case of Seifert fibred 3-manifolds, more specifically the product of our algebraic curve (Riemann surface) with the circle.

The full power of these ideas appears when one considers relative theories, including punctures on the Riemann surface, Lagrangian branes in the Calabi–Yau threefold, and S^1-invariant links in the Seifert fibered 3-manifold. Surprisingly, in the relevant refined formulae due respectively to Hausel–Letellier–Villegas [24], Iqbal–Kozcaz–Vafa [27] and Aganagic–Shakirov [2], Macdonald polynomials play a fundamental role. This indicates the presence of some, hitherto unknown, double affine Hecke algebra symmetries.

Finally, in this relative case the refined Z_{CS} is expected by Aganagic–Shakirov to be intimately related to Khovanov–Rozansky invariants of links in three-manifolds. Recent mathematics results by Oblomkov–Shende [40], Gorsky–Oblomkov–Rasmussen–Shende [21], Gorsky–Mazin [22], Migliorini–Shende [37] and Maulik–Yun [36] are suggesting deep connections between Khovanov–Rozansky invariants, Macdonald polynomials and the cohomology of certain Hitchin fibers. One hopes that these connections also acquire an explanation in the general conjectural framework suggested in this overview.

2 Recent Developments and Open Problems

The study of Donaldson–Thomas invariants [17, 49] and their refined versions has been one of the major sources of inspiration in this field. However, much of the work on DT invariants has been following [34], and is concerned with rank ≤ 1 invariants. Recently, the quantum cohomology of higher rank moduli spaces of sheaves on the affine plane has been computed by Maulik and Okounkov [35]; this should give information on higher rank DT theory. The refined theory especially remains to be explored, as well as connections to theories on surfaces via geometric engineering.

The conjecture of Oblomkov–Shende [40], relating the geometry of its Hilbert scheme of points to the HOMFLY polynomial of the associated algebraic link, has recently been settled by Maulik [33]. Refinements of the conjecture [41] however remain to be studied, as well as the deep conjectured connections [21] to the representation theory of the rational DAHA.

The Cohomological Hall Algebra (CoHA) of Kontsevich–Soibelman is an algebra structure defined on cohomological refinements of DT invariants. Efimov [18] and more recently Davison [12, 13] have proved some exciting structural results about the CoHA in specific cases. One outstanding problem concerns how to understand geometrically the generators of various CoHAs. This could then be used to attack questions about hyperkähler Kirwan subjectivity. An example is a recent result of Zongbin Chen [8]. A deeper understanding of CoHAs, especially purity [13], has the potential to lead to new results, potentially leading to a new TQFT in the framework of multiplicative COHA's. In a different direction, the action of CoHA on the cohomology of moduli of framed sheaves [48, 47] should lead to new results about moduli, as well as new connections with geometric representation theory.

Recently, Brav–Bussi–Dupont–Joyce–Szendroi [6] and Li–Kiem [32] have constructed a global perverse sheaf of vanishing cycles on the moduli space of sheaves on a Calabi-Yau threefold, whose cohomology gives the cohomological refinement of the associated DT invariant. Li–Kiem used this to give the first fully geometric, mathematical definition of Gopakumar-Vafa invariants, mirroring the definition in physics. While ealier attempts at a geometric definition were made by Hosono, Saito, and Takahashi, Li and Kiem's proposal is the first which is not incompatible with known Gromov-Witten invariants. A very interesting case to test the Li-Kiem proposal is for local curves. In this case, the Gromov-Witten invariants are fully known [5] and consequently, there are full predictions for the Gopakumar-Vafa invariants. The relavant moduli space is topologically equivalent to the moduli space of stable bundles on a smooth curve — a very well studied classical object — however, the associated perverse sheaf is highly non-trivial and is an interesting new structure on the moduli space. Computing the hypercohomology of this perverse sheaf (required to test the Li-Kiem proposal) is an open problem.

3 Presentation Highlights

Monday 3 June

The week started off with the first of two talks by **Andrei Okounkov**, on his joint work with Nekrasov, entitled *M-theory and DT-theory*. This work aims to define a new set of enumerative invariants arising from M-theory on on a five-dimensional Calabi-Yau manifold Z. While this theory has not been fully constructed yet, the speaker explained several aspects of this M-theoretic moduli space: it should be a theory of maps from 1-dimensional schemes to Z, with a very specific stability condition, including the fact that the source curve should be Cohen–Macaulay. It should be closely related, but not identical, to a known theory of curve counting defined by Honsen.

Under a suitable torus action, this new theory should reduce to less exotic theories. For example, taking a torus action on Z with fixed-point set being a threefold X, one should conjecturally obtain K-theoretic Donaldson–Thomas invariants of X. Taking more complicated fixed point sets, and different torus actions, one should obtain relations between different counting theories, such as Nekrasov-type theories on algebraic surfaces and DT theory on threefolds, shedding new light in particular on geometric engineering.

Next we heard a talk by **Sheldon Katz** on *Equivariant stable pair invariants and refined BPS indices*. Here, the speaker discussed his joint work with Klemm and Choi, involving localization calculations of refined DT invariants, building on the aforementioned work of Nekrasov–Okounkov, connecting them also to string-theoretic considerations. In particular, he explained a conjectural product formula for the generating function of refined DT invariants in terms of refined BPS indices. Computations can be based on a virtual Bialynicki-Birula decomposition of the stable pairs moduli spaces. This gives an agreement with known results for the Calabi–Yau threefold geometry known as local $\mathbb{P}^1$. He also sampled some new computations for local $\mathbb{P}^2$.

Dominic Joyce's talk *Categorification of Donaldson–Thomas theory using perverse sheaves*, explaining his work done with several groups of collaborators, started with a discussion of results in derived algebraic geometry: the derived structure of moduli of sheaves on a Calabi–Yau threefolds due to Pantev, Toen, Vaquie and Vezzosi. He then explained a Darboux theorem, giving a pleasant classical local model for such moduli spaces. This then leads to natural perverse sheaves on these moduli spaces, whose cohomology defines a cohomological refinement of DT theory.

Zheng Hua gave a talk entitled *Spin structure on moduli space of sheaves on CY 3-folds*, where he discussed some recent progress on the existence of Konstevich–Soibelman orientation data on moduli spaces of sheaves on CY 3-folds, necessary for defining any refined version of Donaldson–Thomas invariants. His work uses a gauge-theoretic construction of the relevant moduli spaces.

Sven Meinhardt's talk *Motivic DT-invariants of* (-2)*-curves*, based on work in collaboration with Davison, gave a concrete but substantial geometric example where refined DT invariants can be computed: small resolutions of certain singular 3-folds. He first introduced the geometry of contractible curves, and sketched how they show up in resolutions of singular 3-folds. After that, he gave an alternative non-commutative resolution of the same threefold singularity using quivers with potential. Finally, he wrote down the refined DT invariants in this set of examples.

Tuesday 4 June

Alexei Oblomkov's talk *Topology of planar curves, knot homology and representation theory of Cherednik algebras* was centered around the conjectures relating Khovanov–Rozansky homology of the link of a planar singularity to the homology of Hilbert schemes on the corresponding plane curve (due to Oblomkov, Rasmussen and Shende). He also presented compelling evidence for a conjecture that, in the case of torus knots, the homology of the link carries the action of a Cherednik algebra (joint work also involving Gorsky).

Luca Migliorini's talk *Support theorems for Hilbert schemes of families of planar curves*, based on joint work with Shende and Viviani, discussed support loci for summands of the pushforward of the constant sheaf on the relative Hilbert scheme in families of curves. Given a family of curves with at worst planar singularities over a base A, one can look at its associated relative Hilbert scheme of a fixed length. By the Decomposition Theorem of Beilinson–Bernstein-Deligne, if the total space of the relative Hilbert scheme is nonsingular, the pushforward of the constant sheaf to A splits into a direct sum of intersection cohomology sheaves (plus shifts); it is an interesting problem to determine them or at least their supports. If the curves in the family are all integral, a result independently due to Maulik Yun and MiglioriniShende ensures that the sheaves are all supported on the whole of A. This result, an analogue of Ngos support theorem, leads to a generalization of the MacDonald formula, relating the cohomology of the Hilbert schemes of a singular curve to that of its compactified Jacobian. In the reducible case this is no longer true, as seen in easy examples. The speaker discussed some new and interesting phenomena that arise in this situation. He also described a slightly more general set-up to study the supports associated with a map, and introduce some loci associated with a map, which they call higher discriminants.

As an additional lunchtime extra, we had the second talk by **Andrei Okounkov** this day. He explained some more examples of his construction, for example the case where the fivefold arises as the direct sum of four line bundles on an algebraic curve. The arising partition function gives a substantial generalization of MacDonald's famous formula on the topological invariants of symmetric products of the curve.

In the afternoon, **Vivek Shende** gave a talk entitled *Legendrian knots and constructible sheaves* on joint work with David Treumann and Eric Zaslow. He explained how to associate to a Legendrian knot at infinity in the cotangent bundle of a surface, the subcategoryof the Fukaya category consisting of Lagrangian branes ending on the knot. The Nadler-Zaslow dictionary relates these to constructible sheaves with singular support controlled by the knot diagram. When the surface is an annulus and the diagram is the closure of a positive braid, he determined the moduli spaces of rank n objects in this category and showed that they are partial compactifications of the rank n character variety on a Seifert surface for the knot. These moduli spaces map to the space of local systems on a circle; the spectral sequence associated to the weight filtration on the pushforward of the constant sheaf recovers the HOMFLY homology of the knot.

Anatoly Preygel gave a talk entitled *Higher structures on Hochschild invariants of matrix factorizations*. The dg-categories of matrix factorizations categorify several linear algebraic singularity invariants: functions on the (derived) critical locus, and vanishing cycles. The speaker explained how the Hochschild invariants of matrix factorizations can be identified with classical linear-algebraic invariants via a formality theorem of Dolgushev–Tsygan–Tamarkin.

Andrew Morrison's talk *Asymptotics of 3D partitions and refined invariants* studied the asympotics of refined DT invariants. He used the Hardy–Littlewood circle method to analyze the bivariate asymptotics of a q-deformation of MacMahon's generating series introduced by Behrend–Bryan–Szendroi.

Wednesday 5 June

Ezra Getzler gave a talk entitled *Derived stacks*, in which he explained his new approach, jointly with Behrend, to defining derived stacks. He explained how they are rewriting the basic theory of higher stacks in the language of categories of fibrant objects. This allows them to develop the theory with a very small set of basic axioms.

An application of refined Donaldson-Thomas theory was explained in the talk by **Ben Davison** entitled *Purity of critical cohomology for graded potentials and quantum cluster positivity*. He discussed a purity result on the Hodge structure on the cohomology of vanishing cycles for certain $\mathbb{C}^*$-equivariant functions on smooth varieties, and showed how this theory can be applied to prove some cases of a conjecture in the theory of the cluster algebras of Fomin and Zelevinsky.

Vittoria Bussi spoke about joint work with Joyce and Meinhardt entitled *On motivic vanishing cycles of derived critical loci*. She how to use the results in Joyce's talk, discussed above, to attach virtual motives, elements in in a modified version of the motivic Grothendieck ring of varieties, to DT-style moduli spaces. This fills in some details in the motivic DT project of Kontsevich and Soibelman. The construction also attaches new invariants to intersections of spin Lagrangians in an algebraic symplectic manifold.

Thursday 6 June

Duiliu-Emanuel Diaconescu gave a talk entitled *Parabolic refined invariants and MacDonald polynomials*, in which he studied a conjecture of Hausel, Letellier and Rodriguez-Villegas on the cohomology of character varieties with marked points, identifying their formula with a Gopakumar–Vafa expansion in the refined stable pair theory of local orbifold curves, in turn related by geometric engineering to K-theoretic invariants of nested Hilbert schemes. MacDonald polynomials appear naturally in this framework via Haiman's geometric construction based on the isospectral Hilbert scheme.

A closely related talk by **Emmanuel Letellier**, entitled *Counting geometrically indecomposable parabolic bundles over the projective line*, discussed a formula for counting (over finite fields) the number of isomorphism classes of geometrically indecomposable parabolic structures (of a given type) on a given vector bundle over $\mathbb{P}^1$. In the case of trivial bundles, these formulae recover some motivic DT-invariants. The ultimate goal of the project is to find an explicit formula for the Poincaré polynomial of the moduli space of stable parabolic Higgs bundles over the projective line.

Andras Szenes explained his joint work with Hausel in the talk *Equivariant intersection theory of Higgs moduli spaces*. It started by first explaining a technique to compute the equivariant intersection numbers of Higgs moduli spaces in order to check conjectures of Moore-Nekrasov-Shatasvili. As an application it concluded with discussion of a new technique on proving the P=W result of [11].

Kai Behrend's talk *Categorification of Lagrangian intersections via deformation quantization* explained what the theory of deformation quantization has to say about Lagrangian intersections. He showed that already the case of first order quantization gives a non-trivial result: namely the construction of a canonical Batalin–Vilkovisky differential on the sheaves of extensions between two Lagrangian half-densities. Extrapolating to the infinite order case, he proposed a construction of a canonical twisted perverse sheaf on the intersection of two Lagrangians inside a complex symplectic manifold, which categorifies the intersection number if the intersection is compact.

Representing the physics side of the subject, **Lotte Hollands** gave a talk on *Surface defects and the superconformal index*. The talk explained how to employ the superconformal index, a very powerful tool in the study of four-dimensional $N = 2$ superconformal gauge theories, to study a general class of half-BPS surface defects in these theories. In her talk, she also made connections to a Verlinde algebra on the dual UV curve, and to four-dimensional N = 2* instanton partition functions in the presence of line operators.

Friday 7 June

The last day's talks discussed recent advances in Gromov–Witten and related theories. Among them, **Vincent Bouchard**'s talk *Mirror symmetry for orbifold Hurwitz numbers* showed that the mirror counterparts to orbifold Hurwitz numbers satisfy toplogical recursion, with spectral curve given by the "r-Lambert curve". He also argued that orbifold Hurwitz numbers can be obtained in the "infinite framing limit" of orbifold Gromov-Witten theory on a certain space.

Dusty Ross' talk was concerned with *The gerby Gopakumar–Marino–Vafa formula*, a result which expresses certain Hurwitz–Hodge integrals on moduli spaces of orbifold curves in terms of loop Schur functions. This leads in particular to the first class of examples of the orbifold Gromov–Witten/Donaldson–Thomas correspondence for targets with curve classes contained in the singular locus.

In memoriam Kentaro Nagao

The organizers and audience were very much looking forward to the announced talk by **Kentaro Nagao** entitled *Cut of a quiver with potential and the cohomological Hall algebra*. The abstract of this talk promised an explaination how the algebraic structure of the cohomological Hall algebra of a quiver with potential would help understand its DT type theory, and a new approach for the algebraic structure of CoHA in the case when the QP has a cut. Unfortunately, Kentaro was prevented from attending this meeting by a recurrence of a long-standing serious illness. Kentaro Nagao passed away not long after this meeting in October 2013.

4 Scientific Progress as a Result of the Workshop

Here we mention some specific conversations that took place at the workshop, and outcomes from these conversations.

Jim Bryan had several fruitful conversations with Dominic Joyce who explained how the local geometry of the moduli space of sheaves constrains the associated perverse sheaf of vanishing cycles. This has been a crucial input into Bryan's ongoing project (with his student Saman Gharib) to compute the Gopakumar-Vafa invariants of local curves via the perverse sheaf.

Dusty Ross had discussions with Jim Bryan and Ben Young at the workshop regarding combinatorial aspects of a research project which eventually appeared as the paper [45].

Dominic Joyce has discussions with a number of people, in particular with Sheldon Katz, which lead to improvements and a new section in the paper [7]. For Sheldon Katz, the talks by Bussi and Joyce were helpful in crystallizing ideas related to motivic DT invariants. Some of these ideas are included in his current work with Choi and Klemm, following up on the work he presented at BIRS. Joyce and Bussi also had several discussions with Kai Behrend about their closely related projects categorifying Lagrangian intersections.

Ben Young and Amin Gholampour continued their collaboration on an existing project on rank 2 DT theory. A paper is in preparation. Gholampour also had discussions with Dagan Karp on their ongoing project on Gromov–Witten theory.

Ben Davison and Balazs Szendroi had discussions with a number of people including Andras Szenes regarding their project on refined DT invariants and cluster positivity, the results of which were incorporated in their preprint [14] released soon after the meeting. Szendroi also had several discissions with Andrei Okounkov concerning higher rank refined DT invariants and geometric engineering. His research student Alberto Cazzaniga is currently working on these ideas.

Tamas Hausel among others discussed with Emanuel Diaconescu, Vivek Shende, Ben Davison and Andrew Morrison. With Diaconescu, following his talk, they discussed the details of the computations in [10], and in particular which computations can be made more rigorously with the techniques of Okounkov-Nekrasov described in Okounkov's talks. With Shende he discussed how to harmonize Shende's results with the P=W conjecture in wild non-abelian Hodge theory; some related arguments can be found in [46]. The discussions with Davison concerned a multiplicative version of the Kontsevitch-Soibelman cohomological Hall algebra, relating to conjectures of Hausel-Villegas [25]. This later appeared in [12]. Following his talk, Morrison explained his techniques to determine the asymptotics of the Behrend-Bryan-Szendroi [4] motivic DT invariants of $Hilb_n(C^3)$, which appeared in [38]. Morrison's work was also discussed in [26] .

Emanuel Diaconescu had very fruitful discussions with Jim Bryan, Tamas Hausel, Emmanuel Letellier, Alexei Oblomkov, Andrei Okounkov and Vivek Shende on mathematical aspects of geometric engineering, which helped shape the results of [10] and led to new open questions on equivariant K-theoretic Donaldson-Thomas invariants. Furthermore, discussions with Emmanuel Letellier and Tamas Hausel have led to new deformation invariance conjectures for stable pair invariants of local orbifold curves based on their results [24] on the cohomology of character varieties with marked points.

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