Geometric aspects of p-adic automorphic forms

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1 Overview of the field

The Langlands program is a vast network of conjectures, meant to provide a bridge between seemingly different areas of mathematics, such as representation theory, harmonic analysis and number theory. At its heart lies the conjectural correspondence between Galois representations and automorphic forms. Over the past two decades, there has been a spectacular amount of progress in the Langlands program, leading to the resolution of major open questions such as Fermat's Last Theorem, the Sato-Tate conjecture and Serre's conjecture. In addition, there have been recent breakthroughs, such as the work of Harris-Lan-Taylor-Thorne and Scholze associating Galois representations to classes in the cohomology of locally symmetric spaces for GL_n [**?**, **?**]. All of these developments have depended crucially on the *p-adic interpolation* of automorphic forms. Further studying p-adic and mod p automorphic forms is likely to lead to new advances in the field.

2 Recent developments and open problems

The Betti cohomology of locally symmetric spaces for GL_n with $\mathbb{Q}_p$ coefficients can be related to automorphic forms on GL_n, by work of Franke. Therefore, the mod p cohomology of the same locally symmetric spaces can naturally be thought of as a space of mod p automorphic forms. Recently, Scholze constructed Galois representations attached to these mod p classes [**?**]. These Galois representations are characterized by matching Frobenius eigenvalues and Satake parameters at unramified places. For applications, for example in order to prove new automorphy lifting theorems as Calegari and Geraghty propose [**?**], one often needs to know more subtle information about these Galois representations. One crucial missing piece of information would be that they satisfy *local-global compatibility* at places above p. When combined with the Calegari-Geraghty method, such a result could have striking consequences, such as modularity or even the Sato-Tate conjecture for elliptic curves over imaginary quadratic fields. (We remark that local-global compatibility at ramified places not dividing p is the subject of Varma's dissertation [**?**], and that she is currently working on the case $l = p$ in the context of [**?**].)

3 Scientific progress made

During this meeting, we focused on investigating a specific instance of local-global compatibility at places above p for torsion classes, namely the *ordinary* case. Our ultimate goal is to understand systems of Hecke eigenvalues occurring in the ordinary part of completed cohomology of locally symmetric spaces for GL_n

(as defined below) and prove that their associated Galois representations are ordinary at p. For a precise statement, see, for example, Conjecture 6.27 of [**?**].

This question turns out to be closely related to a natural question about p-adic automorphic forms on unitary (or symplectic) Shimura varieties. For simplicity, say that we are considering a unitary Shimura variety S for the group $G := U(n,n)$ defined over $\mathbb{Q}$. This example is extremely relevant to us as the cohomology of locally symmetric spaces for GL_n over an imaginary quadratic field contributes to the boundary cohomology of a unitary Shimura variety of this type.

As starting point, we note that (regular algebraic) classical automorphic forms can be thought of either as sections of automorphic vector bundles over S or as systems of Hecke eigenvalues occurring in the Betti cohomology of S. The two notions can be compared in this generality via Faltings' BGG spectral sequence (for elliptic modular forms this is just the classical Eichler-Shimura theory).

In the coherent setting, Hida showes that p-adic automorphic forms can also be realized as certain global sections over the *Igusa tower* $\mathcal{T}$ (by construction $\mathcal{T}$ is the universal space trivializing automorphic vector bundles over an open dense set of S, namely the ordinary locus). On the other hand, when working with Betti or étale cohomology, there is a natural definition of p-adic automorphic forms as classes which occur in the *completed cohomology* of locally symmetric spaces:

$$\tilde{H}^*(\mathbb{Z}_p) := \varprojlim_n (\tilde{H}^*(\mathbb{Z}/p^n)), \text{ for } \tilde{H}^*(\mathbb{Z}/p^n) := \varinjlim_{K_p} H^*(\mathcal{S}_{K^pK_p}, \mathbb{Z}/p^n),$$

where the direct limit is over all open compact subgroups $K_p \subseteq G(\mathbb{Q}_p)$.

One of the key ingredients in Scholze's construction of Galois representation is a p-adic comparison (almost) isomorphism between étale and coherent cohomologies of perfectoid spaces. In our context, say for simplicity with mod p coefficients, it relates $\tilde{H}^*(\mathbb{F}_p)$, thought of as the mod p étale cohomology of the perfectoid Shimura variety $\mathcal{S}_{K^p}$, to the mod p coherent cohomology of $\mathcal{S}_{K^p}$. A natural question is whether the two notions of ordinary classes (on the étale and on the coherent side) can be matched under this comparison.

More precisely, we investigated the following question. Let $\mathcal{S}_{K^p,1}$ be the $\Gamma_1(p^\infty)$-tower of Shimura varieties, and $H^{i,\mathrm{ord}}(\mathcal{S}_{K^p,1}, \mathbb{F}_p)$ denote the part of its cohomology where the U_p operator acts invertibly (we refer to this space as the ordinary part of its cohomology). Also let $H^{0,\mathrm{ord}}(\mathcal{T}, \mathcal{O}_\mathcal{T})$ denote the part of the space of global sections over the Igusa tower $\mathcal{T}$ where the U_p operator acts invertibly.

Conjecture 1. *Every system of Hecke eigenvalues which occurs in the space $H^{i,\mathrm{ord}}(\mathcal{S}_{K^p,1}, \mathbb{F}_p)$ is the reduction modulo p of a system of Hecke eigenvalues occurring in $H^{0,\mathrm{ord}}(\mathcal{T}, \mathcal{O}_\mathcal{T})$.*

We note that Hida proves that the systems of Hecke eigenvalues which occur in $H^{0,\mathrm{ord}}(\mathcal{T}, \mathcal{O}_\mathcal{T})$ correspond to classical automorphic forms on $U(n,n)$ which are ordinary at p. Their associated Galois representations are known to satisfy local-global compatibility at $l = p$ by work of Caraiani [**?**]. Conjecture **??** is therefore closely related to our main goal. As further motivation, we also remark that Conjecture **??** could be a first step towards a general overconvergent Eichler-Shimura theory.

In the case of modular curves, related results have been proven by Cais [**?**] and Wake [**?**]. Both their works heavily rely on the study of the geometry of integral models. Since integral models are harder to work with in the higher-dimensional setting when there is bad reduction, we instead approached this question by reformulating it in terms of the perfectoid Shimura variety $\mathcal{S}_{K^p}$ and its associated Hodge-Tate period domain $\mathcal{F}\ell_G$. During our focused research group, we understood a number of ingredients which should play a key part in the proof of Conjecture **??**.

3.1 The ordinary part of completed cohomology

In order to relate Hida's ordinary cohomology $H^{i,\mathrm{ord}}(\mathcal{S}_{K^p,1}, \mathbb{F}_p)$ to the mod p étale cohomology of the perfectoid Shimura variety $\mathcal{S}_{K^p}$ we used Emerton's theory of *ordinary parts* [**?**]. We checked that Emerton's ordinary parts functor, when applied to the completed cohomology $\tilde{H}^*(\mathbb{F}_p)$ recovers precisely the spaces $H^{*,\mathrm{ord}}(\mathcal{S}_{K^p,1}, \mathbb{F}_p)$. For a general Shimura variety, the relationship is via the Hochschild-Serre spectral sequence; this means we can reinterpret the spaces we are interested in as the equivariant cohomology of $\mathcal{S}_{K^p}$.

3.2 The geometry of the Hodge-Tate period domain

The Hodge cocharacter μ defining the Shimura variety S determines a parabolic subgroup $P_\mu \subseteq G$. The perfectoid Shimura variety $\mathcal{S}_{K^p}$ is equipped with a map of adic spaces

$$\pi_{HT} : \mathcal{S}_{K^p} \to \mathcal{F}\ell_G,$$

where $\mathcal{F}\ell_G$ is the adic space associated to the variety G/P_μ. (The fact that any Shimura variety of Hodge type admits such a period morphism is forthcoming joint work of Caraiani and Scholze.) The idea is that $\mathcal{S}_{K^pK_p}$ is a moduli space of abelian varieties equipped with extra structures; this map is induced by the Hodge-Tate filtration on the p-adic étale cohomology of the family of abelian varieties pulled back to $\mathcal{S}_{K^p}$. We call $\mathcal{F}\ell_G$ the *Hodge-Tate period domain*. A key property is that the map π_{HT} is equivariant for the natural action of $G(\mathbb{Q}_p)$ on both spaces.

To understand the ordinary part of completed cohomology, we studied an analogue of the U_p-operator acting on the equivariant cohomology of $\mathcal{F}\ell_G$. We showed that this operator contracts a large open subset of $\mathcal{F}\ell_G$ to the *anticanonical* locus - a subset of the ordinary locus which is particularly well-behaved. Moreover, certain preliminary computations lead us to expect that our U_p acts topologically nilpotently on the complement of this large open subset. These two observations together imply that ordinary systems of Hecke eigenvalues only contribute to the cohomology of the anticanonical locus.

3.3 The anticanonical tower and the Igusa tower

Both the anticanonical tower and the Igusa tower $\mathcal{T}$ live over the ordinary locus of the integral model of our Shimura variety S with hyperspecial level at p. Using the moduli-theoretic description of both spaces, we checked that the anticanonical tower can be identified with the perfection of the Igusa tower. This means that we can trace systems of Hecke eigenvalues which occur as global sections over the anticanonical tower to systems of Hecke eigenvalues corresponding to ordinary (classical) automorphic forms.

3.4 Logistics

In order to bring everyone up to speed with the various aspects of the questions described above, we scheduled a number of expository talks in the beginning of the meeting. The topics were Hida's theory of ordinary cohomology, Emerton's theory of ordinary parts, Scholze's theory of perfectoid spaces, his construction of the anticanonical tower over the ordinary locus of Siegel modular varieties, the Hodge-Tate period map and p-adic comparison theorems. In addition, as we made progress on various topics, we assembled a set of working notes, to which we all have access via an online repository and which we intend to refine in the coming months.

4 Outcome of the meeting

We are hopeful that the ideas outlined above should play a key part in the proof of Conjecture **??**. We are writing a joint paper exploring this topic as well as the related question of local-global compatibility at $l = p$ for ordinary torsion classes.

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