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Godement-Jacquet Theory Revisited

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BIRS Workshop

14w5001 "The Future of Trace Formulas"

Outline

1. History

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2. **Review of Godement-Jacquet theory**

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2. Review of Godement-Jacquet theory
3. Braverman-Kazhdan: the basic functions

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Some references

1. R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Springer LNM 260, 1972.
2. I. Satake. *Theory of spherical functions on reductive algebraic groups over p-adic fields*. Publ. IHES (18), 1963.
3. A. Braverman and D. Kazhdan. *γ-functions of representations and lifting*. GAFA special volume, Part I, 2000.
4. L. Lafforgue. *Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires*, `http://www.ihes.fr/~lafforgue/math/COURS2013.pdf`.
5. Casselman's notes on `http://www.math.ubc.ca/~cass` .
6. W.-W. Li, Basic functions and unramified local L-factors for split groups, arXiv:1311.2434.
7. The previous lectures!

An inaccurate history

Main concern

Understand the automorphic L-functions (meromorphic continuation, functional equation, bounds, etc.)

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- Abelian L-functions: Hecke.
- Paraphrase in terms of harmonic analysis on $\mathrm{GL}(1) \subset \mathbf{A}^1$: Matchett (1946), Tate (1950).
  - Zeta integrals + adélic setup,
  - Fourier transform on the additive $\mathbf{A}^1$,
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- Abelian L-functions: Hecke.
- Paraphrase in terms of harmonic analysis on $\mathrm{GL}(1) \subset \mathbf{A}^1$: Matchett (1946), Tate (1950).
  - Zeta integrals + adélic setup,
  - Fourier transform on the additive $\mathbf{A}^1$,
  - Poisson summation formula.
- Standard L-function for $\mathrm{GL}(n)$: Tamagawa+(Godement-Jacquet) ($<$ 1972). Idea:
  - embed $\mathrm{GL}(n) \subset M_n \simeq \mathbf{A}^{n^2}$;
  - Schwartz space of M_n, local and global;
  - zeta integrals + Fourier transform + Poisson formula.

  Includes the case of central simple algebras, i.e. inner forms of $\mathrm{GL}(n)$.

In the 60's, Satake, Shimura and Tamagawa considered the similitude groups such as

$$
\begin{tikzcd}
\mathrm{GSp}(2n) \arrow[r, hook, "\text{open, dense}"] \arrow[d, equal] & \mathrm{MSp}(2n) \arrow[d, equal] \\
\{g \in \mathrm{GL}(2n) : {}^*gg = \text{scalar}\} & \{g \in M_{2n} : {}^*gg = \text{scalar}\}
\end{tikzcd}
$$

where $g \mapsto {}^*g$ is the adjoint map w.r.t. some symplectic form. Here:

- $(\mathrm{MSp}(2n), \cdot)$ is a reductive algebraic monoid;
- $\mathrm{GSp}(2n)$ is its unit group.

$\mathrm{MSp}(2n)$ is no longer linear: it is a cone!

Try to imitate the previous construction over a non-archimedean local field F, with the Schwartz space replaced by $C_c^\infty(\mathrm{MSp}(2n, F))$. Say take $n = 2$.

Expectations

Obtain THE L-functions $L(s, \pi, \mathrm{spin})$ for the irreps of $\mathrm{GSp}(2n)$;
$\mathrm{spin} : \mathrm{GSpin}(5, \mathbb{C}) \to \mathrm{GL}(4, \mathbb{C})$ is the spinor representation.

Accidental isomorphism: it is also the standard representation of $\mathrm{GSp}(4, \mathbb{C})$.

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Problems

1. Where is the Fourier transform?
2. Zeta integrals do not yield the "correct" L-factor in the unramified case.

This program was largely forgotten until the works of Braverman-Kazhdan, L. Lafforgue, Ngô, Casselman...... after 1999.

Local functional equation

Consider

- F: local field,
- $\mathrm{GL}(n) \hookrightarrow M_n$,
- $\mathcal{S}(M_n)$: the space of Schwartz functions on $M_n(F) \simeq \mathbf{A}^{n^2}(F)$, viewed as functions on $\mathrm{GL}(n, F)$;
- for any $\phi \in \mathcal{S}(M_n)$ and $s \in \mathbb{C}$, let $\phi_s := |\det|^s \cdot \phi$,
- for any admissible representation (π, V) of $\mathrm{GL}(n, F)$, let $\pi_s := |\det|^s \otimes \pi$.

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Zeta integrals

Assuming absolute convergence (say $\mathrm{Re}(s) \gg 0$), zeta integrals are (essentially) matrix coefficients of

$$\pi(\phi_s) = \pi_s(\phi) : V \to V.$$

Fourier transform

Fix $\psi : F \to \mathbb{C}$: $\rightsquigarrow$ normalization of measures.

Definition

Let $K_0 := |\det|^n \cdot \psi(\mathrm{tr}(\cdot)) : M_n(F) \to \mathbb{C}$. Set

$$\mathcal{F}_0 : \phi \longmapsto |\det|^{-n} (K_0 * {}^\iota\phi)$$

for all $\phi \in \mathcal{S}(M_n)$, where ${}^\iota\phi(x) = \phi(x^{-1})$.

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Observation

Can check that

$$\mathcal{F}_0\phi : x \longmapsto \int_{M_n(F)} \psi(\mathrm{tr}(xy))\phi(y)\mathrm{d}y.$$

Hence $\mathcal{F}_0 : \mathcal{S}(M_n) \to \mathcal{S}(M_n)$.

Invariance of $\mathcal{K}_0$: it factors through the adjoint quotient

$$\mathrm{GL}(n) \twoheadrightarrow \mathrm{GL}(1)^n/\mathfrak{S}_n.$$

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$$\mathrm{GL}(n) \twoheadrightarrow \mathrm{GL}(1)^n / \mathfrak{S}_n.$$

Local functional equation

For π, ϕ as before,

$$\pi(\underbrace{\mathcal{K}_0 * {}^{\iota}\phi}_{=|\det|^n \cdot \mathcal{F}_0 \phi}) = \underbrace{\pi(\mathcal{K}_0)}_{\text{invariance} \implies \text{scalar}} \pi({}^{\iota}\phi) = \gamma(\pi, \psi)\check{\pi}(\phi).$$

Convergent for s in some half-plane, and $\gamma(\pi_s, \psi)$ has meromorphic continuation to all s.

Taking matrix coefficients yields the usual functional equation for local zeta integrals!

L-functions

Definition

Set $L\left(s - \frac{n-1}{2}, \pi\right)$ to be the gcd of $\{\pi_s(\phi) : \phi \in \mathcal{S}(M_n)\}$. Meromorphic in s.

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Remark. Define $K := \psi(\mathrm{tr}(\cdot))|\det|^{n/2}$ so that

$$\mathcal{F} : \phi \mapsto K * {}^{\iota}\phi$$

preserves $|\det|^{n/2}\mathcal{S}(M_n)$. Some advantages:

- the Fourier transform $\mathcal{F}$ looks cute;
- taking gcd $\left\{\pi(\phi) : \phi \in |\det|^{n/2}\mathcal{S}(M_n)\right\}$ yields the central value $L(\frac{1}{2}, \pi)$;
- similarly, we get the central value of γ-factors.

Justification

Look at the unramified case. Take

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Look at the unramified case. Take

$$\phi^\circ := \mathbf{1}_{M_n(\mathfrak{o}_F)} \in \mathcal{S}(M_n).$$

1. $\mathcal{F}_0 \phi^\circ = \phi^\circ$;
2. recall: $L(s, \pi) = \det(1 - c q_F^{-s})^{-1}$ for unramified π with Satake parameter $c \in \mathrm{GL}(n, \mathbb{C})_{\mathrm{ss}}/\mathrm{conj}$;
3. for unramified π, the restriction of $\pi_s(\phi^\circ)$ to the spherical vectors is the scalar $L(s - \frac{n-1}{2}, \pi)$;

Conclusion: zeta integrals yield the desired L-factors.

Summary

Local inputs

1. Schwartz space $\mathcal{S}(M_n)$. Requirement: the gcd of $\{\pi_s(\phi) : \phi \in \mathcal{S}(M_n)\}$ yields L-factors for $\mathrm{Re}(s) \gg 0$.
2. (Fourier transform $\mathcal{F}_0$) $\leftrightarrow$ (kernel $\mathcal{K}_0$) $\leftrightarrow$ (γ-factor $\gamma(\pi, \psi)$).

The ϵ-factors are secondary objects in this story!

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Global inputs

To obtain the global functional equation, we need the Poisson formula:

$$\sum_{\gamma \in M_n(F)} \phi(\gamma) = \sum_{\gamma \in M_n(F)} \mathcal{F}_0\phi(\gamma)$$

for F: global field, $\phi \in \mathcal{S}(M_n(\mathbb{A}_F))$ and normalized measures.

In [Satake 1963]:

- $\mathrm{GSp}(4) \hookrightarrow \mathrm{MSp}(4)$ instead of $\mathrm{GL}(n) \hookrightarrow M_n$;
- $\nu : \mathrm{GSp}(4) \to \mathrm{GL}(1)$ (the similitude character) instead of det;
- $\mathcal{S}(\mathrm{MSp}(4))$ instead of $\mathcal{S}(M_n)$, with $\phi^\circ := \mathbf{1}_{\mathrm{MSp}(4,\mathfrak{o}_F)}$ in the unramified case;
- the spinor L-function for GSp(4) instead of the standard L-function for $\mathrm{GL}(n)$.

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In the unramified case $\pi_s(\phi^\circ)$ does not give L-functions: the denominator matches but the numerator becomes messy!

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Question

To generalize Godement-Jacquet, one should begin by pinning down the Schwartz space $\ni \phi^\circ$ in the unramified case via unramified L-factors.

The setup of Braverman-Kazhdan (2000)

F: non-archimedean local field. Everything being unramified.

$$1 \to G_0 \to G \xrightarrow{\det_G} \mathrm{GL}(1) \to 1$$

where G is split and G_0 is its derived subgroup.
The dual side. Fix a "transfer representation" (ρ, V) so that

$$\begin{tikzcd} \hat{G} \arrow[r, "\rho"] & \mathrm{GL}(n, \mathbb{C}) \\ \mathrm{GL}(1, \mathbb{C}) \arrow[u, hook, "\widehat{\det_G}"] \arrow[r, equal] & \mathrm{GL}(1, \mathbb{C}) \arrow[u, hook] \end{tikzcd}$$

commutes. Let $K := G(\mathfrak{o}_F)$: hyperspecial.

Let $\hat{T}$ be the "universal maximal torus" of $\hat{G}$. The abstract Weyl group W acts on $\hat{T}$.

L-factor

Let π: unramified of Satake parameter $c \in \hat{G}_{\mathrm{ss}}/\mathrm{conj}$. Set

$$L(s, \pi, \rho) := \det(1 - \rho(c) q_F^{-s} | V)^{-1}$$

Then $L(0, \cdot, \rho)$ is rational on $\hat{G} /\!/ \hat{G} \simeq \hat{T}/W$.

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The basic function

We seek for $f_\rho : K \backslash G(F)/K \to \mathbb{C}$ so that

$$\mathrm{tr}(\pi(f_{\rho,s})) = \underbrace{\pi(f_{\rho,s}) = \pi_s(f_\rho)}_{\text{on spherical vectors}} = L(s, \pi, \rho)$$

for unramified π and $\mathrm{Re}(s) \gg 0$, where $\pi_s := |\det_G|^s \otimes \pi$ and $f_{\rho,s} := |\det_G|^s f_\rho$.

The Satake isomorphism $\mathscr{S} : C_c(K\backslash G(F)/K) \to \mathbb{C}[\hat{T}/W]$ is defined over $\mathbb{Z}[q_F^{\pm 1/2}]$.

Idea: try to invert $L(0, \cdot, \rho) \in \mathbb{C}(\hat{T}/W)$.

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$$\det(1 - \rho(c) q_F^{-s} | V) = \sum_{k \geq 0} \mathrm{tr}(\mathrm{Sym}^k \rho))(c) q_F^{-ks}$$

$$f_{\rho,k} := \mathscr{S}^{-1} \left(\mathrm{tr}(\mathrm{Sym}^k \rho))(\cdot) \right),$$

$$\mathrm{Supp}(f_{\rho,k}) \subset \left\{ x \in G(F) : v_F(\det_G(x)) = k \right\},$$

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Goal: a geometric description, cf. the Godement-Jacquet case.

- $X_*(T)_- = X^*(\hat{T})_-$ is the anti-dominant chamber (i.e. dominant for B^-), and $\leq$ is the anti-dominant Bruhat order;
- $\chi_\mu \in \mathbb{C}[\hat{T}/W]$ is the character of the irrep of $\hat{G}$ with extremal weight $\mu \in X_*(T)$;
- our q-partition function $\mathcal{P}(\cdot; q) \in \mathbb{Z}[q]$:

$$\prod_{\text{root } \alpha \underset{B}{>} 0} \left(1 - qe^{-\alpha^\vee}\right)^{-1} = \sum_\nu \mathcal{P}(\nu; q) e^\nu.$$

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Lusztig's q-analogue/Kostka-Foulkes polynomial

For $\mu, \lambda \in X_*(T)_-$,

$$\begin{aligned} m^\mu_\lambda(q) &:= \sum_{w \in W} (-1)^{\ell(w)} \mathcal{P}(w(\lambda + \check{\rho}_{B^-}) - (\mu + \check{\rho}_{B^-}); q) \\ &= q^{\langle \rho_{B^-}, \lambda - \mu \rangle} P_{n_\mu, n_\lambda}(q^{-1}) \quad \text{(KL-polynomials)} \end{aligned}$$

Kato's formula

For every $\lambda \in X_*(T)_-$,

$$\mathscr{S}^{-1}(\chi_\lambda) = \sum_{\substack{\mu \in X_*(T)_- \\ \mu \leq \lambda}} m_\lambda^\mu(q_F^{-1}) \delta_{B^-}^{\frac{1}{2}}(\mu(\varpi_F)) \mathbf{1}_{K\mu(\varpi_F)K}.$$

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Two apparent difficulties for applying this to f_ρ Take a look:

1. must know the decomposition of every $\mathrm{Sym}^k(\rho)$ — the usual recipe for $\otimes$-multiplicities (via crystal bases, etc.) does not apply;
2. must know the relevant m_λ^μ or the KL-polynomials.

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Question

Can we find out some structure in the coefficient of $\mathbf{1}_{K\mu(\varpi_F)K}$?

Ngô's conjectures

- Attach a reductive monoid M_ρ to ρ, containing G as its unit group (Vinberg's approach, assuming G_{der} simply connected, etc.)
- Consider the case $\mathfrak{o}_F = \mathbb{F}_q[\![t]\!]$.

Conjecture

The function f_ρ (or $f_{\rho,s}$ for some s...) comes from a certain $\mathrm{IC}(M_\rho(\mathfrak{o}_F))$ via the function-sheaf dictionary.

Note: M_ρ is usually singular!

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Aside

There is a such an interpretation of $m^\mu_\lambda(q)$ for a fixed λ (Kazhdan-Lusztig, 1980).

- For the case $G = \mathrm{GL}(n)$, $\rho = \mathrm{Std} : \mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C})$: the monoid is just M_n.
- For the case $G = \mathrm{GSp}(4)$, $\rho = \mathrm{spinor} : \mathrm{GSpin}(5, \mathbb{C}) \to \mathrm{GL}(4, \mathbb{C})$: the monoid should be $\mathrm{MSp}(2n)$ Take a look.

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The theory of reductive monoids was

1. initiated by Putcha and Renner around 1980,
2. taken up by Vinberg (1995), and
3. recast in the framework of spherical varieties by Rittatore, Brion et al. ($\geq$ 1998)

[Connected reductive monoids with unit group G] = [($G \times G$)-equivariant affine embeddings]!

Invariant-theoretic constructions

Geometric interpretation of m_λ^μ on the dual side: by the Dutch school (Hesselink, Broer... from the late 70's) and R. Brylinski (1989).
Setup. $\hat{B} = \hat{T}\hat{U}$: a Borel subgroup of $\hat{G}$.

$$\begin{tikzcd} & \hat{G} \overset{\hat{B}}{\times} \hat{\mathfrak{u}} \arrow[dl, "p"'] \arrow[dr, "\text{Springer resolution}"] & \\ \hat{G}/\hat{B} & & \hat{\mathfrak{g}}_{\mathrm{nil}} \end{tikzcd}$$

(all equivariant).

For $\lambda, \mu \in X^*(\hat{T}) = X_*(T)$, let

- $V(\lambda)$ be the irrep of $\hat{G}$ with extremal weight λ;
- $\mathscr{L}(\mu)$ be the $\hat{G}$-linearized invertible sheaf on $\hat{G}/\hat{B}$ attached to μ.

Note: $\mathbb{G}_m$ acts on $\hat{G} \overset{\hat{B}}{\times} \hat{\mathfrak{u}}$ by $(z, v) \mapsto z^{-1}v$ along the fibers of p.

Theorem

For $\mu, \lambda \in X_*(T)_-$, the $\mathbb{Z}_{\geq 0}$-graded ($\rightsquigarrow q$) Poincaré series of the "covariant"

$$\mathrm{Hom}_{\hat{G}}\left(V(\lambda), \Gamma\left(\hat{G} \overset{\hat{B}}{\times} \hat{\mathfrak{u}}, p^*\mathscr{L}(\mu)\right)\right)$$

equals $m_\lambda^\mu(q)$.

Crucial ingredient: a vanishing theorem due to Grauert-Riemenschneider-Kempf in characteristic 0.

Generalized Kostka-Foulkes polynomials

The q-partition function for the positive roots relative to $\hat{B}^-$ is defined by

$$\prod_{\text{roots } \alpha \underset{B}{>} 0} \left(1 - qe^{-\alpha^\vee}\right)^{-1} = \sum_{\nu} \mathcal{P}(\nu; q) e^{\nu}.$$

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Stembridge + (Panyushev (2010))

Replace $\left\{\text{roots } \alpha \underset{B}{>} 0\right\}$ by a multiset Ψ satisfying

1. Ψ contained in some strictly convex cone,
2. $\Psi =$ weights of a $\hat{B}$-stable subspace N of some finite-dimensional $\hat{G}$-module V.

Can well-define $\mathcal{P}_{\Psi}(\cdot; q)$ and $m^{\mu}_{\lambda,\Psi}(q)$! Recall

Again, we have equivariant morphisms

$$
\begin{tikzcd}
 & \hat{G} \overset{\hat{B}}{\times} N \arrow[dl, "p"'] \arrow[dr, "\text{“collapsing”}"] & \\
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Under some conditions on p, μ etc., $m^{\mu}_{\lambda,\Psi}(q)$ becomes the Poincaré series of

$$\mathrm{Hom}_{\hat{G}}\left(V(\lambda), \Gamma\left(\hat{G} \overset{\hat{B}}{\times} N, p^{*}\mathscr{L}(\mu) \right) \right)$$

(Details omitted)

Back to the basic function

Use Kato's formula to write

$$f_\rho = \sum_{\mu \in X_*(T)_-} c_\mu(q_F) \delta_{B^-}^{\frac{1}{2}}(\mu(\varpi_F)) \mathbf{1}_{K\mu(\varpi_F)K}$$

with $c_\mu(q) \in \mathbb{Z}[q^{-1}]$ expressed in terms of

- decomposition of $\mathrm{Sym}^k \rho$ ($k = \det_G(\mu)$),
- the Kostka-Foulkes polynomials $m_\lambda^\mu(q)$.

Recall: f_ρ and Kato's formula...

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Recall: f_ρ and Kato's formula...

To understand(?) c_μ, take

$$\Psi := \left\{ \alpha^\vee : \alpha \underset{B}{>} 0, \text{ root} \right\} \sqcup \{\text{weights of } \rho\}$$

to define generalized Kostka-Foulkes polynomials $m_{\lambda,\Psi}^\mu(q)$.

Theorem

For every $\mu \in X_*(T)_-$ *such that* $\det_G(\mu) \geq 0$,

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Theorem

$\mathbf{P}$ *is rational.*

Rationale. $\mathbf{P}$ is the Poincaré series of an affine variety $\mathrm{Spec}(\mathcal{Z}^{\hat{G}})$ with $\hat{T} \times \mathbb{G}_m$-action.

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The cone generated by Ψ defines a normal reductive monoid M_ρ by Luna-Vust theory (colors = simple coroots).

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- At the classical limit $q \to 1$:

$$\sum_\mu c_\mu(1)e^\mu = \sum_\mu \mathrm{mult}(\mathrm{Sym}(\rho)|_{\hat{T}} : \mu)e^\mu.$$

Get the weight-multiplicities of the symmetric algebra $\mathrm{Sym}(\rho)$ — not so hard...

- At the limit $q \to \infty$

$$\sum_\mu c_\mu(0)e^\mu = \sum_{\mu:\hat{B}^-\text{weights of }\mathrm{Sym}(\rho)} \mathrm{mult}(\cdots)e^\mu.$$

Get the decomposition of $\mathrm{Sym}(\rho)$ into irreducibles.

Questions

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- Finer structural properties about $f_{\rho,s}$ or **P**, such as functional equations? Cf. the case of $m_\lambda^\mu(q)$.
- Efficient computations? (Casselman, ...)

Outlooks

To further the Braverman-Kazhdan program:

1. Try to define the Schwartz space $C_c^\infty(G(F)) \subset \mathcal{S}_\rho \subset C^\infty(G(F))$ using our knowledge about f_ρ.
2. Try to define the "evaluation maps" $\mathcal{S}_\rho \ni f \mapsto f(\gamma)$, for $\gamma \in (M_\rho \smallsetminus G)(F)$. Needed for stating the Poisson formula for ρ.
3. The case $F = \mathbb{R}$? Define $\mathcal{S}_\rho$ by prescribing asymptotic behaviours. Try differential equations, etc.
4. Try to define the kernel $\mathcal{K}$ and the Fourier transform $\mathcal{F} : \mathcal{S}_\rho \to \mathcal{S}_\rho$. Maybe the real case is worth a try since the Subrepresentation Theorem furnishes explicit γ-factors.
5. Finally, formulate a precise Poisson formula for ρ.

And then... study those automorphic L-functions!

Bonus

Some other uses of basic functions (just to mention a few):

1. Plug into in the trace formula (cf. Matz's thesis).
2. L. Lafforgue's program — the Poisson formula plays a crucial rôle there.
3. And... the previous lectures.