Topological Recursion for Hitchin Fibrations and Quantum Curves

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Outline

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General Definitions and Notations

- Denote by C a smooth projective curve of **genus** $g = g(C)$ **at least** 2
- K_C denotes the canonical sheaf (sections are holomorphic one forms)
- Let E be a **holomorhic** vector bundle of rank r and degree d over C and $\phi \in H^0(C, End(E) \otimes K_C)$

Goal:

- Extend the theory of Eynard-Orantin in the context of Hitchin systems
- This procedure quantizes the spectral curves of Hitchin fibrations

Stable Higgs pairs

- Define (E, ϕ) a stable (semistable) **Higgs pair** if for any ϕ-invariant subbundle F of E then

  $$\deg F/rankF < \deg E/rankE$$

  ($\leq$ for semistable)
- Denote by $\mathcal{U}_C(r, d)$ the moduli space of stable vector bundles of rank r and degree d
- Denote by $\mathcal{H}_C(r, d)$ the moduli space of stable Higgs pairs (E, ϕ) of rank r and degree d
- $T^*(\mathcal{U}_C(r, d)) \subset \mathcal{H}_C(r, d)$ open dense, complement has codimen 2 or higher.

Spectral curve

The **Hitchin map**

$$\begin{array}{c} \mathcal{H}_C(r,d) \\ \Big\downarrow H \\ V^* := \oplus_{i=1}^r H^0(C, K_C^i) \end{array}$$

For a Higgs pair (E, Φ) let s denote the **spectral data**

$$s := ((-1)^i tr(\wedge^i \phi)) \in V^*$$

The Hitchin map sends

$$(E, \phi) \stackrel{H}{\to} s$$

The characteristic polynomial for each Higgs pair with spectral data s, defines a **spectral curve** denoted by Σ (it depends on s).

General Properties of Σ

- Σ is non-singular for generic s
- Σ is a curve inside the cotangent bundle T^*C of genus

$$g(\Sigma) = r^2(g-1)+1$$

- The fiber over a generic point

$$H^{-1}(s) = Jac(\Sigma)$$

- There is a degree r cover

$$\begin{array}{c} \Sigma \\ \big\downarrow{\scriptstyle \pi} \\ C \end{array}$$

Rank 2 simplification

We will focus next on $r = 2$ and $d = 0$.

- We reduce each (E, ϕ), by a symplectic change of coordinates, to $tr(\phi) = 0$.
- Notice

$$\begin{array}{c} \mathcal{H}_C(2,0) \\ \Big\downarrow H \\ V^*_{SL} := H^0(C, K^2_C) \ni s = s_2 \end{array}$$

is just $(E, \phi) \stackrel{H}{\to} det(\phi)$

- Identify the total space of the canonical bundle, K_C, with the cotangent bundle T^*C.
- There is a tautological 1-form $\eta \in H^0(T^*C, \pi^* K_C)$ on T^*C, defined by

$$\begin{array}{ccc} T^*C & \longleftarrow & \pi^* K_C \\ {\scriptstyle \pi}\big\downarrow & & \\ C & \longleftarrow & K_C. \end{array}$$

- $-d\eta$ is the natural holomorphic symplectic form on T^*C locally given by ydx.
- η gives a 1 form on the spectral curve Σ

$$i^*\eta \in H^0(\Sigma, K_\Sigma)$$

Properties of $i^*\eta$

- $i^*\eta$ is a holomorphic 1-form on Σ
- $i^*\eta$ has $2g(\Sigma) - 2 = 8g - 8$ zeros
- The zero divisor $i^*(\eta)$ of is just $C \cap \Sigma$
- The ramification points of π is the set $C \cap \Sigma$ denote it

$$C \cap \Sigma := \{P_1, \ldots, P_{4g-4}\}$$

- $i^*\eta$ has a double zero at each ramification point

Sympectic basis

- Choose a symplectic basis for $H_1(\Sigma, \mathbb{Z})$ and $H_1(C, \mathbb{Z})$
- Let $H_1(C, \mathbb{Z}) :=< A_1, \ldots, A_g, B_1, \ldots, B_g >$
- Choose A-cycles for Σ with

$$\pi_* a_j = \pi_* a_j^* = A_j$$

- Connecting ramification points P_{2i} and P_{2i+1} with an orientation completes the choices for A-cycles basis on Σ to

$$a_1, \ldots, a_g, a_1^*, \ldots, a_g^*, \alpha_1, \ldots, \alpha_{2g-3}$$

- Let σ be an **involution** of the cotangent bundle that is locally defined as

$$(x, y) \stackrel{\sigma}{\to} (x, -y)$$

- The involution acts on the tautological 1 form $\sigma(\eta) = -\eta$

Proposition (see Tata Lectures by Mumford)

On a compact Riemann surface, C, with distinct points a and b, there exists a unique meromorphic one form, $w^{a-b}(z)$, such that

1. $w^{a-b}(z)$ is holomorphic on C except at $z = a$ and $z = b$
2. $w^{a-b}(z)$ has a simple pole at a with residue 1
3. $w^{a-b}(z)$ has a simple pole at b with residue -1
4. $\oint_{A_i} w^{a-b}(z) = 0$ for every A cycle

Define

$$B_C(x_1, x_2) := d_{x_1} w^{x_1 - b}(x_2)$$

a 2-form on $C \times C$ with doble poles along the diagonal

Topological recursion

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ i\ } & \overline{T^*C} \\ & & \Big\downarrow \pi \\ & & C \end{array}$$

- The **topological recursion** is an inductive mechanism to construct meromorphic n-linear forms $W_{g,n}$ on the Hilbert scheme $\Sigma^{[n]} = \text{Hilb}^n(\Sigma)$ of n points on the curve Σ.
- We consider a ramified covering

$$\pi \circ i : \Sigma \longrightarrow C$$

with the ramification divisor $R \subset \Sigma$.

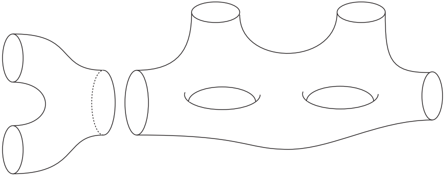
Topological recursion

1. $W_{0,1} := i^*\eta$.
2. $W_{0,2} := d_1\omega^{z_1-a}(z_2)$.
3. For $2g-2+n>0$, let $\tilde{z}=$ Galois conjugate with respect to the involution σ. Define

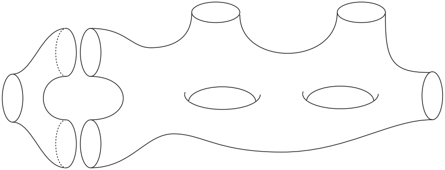
$$W_{g,n}(z_1,\ldots,z_n) := \frac{1}{2\pi i}\sum_{p\in R}\oint_{\gamma_p}\frac{\omega^{\tilde{z}-z}(z_1)}{\eta(\tilde{z})-\eta(z)} \times \Bigg[W_{g-1,n+1}(z,\tilde{z},z_2,\ldots,z_n) + \sum_{g_1+g_2=g,\, I\sqcup J=\{2,\ldots,n\}}^{\text{no (0,1) terms}} W_{g_1,|I|+1}(z,z_I)\,W_{g_2,|J|+1}(\tilde{z},z_J)\Bigg].$$

Topological Recursion = Degeneration on $\overline{\mathcal{M}}_{g,n}$

$$(g,n) \Longrightarrow (g, n-1)$$

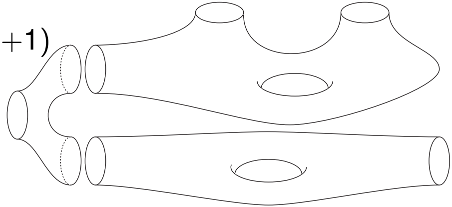


$$(g,n) \Longrightarrow (g-1, n+1)$$



$$(g,n) \Longrightarrow (g_1, n_1+1)+(g_2, n_2+1)$$

$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 - 1 \end{cases}$$



Proposition

1. $W_{g,n}(z_1, \ldots, z_n)$ are symmetric n-forms on Σ
2. $W_{g,n}(z_1, \ldots, z_n)$ has poles only at the ramification divisor R
3. $W_{g,n}(z_1, \ldots, z_i, \ldots, z_n) = -W_{g,n}(z_1, \ldots, \tilde{z}_i, \ldots, z_n)$ for every i and $2g - 2 + n \geq 1$

Define the functions $F_{g,n}(z_1, \ldots, z_n)$ on Σ^n subject to

1. $d_1 \ldots d_n F_{g,n} := W_{g,n}(z_1, \ldots, z_n)$
2. $F_{g,n}(z_1, \ldots, z_i, \ldots, z_n) = -F_{g,n}(z_1, \ldots, \tilde{z}_i, \ldots, z_n)$ for every i and $2g - 2 + n \geq 1$

Theorem 1 (D.-Mulase)

For $2g-2+n \geq 2$ the $F_{g,n}$ are uniquely determined by the following recursion

$$d_1 F_{g,n}(z_1,\dots,z_n) = -\sum_{j=2}^{n}\Big\{\frac{\omega_s^{z_j-\tilde{z}_j}(z_1)}{2\eta(z_1)}\cdot d_1 F_{g,n-1}\big(z_{[\hat{j}]}\big) - \frac{\omega_s^{z_j-\tilde{z}_j}(z_1)}{2\eta(z_j)}\cdot d_j F_{g,n-1}\big(z_{[\hat{1}]}\big)\Big\} - \frac{1}{2\eta(z_1)} d_{u_1} d_{u_2} \times \left[F_{g-1,n+1}\big(u_1,u_2,z_{[\hat{1}]}\big) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I) F_{g_2,|J|+1}(u_2,z_J) \right]\Bigg|_{\substack{u_1=z_1\\ u_2=z_1}}$$

Main Ingredients for Theorem

Using the **geometry of Riemann Surfaces** one easily estabilshes

- $W_{g,n}$ has poles at the ramification divisor R
- Right hand side has poles at R, z_i and $\tilde{z}_i$
- Any meromorphic function f on Σ satisfies

$$\sum Res(f) = 0$$

- This allows to **change contour of integration.** Integrate the $W_{g,n}$ to get the recursion for $F_{g,n}$.

We must remark that this is a global computation.

The spectral curve of the Hitchin system, $\mathcal{H}_C(2,0)$, is quantizable i.e.

Theorem 2 (D.- Mulase)

There is a second-order differential operator $P(x, \hbar d/dx)$ such that

- $$P\left(x, \hbar \frac{d}{dx}\right) \exp\left(\sum_{g,n} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}(x, \ldots, x)\right) = 0.$$
- The semi-classical limit of the above equation (the quantum curve) recovers the spectral curve $\Sigma \subset T^*C$.

Here, $P\left(x, \hbar \frac{d}{dx}\right) := (\hbar \frac{d}{dx})^2 + s_2(x)$

Main Ingredients of Proof

Idea: The Eynard-Orantin topological recursion gives a method to quantize the spectral curves

- First define $S_{m+1} := \sum_{2g-2+n=m} \frac{1}{n!} F_{g,n}$ and $F := \sum_{m=0}^{\infty} S_m \hbar^{m-1}$
- Using **principal specialization** one proves that the recursion of the free energies implies a recursion of S_m
- On the other hand by expansion as powers of $\hbar$ the quantum curve equation gives the same recursion provided that $S_0 := F_{0,1}$ and S_1 satisfies the consistency condition $\frac{d^2}{dx^2} S_0 + 2 \frac{d}{dx} S_0 \frac{d}{dx} S_1 = 0$. Its semi-classical approximation gives the spectral curve.

General Philosophy

Let us summarize the most important points:

1. This work reformultes the Eynard-Orantin theory that can be defined over a base curve, C, a smooth Riemann Surface of genus $g(C) \geq 2$
2. The spectral curve of the Eynard-Orantin theory is precisely the spectral curve of the Hitchin systems
3. The topological recursion should be considered globally; the recursion kernel and unstable differentials $W_{0,1}$ and $W_{0,2}$ are determined by the geometry of the spectral curve
4. The topological recursion gives a method to quantize the spectral curve of the Hitchin systems

Still a long way to go

- We extended topological recursion over arbitratry Riemann Surfaces; at this point we don't have any example of our theory
- Most examples computed in the topological recursion have the base curve C a rational curve
- Hitchin construction doesn't apply to a rational curve mainly because the pluricanonical bundle $K_{\mathbb{P}^1}^i$ has no sections

A More General Framework

- Consider any projective algebraic curve C and with n marked points p_j
- We denote divisor $D = \sum_{j=1}^{n} m_j p_j$ with $m_j > 0$
- An algebraic vector bundle $E \longrightarrow C$ of rank 2.
- A meromorphic Higgs field

$$\phi : E \longrightarrow E \otimes K_C(D)$$

with poles along the divisor D

Geometry of $\overline{T^*C}$

- The cotangent bundle T^*C is the total space of K_C
- The **compactified cotangent bundle** of C is a ruled surface over C defined by

$$\overline{T^*C} := \mathbb{P}\left(K_C \oplus \mathcal{O}_C\right)$$

- The tautological form η extends on $\overline{T^*C}$ as a **meromorphic** 1-form with simple poles along the divisor at infinity, call it s.
- We construct the spectral curve in the compactified cotangent bundle

$$\Sigma := \{\det(\eta - \phi) = 0\} \subset \overline{T^*C}.$$

Discriminant Divisor

Notice unlike the holomorphic case, meromorphic Higgs bundles can not be reduced to traceless ones; a different analysis is necessary. Since

$$\det(\eta-\phi)=\left(\eta-\frac{1}{2}\mathrm{tr}(\phi)\right)^2-\frac{1}{4}\mathrm{tr}(\phi)^2+\det(\phi),$$

we define

$$\Delta=\left(\frac{1}{4}\mathrm{tr}(\phi)^2-\det(\phi)\right)$$

as a divisor on C.

We also introduce an involution on $\overline{T^*C}$ by

$$\sigma:\left(x,y-\frac{1}{2}\mathrm{tr}(\phi(x))\right)\longmapsto\left(x,-y+\frac{1}{2}\mathrm{tr}(\phi(x))\right).$$

- Obviously $\deg K_C = 2g - 2$
- Let F denote the class of a fiber of $\pi : \overline{T^*C} \longrightarrow C$, and B the 0-section of T^*C.
- Notice $Pic(\overline{T^*C}) = \langle F, B \rangle$ where

$$F^2 = 0, FB = 1, B^2 = 2g - 2$$

- We identify the spectral curve, Σ, as a divisor in the ruled surface $\overline{T^*C}$.
- s is a divisor of the form $(2 - 2g)F + B$
- When the discriminant divisor Δ is reduced, Σ is non-singular

Spectral curve of a meromorphic Higgs field

- We present in detail the case when the discriminant divisor Δ is reduced
- We present some cases when Δ is not reduced.
- Note that $\Sigma \cdot s = n$ while $\Sigma \cdot F = 2$
- This identifies the spectral curve with the divisor

$$\Sigma = nF + 2B$$

- In particular, $\Sigma \cdot B = 4g - 4 + n$ i.e. Σ has $4g - 4 + n$ zeros

- The canonical line bundle of $\overline{T^*C}$ is

$$K_{\overline{T^*C}} = -2B + (4g-4)F$$

- Recall the arithmetic genus formula

$$p_a(\Sigma) = \frac{1}{2}\Sigma \cdot (\Sigma + K) + 1$$

- The spectral curve Σ has genus

$$p_a(\Sigma) = 4g - 3 + n$$

- We constructed a double cover

$$\Sigma \stackrel{\pi \circ i}{\to} C$$

- This map is simply ramified at $4g - 4 + 2n$ points (i.e., $|\Sigma \cdot s + \Sigma \cdot B|$)
- The intersection

$$|\Sigma \cap C| = 4g - 4 + n$$

- The involution of T^*C extends to an involution, σ, of the ruled surface $\overline{T^*C}$
- The divisor $\left(\eta - \frac{1}{2}\mathrm{tr}\phi\right)$ is invariant under σ and $\Sigma^\sigma = R$

Theorem

Theorem [D.-Mulase]

Under the above geometric setting Theorem 1 and Theorem 2 hold.

Classical example (= Hermite differential equation)

- The curve $C = \mathbb{P}^1$, the vector bundle $E = \mathcal{O}(2) \oplus \mathcal{O}$
- The Higgs field $\phi = \begin{pmatrix} 0 & 1 \\ -1(dx)^2 & -xdx \end{pmatrix}$,

where xdx is the unique 1-form on $\mathbb{P}^1$ with a simple zero at $x = 0$ and an order 3 pole at $x = \infty$. The spectral curve in the affine coordinate of the Hirzebruch surface $\overline{T^*\mathbb{P}^1} = \mathbb{F}_2$ is

$$y^2 + xy + 1 = 0,$$

which has a rational double point in $\mathbb{F}_2$ at $(x, y) = (\infty, \infty)$. We remark that ϕ is not traceless.

What is the spectral curve?

- The spectral curve, Σ, can be identified with a divisor

$$4F + 2B$$

with a mild singulrity; a double point

- Blow up the rational surface $\mathbb{F}_2$ at the singularity
- Let E be the exceptional divisor
- Denote by $\widehat{\Sigma}$ the proper transform of Σ

$$\widehat{\Sigma} = 4F + 2B - 2E$$

- $\widehat{\Sigma}$ is a smooth rational curve on $Bl\mathbb{F}_2$, a double cover of $\mathbb{P}^1$ with 2 ramification points.

What does this spectral curve compute for us?

The Main Theorem applies for this spectral curve. Its quantization is (the Hermite differential equation)

$$\left(\hbar^2 \frac{d^2}{dx^2} + \hbar x \frac{d}{dx} + 1\right) \Psi(x,\hbar) = 0.$$

$$\psi(x,\hbar) = \exp\left(\sum_{2g-2+n\geq -1} \frac{1}{n!} \hbar^{2g-2+n} F^C_{g,n}(x,\ldots,x)\right)$$

$$F^C_{g,n}(x_1,\ldots,x_n) = \sum_{\mu_1,\ldots,\mu_n>0} \frac{C_{g,n}(\mu_1,\ldots,\mu_n)}{\mu_1\cdots\mu_n} \prod_{i=1}^{n} x_i^{-\mu_i},$$

where $C_{g,n}(\mu_1,\ldots,\mu_n)$ is the Catalan number of genus g and n labeled vertices that counts the number of **cellular graphs.**

Catalan Recursion

The $(g, n) = (0, 1)$ case is the original Catalan numbers:

$$C_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m} = C_m.$$

$$\begin{aligned} C_{g,n}(\mu_1, \ldots, \mu_n) &= \sum_{j=2}^{n} \mu_j C_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_n) \\ &+ \sum_{\alpha+\beta=\mu_1-2} \Bigg[C_{g-1,n+1}(\alpha, \beta, \mu_2, \ldots, \mu_n) \\ &+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\}}} C_{g_1,|I|+1}(\alpha, \mu_I)\, C_{g_2,|J|+1}(\beta, \mu_J) \Bigg]. \end{aligned}$$

The $(g, n) = (0, 1)$ case reduces to the classical formula

$$C_m = \sum_{a+b=m-1} C_a\, C_b.$$

Quantization of the Catalan recursion

Start with the Catalan recursion

$$C_m = \sum_{a+b=m-1} C_a \, C_b.$$

Define a generating function (or the discrete **Laplace transform**)

$$y(x) = -\sum_{m=0}^{\infty} C_m \, x^{-(2m+1)}.$$

Then $y^2 + xy + 1 = 0$. Its quantization is the Hermite differential equation

$$\left(\hbar^2 \frac{d^2}{dx^2} + \hbar x \frac{d}{dx} + 1\right) \Psi(x, \hbar) = 0.$$

Geometry of the Airy function

- The curve $C = \mathbb{P}^1$ and the vector bundle $E = \mathcal{O}(2) \oplus \mathcal{O}$.
- The Higgs field $\phi = \begin{pmatrix} 0 & 1 \\ x(dx)^2 & 0 \end{pmatrix}$,

where $x(dx)^2$ is the unique quadratic differential on $\mathbb{P}^1$ with a simple zero at $x = 0$ and an order 5 pole at $x = \infty$. The spectral curve in the affine coordinate of $\overline{T^*\mathbb{P}^1} = \mathbb{F}_2$ is

$$y^2 - x = 0,$$

which has a quintic cusp singularity in the Hirzebruch surface $\mathbb{F}_2$ at $(x, y) = (\infty, \infty)$.
As a divisor on $\mathbb{F}_2$ it is just $5F + 2B$ with infinitely near singularity

The Airy differential equation = the simplest quantum curve

$$\left(\hbar^2 \frac{d^2}{dx^2} - x\right) Ai(x,\hbar) = 0.$$

What doest this compute?

$$Ai(x,\hbar) = \frac{1}{2\sqrt{\pi}} \exp\left(\sum_{2g-2+n\geq -1}^{\infty} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}(x) \right)$$

$$F_{g,n}(x) = \frac{(-1)^n}{2^{2g-2+n}} x^{-\frac{6g-6+3n}{2}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i - 1)!!$$

The geometric setting

- A meromorphic Higgs field with poles along a non-reduced divisor $D = \sum m_j p_j$, $m_j > 0$. The the spectral curve

$$\begin{array}{ccccc} \widehat{\Sigma} & \xrightarrow{\nu} & \Sigma & \xrightarrow{i} & \overline{T^*C} \\ & & & & \big\downarrow \pi \\ & & & & C \end{array}$$

- The **topological recursion** constructs meromorphic n-linear forms $W_{g,n}$ on the Hilbert scheme $\widehat{\Sigma}^{[n]} = \mathrm{Hilb}^n(\widehat{\Sigma})$ of n points on the normal curve $\widehat{\Sigma}$ for $g \geq 0$.
- We consider a ramified covering

$$\hat{\pi} = \pi \circ i \circ \nu : \widehat{\Sigma} \longrightarrow C$$

with the ramification divisor $R \subset \widehat{\Sigma}$.

Main Theorem

Main Theorem [D-Mulase]

Under the above geometric setting, the residues of the topological recursion applied to the non-singular $\widehat{\Sigma}$ can be globally calculated. As a consequence,

- There is a unique $F_{g,n}$ such that $d_1 \cdots d_n F_{g,n} = W_{g,n}$.
- There is a second-order differential operator $P(x, \hbar d/dx)$ such that

$$P\left(x, \hbar \frac{d}{dx}\right) \exp\left(\sum_{g,n} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}(x, \ldots, x)\right) = 0.$$

The semi-classical limit of the above equation (the quantum curve) recovers the spectral curve $\Sigma \subset \overline{T^*C}$.

Remark

1. This theorem provides the first mathematical examples of quantum curves defined on algebraic curves of arbitrary genera. The previously constructed examples are all defined on the rational curve.
2. The covering $\widehat{\Sigma} \longrightarrow C$ being Galois is important for the construction of the quantum curve.

Thank you for your attention!