Ancient solutions to geometric flows

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Outline

- We will discuss ancient or eternal solutions to geometric flows, that is solutions that exist for all time

$$-\infty < t < T \qquad \text{where} \quad T \in (-\infty, +\infty].$$

- Such solutions appear as blow up limits near a singularity.

  We will discuss:

- the problem classification of ancient or eternal solutions (mean curvature flow, Ricci flow, Yamabe flow), and
- methods of constructing new ancient solutions from the gluing of two or more self-similar solutions (Yamabe flow).

Ancient and Eternal solutions

- Definition: A solution to a parabolic equation is called ancient if it is defined for all time $-\infty < t < T$ and eternal if it is defined for all time $-\infty < t < +\infty$.
- The classification of ancient and eternal solutions often plays a crucial role in understanding the singularities of the flow.
- Solitons (self-similar solutions) are examples of ancient or eternal solutions and often models of singularities.
- However, there exists other interesting ancient or eternal solutions that are not solitons and they often can be visualized as obtained from the gluing of one or more solitons.
- Definition: An ancient solution is called of type I if:

$$\limsup_{t \to -\infty} \left(|t| \max_M R(\cdot, t) \right) < \infty.$$

Otherwise, it is called of type II.

Liouville type Theorem for the heat equation

- Let M be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}\,(M) \geq 0$.
- Yau - 1975: Any positive harmonic function u on M must be constant.
- This is the analogue of Liouville's Theorem for harmonic functions on $\mathbb{R}^n$.
- Question: Does the analogue of Yau's theorem hold for positive solutions of the heat equation

$$u_t = \Delta u \qquad \text{on } M?$$

- Answer: No. Example $u(x,t) = e^{x+t}$ on $M := \mathbb{R}^n$.

Liouville type Theorem for the heat equation

- Souplet and Zhang - 2006:

  (a) Let u be a positive ancient solution to the heat equation on $M \times (-\infty, T)$ such that

  $$u(x,t) = e^{o(d(x)+\sqrt{|t|})} \qquad \text{as } d(x) \to \infty.$$

  Then u is a constant.

  (b) Let u be an ancient solution to the heat equation such that

  $$u(x,t) = o(d(x) + \sqrt{|t|}) \qquad \text{as } d(x) \to \infty.$$

  Then u is a constant.

- Proof: By using a local gradient estimate on large appropriately scaled parabolic cylinders.

The sub-critical semi-linear heat equation

- Consider nonnegative solutions $u(t) \in H^1(\mathbb{R}^n)$ of equation

$$u_t = \Delta u + u^p \quad \text{on } \mathbb{R}^n \times (0, T), \quad 1 < p < \frac{n+2}{n-2}.$$

- u blows up in finite time T if $\lim_{t \to T} \|u(t)\|_{H^1(\mathbb{R}^n)} = +\infty$.
- Recall: $\|u\|_{H^1(\mathbb{R}^n)} := \big(\int_{\mathbb{R}^n} u^2 + |Du|^2 \, dx \big)^{1/2}$.
- Fillipas, Giga, Herrero, Kohn, Merle, Velazquez, Zaag etc: analyzed the blow up behavior behavior of the solution $u(t)$ near a singular point (a, T).
- Self-similar scaling:

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x - a}{\sqrt{T - t}}, \; s = -\log(T - t).$$

- Giga, Kohn - 1985: We have $\|w(s)\|_{L^\infty(\mathbb{R}^n)} \leq C$, $s > -\log T$.

The rescaled semi-linear heat equation

- The rescaled solution satisfies the equation

$$(\star) \qquad w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

- To analyze the blow up behavior of u we need to understand the long time behavior of w as $s \to +\infty$.
- If $s_k \to +\infty$ then (passing to a subsequence)

$$w_k(y,s) := w(y, s+s_k)$$

converges to an eternal solution W of equation $(\star)$.

- To understand the blow up behavior of u we need to classify the bounded eternal solutions of $(\star)$.

Eternal solutions of the semi-linear heat equation

- Consider a bounded nonnegative eternal solutions of equation

$$(\star) \quad w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p \quad \text{on } \mathbb{R}^n \times \mathbb{R}.$$

- Steady states: $w = 0$ or $w = \kappa$, with $\kappa := (p-1)^{-\frac{1}{(p-1)}}$.
- $\lim_{s \to \pm\infty} w(\cdot, s) =$ steady state.
- Solutions independent of y: $\phi(s) = \kappa(1 + e^s)^{-\frac{1}{(p-1)}}$.
- Theorem (Giga-Kohn '87 and Merle-Zaag '98)
  If w is bounded nonnegative eternal solution of $(\star)$ then

$$w = 0 \ \text{ or } \ w = \kappa \ \text{ or } \ w(s) = \phi(s - s_0).$$

Remaks

- The main main complexity of this result is the classification of the eternal solutions that connect the steady states $w_{-\infty} = \kappa$ and $w_{+\infty} = 0$. These solutions are shown to be independent of y, hence given by $\phi(s)$.
- Its proof is heavily involved and strongly relies on analyzing the behavior of the solution w near $\tau \to -\infty$ in terms of its projections on the positive, zero and negative eigenspaces of the linearized operator $Lv := \Delta v - \frac{1}{2} y \cdot \nabla v + v$ at $w = \kappa$.
- Note that in terms of $v := w - \kappa$, the equation takes the form

$$v_\tau = Lv + f(v)$$

with superlinear error $f(v) := (v + \kappa)^p - \kappa^p - p\kappa^{p-1} v$.

- Similar equations often arise in the analysis of singularities of neck-pinches in the mean curvature flow and the Ricci flow.

Ancient solutions to the curve shortening flow

- Let Γ_t be a family of closed curves which is an embedded solution to the Curve shortening flow, i.e. the embedding $F : \Gamma_t \to \mathbb{R}^2$ satisfies

$$\frac{\partial F}{\partial t} = -\kappa\,\nu$$

  with κ the curvature of the curve and ν the outer normal.

- Gage - 1984, Gage and Hamilton - 1996: if Γ_0 is convex, then the CSF shrinks Γ_t to a round point.
- Grayson - 1987: if Γ_0 is any embedded curve in $\mathbb{R}^2$, then the solution Γ_t to the CSF does not develop singularities before it becomes strictly convex.
- Problem: Classify the ancient compact embedded solutions to the Curve shortening flow.

The evolution of the curvature

- Let Γ_t is an ancient convex solution to the CSF which defined on $I = (-\infty, T)$ and shrinks to a point at T.
- The curvature κ of Γ_t evolves, in terms of its arc-length s, by

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

- If θ is the *angle between the tangent vector of* Γ_t *and a fixed direction* ω, then on convex curves $\kappa = \kappa(\theta, t)$ satisfies

$$\kappa_t = \kappa^2\, \kappa_{\theta\theta} + \kappa^3.$$

- We introduce the pressure function $p = \kappa^2$ which evolves by

$$p_t = p\, p_{\theta\theta} - \frac{1}{2}\, p_\theta^2 + 2\, p^2.$$

Examples and the Result

- Type I solution (contracting circles): $p(\theta, t) = \frac{1}{2(T-t)}$
- Type II solution (Angenent ovals): For all $\lambda > 0$ and $\theta_0 \in \mathbb{R}$:

$$p(\theta, t) = \lambda\left(\frac{1}{1 - e^{-2\lambda(T-t)}} - \sin^2(\theta + \theta_0)\right), \quad t < 0$$

  As $t \to -\infty$ the Angenent ovals look like two grim reaper solutions to the CSF glued together.

- Theorem (D., Hamilton, Sesum)
  The only ancient convex solutions to the CSF are the contracting spheres or the Angenent ovals.
- Proof: It is based on various monotonicity formulas and the fact that as $t \to T$ any solution becomes circular.

Non-Convex ancient solutions

- Question: Do they exist non convex compact embedded solutions to the curve shortening flow ?
- Angenent - 2011: Presents a YouTube video of an ancient solution to the CSF built out from one Yin-Yang spiral and one Grim Reaper.
- It has exactly two inflection points, until it becomes convex.
- Proof: at ttp://www.youtube.com/watch?v=8Ez0QoJ3XG8.
- Angenent is currently working on giving a rigorous construction of this !

Open problems - Mean Curvature flow

- Problem: Provide the classification of ancient convex compact solutions of the Mean Curvature flow in dimensions $n \geq 2$.
- Xu-Jia Wang - 2011: Classifies the blow-down in space-time as $t \to -\infty$ of any ancient convex solution to the MCF that sweeps the whole space.
- Huisken & Sinestrari; Haslhofer & Hershkovits 2013: Provide geometric conditions that characterize an ancient convex solution to the MCF as a family of contracting spheres.
- Angenent - 2012: Provides the detailed matched asymptotics of ancient convex compact solutions to MCF that near $t = -\infty$ are (after rescaling) small perturbations of an ellipsoid.
- B. White - 2003; Haslhofer & Hershkovits - 2013: Establish the existence of the Angenent ovals.

- Consider an ancient solution of the Ricci flow

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

  on S^2 that exists for all time $-\infty < t < T$ and becomes singular at time T.
- In dim 2, we have $R_{ij} = \frac{1}{2} R\, g_{ij}$, where R is the scalar curvature.
- R. Hamilton 1988, Chow 1991: After re-normalization, the metric becomes spherical at $t = T$.
- Problem: Classify the ancient compact solutions.

The equations for the conformal factor and the pressure

- Choose a parametrization $g_{S^2} = d\psi^2 + \cos^2\psi\, d\theta^2$ of the limiting spherical metric and parametrize the (RF) by this, i.e. we write $g(\cdot,t) = u(\cdot,t)\, g_{S^2}$.
- If $g(\cdot,t) = u(\cdot,t)\, g_{S^2}$, then the (RF) becomes equivalent to:

$$u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T).$$

- Assume from now on that $T = 0$.
- It is natural to consider the pressure function $v = u^{-1}$ which evolves by

$$\text{(PE)} \qquad v_t = v\,\Delta_{S^2} v - |\nabla v|^2 + 2v^2.$$

Examples of ancient solutions on S^2

- Example of a Type I solution (contracting spheres):

$$v(\psi, \theta, t) = \frac{1}{2(-t)}$$

- Example of a Type II solution is the King solution:

$$v(\psi, \theta, t) = -\mu\,[\coth(2\mu t) - \tanh(2\mu t)\,\sin^2\psi\,]$$

for $t < 0$, with $\mu > 0$.

- As $t \to -\infty$ the King solution looks like two cigar solitions glued together.

The King solutions

- We look for explicit ancient solutions $g_{ij} = u(\cdot, t)\, g_{S^2}$ of the (RF) that are different than the contracting spheres and become singular at time $T = 0$.
- It is simpler to work on $\mathbb{R}^2$ (via stereographic projection) setting $g_{ij} = \bar{u}\, g_{euc}$.
- The function $\bar{u}$ satisfies the logarithmic fast-diffusion equation

$$\text{(LFD)} \qquad \bar{u}_t = \Delta \log \bar{u} \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0).$$

- J.R. King - 1993 looked for radial solutions of equation (LFD) where the pressure function $\bar{v} := \bar{u}^{-1}$ is a polynomial function in $r := |x|$ with coefficients depending on t.

The King solutions

- A direct calculation shows that

$$\bar{v}(x,t) = a(t) + 2\,b(t)\,|x|^2 + a(t)\,|x|^4$$

where either $a(t) = b(t)$ (recovering the contracting spheres) or

$$a(t) = -\frac{\mu}{2}\,\mathrm{csch}(4\mu t), \quad a(t) = -\frac{\mu}{2}\,\coth(4\mu t)$$

for any $\mu > 0$.

- These solutions were independently discovered by P. Rosenau.
- The King solutions are not self-similar.
- As $t \to -\infty$ they look like two cigar (Barenblatt self similar) solutions glued together.

The classification result

Theorem: (D., Hamilton, Sesum)
An ancient solution to the (RF) on S^2 is either one of the contracting spheres or one of the King solutions.

Sketch of proof:

We recall that if $g(\cdot,t) = u(\cdot,t)\, g_{S^2}$ is the evolving metric, then the *pressure* $v = u^{-1}$ satisfies:

$$v_t = v\,\Delta v - |\nabla v|^2 + 2v^2 = R\,v > 0$$

- We show, by establishing sharp a'priori derivative estimates, that $v(\cdot,t) \xrightarrow{C^{1,\alpha}} v_\infty$, as $t \to -\infty$, for all $\alpha < 1$.
- Via a suitable Lyapunov functional we show that

$$R_\infty := \lim_{t\to-\infty} R(\cdot,t) = 0 \quad \text{a.e. on } S^2$$

and

$$v_\infty\,\Delta v_\infty - |\nabla v_\infty|^2 + 2v_\infty^2 = 0 \quad \text{a.e. on } S^2$$

Sketch of proof - Continuation

- We next classify the steady states v_∞ which satisfy

$$v_\infty \, \Delta v_\infty - |\nabla v_\infty|^2 + 2 v_\infty^2 = 0.$$

Main Step: We show that v_∞ has at most two zeros.
We conclude that

$$v_\infty(\psi, \theta) = C \, \cos^2 \psi, \qquad \text{for } C \geq 0$$

namely that the pointwise limit as $t \to -\infty$ is a cylinder.

- If $C = 0$, then v must be a family of contracting spheres.
- If $C > 0$, then v must be one of the King solutions.

The characterization of King solutions

- To capture the King solutions we consider the scaling invariant nonotone quantity

$$Q(x,y,t) := \bar{v}\left[\left(\bar{v}_{xxx} - 3\bar{v}_{xyy}\right)^2 + \left(\bar{v}_{yyy} - 3\bar{v}_{xxy}\right)^2\right]$$

where $\bar{v} := \bar{u}^{-1}$ is the pressure in plane coordinates.

- Using complex variable notation $z = x + iy$, this quantity is nothing but

$$Q = \bar{v}\,|\bar{v}_{zzz}|^2.$$

- The quantity Q is well defined.
- It turns out that $Q \equiv 0$ implies that $\bar{v}$ is one of the King solutions.
- To establish that $Q \equiv 0$ we prove that:
    i. $Q_{\max}(t)$ is decreasing in t (by considering its evolution equation), and
    ii. $\lim_{t\to-\infty} Q_{\max}(t) = 0$.

The 3 dimensional Ricci flow - Open problems

- 3-dim Ricci flow: The equivalent to the King solutions have been shown to exist by Perelman. These solutions are Type II ancient solutions and they are k non-collapsing.
- Other compact solutions in closed form have been found by V.A. Fateev in a paper dated back to 1996.
- Conjecture: The only k non-collapsing ancient compact solutions to the 3-dim Ricci flow are the contracting spheres and the Perelman solutions. In particular they are radially symmetric.
- Brendle, Huisken, Sinestrari - 2011: Provided a pinching curvature condition that characterizes ancient solutions to the 3-dim Ricci as contracting spheres.

Ancient solutions to the Yamabe flow

- Consider an ancient solution g_{ij} of the Yamabe flow $(n \geq 3)$

$$\frac{\partial g_{ij}}{\partial t} = -R\, g_{ij} \qquad \text{on} \ -\infty < t < T$$

  where the evolving metric g_{ij} is conformally equivalent to the standard metric on S^n , namely $g_{ij} = u^{\frac{4}{n+2}}\, g_{S^n}$.

- Question: Can we classify all such ancient solutions ?

The Yamabe flow - Background

- Let (M, g_0) be a compact manifold without boundary of dimension $n \geq 3$. If $g = v^{\frac{4}{n-2}}\, g_0$ is a metric conformal to g_0, the scalar curvature R of g is given in terms of v and the scalar curvature R_0 of g_0 by

$$R = -v^{-\frac{n+2}{n-2}} \left(\bar{c}_n \Delta_{g_0} v - R_0\, v \right).$$

- In 1989 R. Hamilton introduced the Yamabe flow

$$\frac{\partial g_{ij}}{\partial t} = -R\, g_{ij}.$$

  as a parabolic approach to solve the Yamabe problem.
- Brendle - 2007: convergence of the normalized flow to a metric of constant scalar curvature (under some mild technical assumptions in dimensions $n \geq 6$).
- Previous important works: Hamilton '89, Chow '92, Ye '94, del Pino-Saez '2001, Schwetlick-Struwe '2003.

Ancient compact solutions to the Yamabe flow

- King '93: found non-self similar ancient compact solutions to the Yamabe flow in closed form. As $t \to -\infty$ they converge to two cigar solutions joined with a long cylindrical neck.
- They resemble the King Ricci flow solutions, but they are of type I.
- Question 1:
  Are the contracting spheres and the King solutions the only Type I ancient solutions of the Yamabe flow (YF) that are conformal to the standard spherical metric on S^n ?
- Question 2:
  Are there any Type II ancient solutions ?

Towers of bubbles

- The following is joint work with M. del Pino and N. Sesum.
- We construct a class of rotationally symmetric ancient solutions
$$g_{ij} = u^{\frac{n+2}{4}}(\cdot, t)\, g_{S^n}$$
of the Yamabe flow on $-\infty < t < 0$ that become singular at $t = 0$ and near $t = -\infty$ they look (after re-normalization) like n spheres jointed by short necks.
- The curvature operator of these solutions changes sign near $t = -\infty$ and they are Type II ancient solutions.
- We refer to them as towers of bubbles.
- Our construction can be viewed as a parabolic analogue of the gluing of manifolds of constant scalar curvature or constant mean curvature that has been used in the past in various elliptic settings.

The equations

- Notation: From now on we will set $m = \frac{n-2}{n+2}$ and $p = \frac{n+2}{n-2}$.
- Express, for the moment, the metric g_{ij} as $g_{ij} = \bar{u}^{\frac{n+2}{4}} \, g_{euc}$, where g_{euc} denotes the standard Euclidean metric.
- Our problem is equivalent to constructing a radially symmetric solution $\bar{u}(|x|, t)$ of the fast-diffusion equation

$$\text{(FD)} \qquad \bar{u}_t = \Delta \bar{u}^m \qquad \text{on } \mathbb{R}^n \times (-\infty, 0)$$

  with the right asymptotic behavior as $r = |x| \to \infty$.
- The function $\hat{u} := u^m$ satisfies the equation

$$\hat{u}_t^p = \Delta \hat{u} \qquad \text{on } \mathbb{R}^n \times (-\infty, 0)$$

The equation in cylindrical coordinates

- We next introduce cylindrical coordinates setting

$$v(x,\tau) = (-t)^{-\frac{1}{p-1}} r^{\frac{2}{p-1}} \hat{u}(r,t), \;\; r = e^x, \; t = 1 - e^{-\tau}$$

- In these coordinates the equation becomes

$$(v^p)_\tau = v_{xx} + \alpha v^p - \beta v$$

  for certain constants $\alpha > 0$ and $\beta > 0$, depending on dimension n. It is defined for all $(x,\tau) \in \mathbb{R} \times \mathbb{R}$.

- By scaling away the constants, we may assume that $\alpha = \beta = 1$.

The equation in cylindrical coordinates

- Consider from now on our equation in cylindrical coordinates

$$(E) \qquad (u^p)_t = u_{xx} - u + u^p, \quad \text{on } \mathbb{R} \times (-\infty, +\infty)$$

  for some $t_0 \in \mathbb{R}$ and $p = \frac{n+2}{n-2}$.

- Notation: For simplicity we denote τ by t.

- The function $w(x) = \left(\frac{2\,k_n}{e^{\beta x} + e^{-\beta x}}\right)^{1/\beta}$ with $\beta = \frac{2}{n-2}$ and $k_n = (\frac{n}{n-2})^{1/2}$ solves the steady-state equation

$$w_{xx} - w + w^p = 0, \qquad x \in \mathbb{R}$$

  and represents the conformal factor for the standard metric on the unit sphere in cylindrical coordinates.

The ansatz of the non-linear problem

- We construct an ancient solution of (E) of the form

$$u(x,t) = (1+\eta(t))\, z(x,t) + \psi(x,t)$$

  for a suitable small parameter function $\eta(t)$, where

$$z(x,t) = w(x-\xi(t)) + w(x+\xi(t))$$

  and $\psi(x,t) \to 0$ as $t \to -\infty$ in a suitable sense.

- The function $\xi(t)$ is given by

$$\xi(t) = \xi_0(t) + h(t), \quad \xi_0(t) \approx \frac{1}{2}\log(2b|t|)$$

  for a suitable $b > 0$ and a small parameter function $h(t)$.

- Both parameter functions $h(t)$ and $\eta(t)$ will decay in $|t|$, as $|t| \to -\infty$ in a suitable sense.

The ansatz of the non-linear problem

- We notice that $z(\cdot, t)$ is an even function of x and we will impose the condition that $\psi(\cdot, t)$ is an even function as well.
- A direct computation shows that equation (E) for u is equivalent to equation

$$(\star_{NL}) \qquad p z^{p-1}\,\psi_t = \psi_{xx} - \psi + p z^{p-1}\,\psi + z^{p-1}\,E(\psi, h, \eta)$$

  for the unknown perturbation function $\psi(x, t)$.
- Our goal is to construct an ancient solution ψ of the nonlinear equation $(\star_{NL})$ via the fixed point Theorem.
- To this purpose we will first need to invert the corresponding linear problem for any given parameter functions h and η in suitable spaces.

The linear problem

- We first establish the existence of an ancient solution ψ to

$$(\star_L) \qquad p z^{p-1} \psi_t = \psi_{xx} - \psi + p z^{p-1} \psi + z^{p-1} g$$

  in suitable Banach spaces.

- Recall that $z(x,t) = w(x - \xi(t)) + w(x + \xi(t))$.
- We will consider a class of even functions g that decay both in x and t at suitable rates and satisfy certain orthogonality conditions.
- We build an even ancient solution $\psi := T(g)$ of equation $(\star_L)$ such that

$$\|\psi\|_{\star\star} \leq C \, \|g\|_{\star}$$

  and shares the same decay rates and orthogonality conditions.

- $\| \cdot \|_{\star\star}$ in an appropriate weighted L^2 and $W^{2,p}$ norm.

The orthogonality conditions

- Recall our equation

$$(\star_L) \qquad p z^{p-1} \psi_t = \psi_{xx} - \psi + p z^{p-1} \psi + z^{p-1} g$$

on $\mathbb{R} \times (-\infty, t_0]$, for some $t_0 \in \mathbb{R}$.

- We say that $g \in \bar{\mathcal{S}}$, if g satisfies the orthogonality conditions

$$\int_{\mathbb{R}} g(y \pm \xi(t), t)\, w'(y)\, w^{p-1}\, dy = 0, \qquad a.e. \quad t < t_0$$

$$\int_{\mathbb{R}} g(y \pm \xi(t), t)\, w(y)\, w^{p-1}\, dy = 0, \qquad a.e. \quad t < t_0.$$

- Notice that since g is assumed to be an even function in x, the above orthogonality conditions also imply the symmetric conditions $f(y,t) := g(y + \xi(t)) \in \mathcal{S}$.

Back to the non-linear problem

- Our goal is to establish the existence of an ancient solution of the non-linear equation

$$(\star_{NL}) \qquad p z^{p-1}\,\psi_t = \psi_{xx} - \psi + p z^{p-1}\,\psi + z^{p-1}\,E(\psi, h, \eta)$$

on $\mathbb{R} \times (-\infty, t_0]$, for t_0 sufficiently close to $-\infty$, such that $\psi(\cdot, t) \to 0$, as $t \to -\infty$ in a suitable sense.

- This is equivalent to $u := (1+\eta)\,z(x,t) + \psi(x,t)$, with

$$z(x,t) = w(x-\xi) + w(x+\xi), \;\; \xi = \xi_0 + h$$

being the desired ancient solution of the Yamabe flow.

- We will see that $\xi_0(t) \approx \frac{1}{2}\log(2b|t|)$, as $t \to -\infty$.

The auxiliary problem

- We will establish the existence of ψ through an iteration process, where at each step ψ_{k+1} we will be the solution of the linear equation with right hand side $E(\psi_k, h, \eta)$.
- Hence it is essential that ψ_k, $E(\psi_k, h, \eta) \in \bar{\mathcal{S}}$, for all k.
- Because of this, we will first consider the auxiliary system

$$(\star_A) \quad pz^{p-1}\,\psi_t = \psi_{xx} - \psi + pz^{p-1}\,\psi + z^{p-1}\,\bar{E}(\psi, h, \eta)$$

where $\bar{E}(\psi, h, \eta) := E(\psi, h, \eta) - (c_1\,z + c_2\,\bar{z})$ and $c_1 := c_1(t)$, $c_2 = c_2(t)$ are chosen so that

$$\bar{E}(\psi, h, \eta) \in \bar{\mathcal{S}}, \quad \text{if } \psi \in \bar{\mathcal{S}}$$

Recall $z = w(x-\xi) + w(x+\xi)$, $\bar{z} = w'(x-\xi) + w'(x+\xi)$.

Solving the nonlinear problem

- For any given (h, η), we establish the existence of a solution $\psi = \Psi(h, \eta)$ of the auxiliary equation where $c_1(t)$ and $c_2(t)$ are chosen so that

$$\bar{E}(\psi, h, \eta) := E(\psi, h, \eta) - (c_1(t)\, z + c_2(t)\, \bar{z})$$

  satisfies our orthogonality conditions whenever ψ does.
- Thus $\psi = \Psi(h, \eta)$ will define a solution to our original equation if we manage to adjust the parameter functions (h, η) in such a way that $c_1 \equiv 0$ and $c_2 \equiv 0$.
- This is equivalent to choosing (h, η) so that

$$\int_{\mathbb{R}} E(\psi)\, w^p(x + \xi(t))\, dx = 0$$

  and

$$\int_{\mathbb{R}} E(\psi)\, w'(x + \xi(t))\, w^{p-1}(x + \xi(t))\, dx = 0.$$

Solving for h and η

- We conclude that (ξ, η) with $\xi = \xi_0 + h$ satisfy the system

$$(\star\star_1) \quad \dot\eta - \frac{p-1}{p}\eta - \alpha\, e^{-2\xi} = \mathcal{R}_1(\xi, \dot\xi, \eta, \dot\eta)$$

$$(\star\star_2) \qquad \dot\xi + \beta\, e^{-2\xi} = \mathcal{R}_2(\xi, \dot\xi, \eta, \dot\eta).$$

- We observe that the ODE $\dot\xi + \beta\, e^{-2\xi} = 0$ admits the explicit solutions

$$\xi_0(t) := \frac{1}{2}\log(2b|t-\lambda|), \quad \text{for any } \lambda \in \mathbb{R}$$

  where λ is a translating parameter (dilation in the original variables).

- We conclude the existence of $\xi = \xi_0 + h$ and η via a fixed point Theorem.

The Main Result

- The previous discussion leads to the final result:

  Theorem. (*D., M. del Pino, N. Sesum*)

  There exist radially symmetric ancient solutions $g_{ij} = u\, g_{_{cyl}}$ of the Yamabe flow of the form

  $$u(x,t) = (1+\eta)\,[\,w(x-\xi) + w(x+\xi)\,] + \psi(x,t)$$

  with

  $$\xi := \frac{1}{2}\log(2b\,|t|) + h(t)$$

  for suitable parameter functions h and η that decay in $|t|$ as $t \to -\infty$ at a certain rate. This ancient solution is of Type II and its curvature operator changes sign.

- Remark: The result in the previous theorem generalizes to any finite number of k-bubbles.

Discussion

- The appearance of the towers of bubbles indicates that the problem of the classification of all ancient solutions to the Yamabe flow on S^n is delicate if not impossible.
- Question: What are the Type I ancient solutions ?
- Question: Are the contracting spheres and the King solutions the only type I ancient solutions ?
- Question: Are all Type I ancient solutions rotationally symmetric ?

Predictions - The new King solutions

- Question: Are the contracting spheres and the King solutions the only type I ancient solutions ?
- D., King and Sesum: There exist a two parameter family of self-similar solutions $g_\gamma := v_{\gamma,\lambda}^{\frac{4}{n+2}}\, dx^2$ to the Yamabe flow on $\mathbb{R}^n$ that they are complete, non-compact and all behave as cylinders as $|x| \to \infty$.
- $\gamma = 2$ corresponds to the explicit Barenblatt solutions.
- Conjecture: One may construct new ancient King solutions such that as $t \to -\infty$ look as two g_γ solutions joint with a neck.