A priori estimates, existence and Liouville theorems for semilinear elliptic systems with power nonlinearities

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Geometric Aspects of Semilinear Elliptic and Parabolic Equations:
Recent Advances and Future Perspectives

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Motivation

Steady states of R-D systems or standing waves of NLS often solve semilinear elliptic systems with power nonlinearities.
Typical example (nonlinear optics, Bose-Einstein condensates):

$$\left.\begin{aligned} -\Delta u + \omega u &= \lambda u^3 + \beta u v^2 \\ -\Delta v + \omega v &= \lambda v^3 + \beta u^2 v \end{aligned}\right\} \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1}$$

A priori estimates of positive solutions based on scaling arguments require information on the corresponding scaling invariant system

$$\left.\begin{aligned} -\Delta u &= \lambda u^3 + \beta u v^2 \\ -\Delta v &= \lambda v^3 + \beta u^2 v \end{aligned}\right\} \quad \text{in } \mathbb{R}^n. \tag{2}$$

- $\lambda > 0$: Using the variational structure, an optimal Liouville theorem for (2) with general power p was obtained for $n < 5$ in [Q., Souplet 2012]
- $\lambda \leq 0$, $\beta > 0$: Using the structural condition $(f-g)(u-v) \leq 0$ one can show that positive solutions of (2) satisfy $u \equiv v$ [Q., Souplet 2012; Montaru, Souplet, Sirakov 2014; Farina 2014]

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General setting

Under the structural conditions mentioned above, some of the results remain true for more general systems of the form

$$\left.\begin{aligned} -\Delta u &= \lambda u^r v^p + \beta u^{r_1} v^{p_1}, \\ -\Delta v &= \lambda u^q v^s + \beta u^{q_1} v^{s_1}. \end{aligned}\right\} \tag{3}$$

We will consider (3) without any structural assumption.
For simplicity, consider first

$$\left.\begin{aligned} -\Delta u &= u^r v^p, \\ -\Delta v &= u^q v^s. \end{aligned}\right\} \tag{4}$$

Known Liouville theorems for (4) and existence proofs based on scaling usually use **cooperativity** and require **subcriticality conditions** and additional **technical assumptions**, e.g.

$$\Omega \text{ convex}, \quad pq > (1-r)(1-s), \quad \min(p+r, q+s) \geq 1.$$

[Felmer 1994, Mitidieri 1996, Mitidieri, Pohozaev 1999, Bidaut-Véron 2000, Reichel, Zou 2000, Clément, Fleckinger, Mitidieri, de Thélin 2000, Bidaut-Véron, Pohozaev 2001, Zou 2006-2009, Bidaut-Véron, Giacomini 2010, Armstrong, Sirakov 2011, Wang, Hong 2012, ...]

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Our approach enables us to prove a priori estimates and existence for

$$\left.\begin{array}{l}\left.\begin{array}{r}-\Delta u = u^r v^p \\ -\Delta v = u^q v^s\end{array}\right\} \quad \text{in } \Omega \\ \qquad u = v = 0 \qquad\qquad \text{on } \partial\Omega\end{array}\right\} \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is smooth and bounded, $p, q, r, s \geq 0$,
without the additional **technical assumptions**;
it also works for many **noncooperative** systems.

We only need:

– a **nondegeneracy condition** (which is also necessary)
– **subcriticality conditions** (optimal for very weak solutions)

Basic ideas: Advanced bootstrap arguments (using estimates on auxiliary functions $u^a v^{1-a}$) and nonstandard homotopies.

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Nondegeneracy condition

$$pq \neq (1-r)(1-s) \tag{6}$$

Subcriticality conditions

$$\left.\begin{array}{r} r,\ s,\ \min(p+r, q+s) < p^* := \frac{n+1}{n-1}, \\ \text{if } \ pq > (1-r)(1-s) \ \text{ then } \ \max(\alpha, \beta) > \frac{2}{p^*-1} \end{array}\right\} \tag{7}$$

where

$$\alpha := 2\frac{p+1-s}{pq-(1-r)(1-s)}, \qquad \beta := 2\frac{q+1-r}{pq-(1-r)(1-s)}$$

are the scaling exponents:

If u, v are solutions of the equations in (5) then so are

$$U(x) = \lambda^\alpha u(\lambda x), \quad V(x) = \lambda^\beta v(\lambda x).$$

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Very weak solutions:

$$u, v \in L^1(\Omega), \quad u^r v^p,\ u^q v^s \in L^1(\Omega, \delta), \quad \delta(x) := \mathrm{dist}\,(x, \partial\Omega),$$

$$\left.\begin{array}{l} \displaystyle\int_\Omega u(-\Delta\varphi)\,dx = \int_\Omega u^r v^p \varphi \\ \displaystyle\int_\Omega v(-\Delta\varphi)\,dx = \int_\Omega u^q v^s \varphi \end{array}\right\} \quad \forall \varphi \in C^2(\overline{\Omega}),\ \varphi|_{\partial\Omega} = 0$$

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Theorem 1 [Q. 2014]

Assume (6) *and* (7). *Then there exists a positive classical solution of* (5). *In addition, there exists* $C = C(\Omega, n, p, q, r, s) > 0$ *such that* $\|u\|_\infty + \|v\|_\infty < C$ *for any positive very weak solution of* (5).

Remark: The critical exponent p^* in (7) can be replaced by $p^{**} := \frac{n}{n-2}$ if Ω is convex and we consider classical solutions only.

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Idea in the scalar case

$$\left.\begin{aligned} -\Delta u = u^p \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned}\right\}$$

Assume $p > 1$.
Choosing $\varphi := \varphi_1$ in $\quad \int_\Omega u(-\Delta\varphi)\,dx = \int_\Omega u^p\varphi\,dx \quad$ yields

$$\lambda_1 \int_\Omega u\varphi_1\,dx = \int_\Omega u^p\varphi_1\,dx \geq C\Big(\int_\Omega u\varphi_1\,dx\Big)^p,$$

which implies a bound for u in $L^1(\Omega,\delta)$.

If $p < p^*$ then a bootstrap argument in $L^q(\Omega,\delta)$ yields a bound in L^∞.

If $p \geq p^*$ then there exist unbounded very weak solutions.

Similar approach works for many systems (see [Bidaut-Véron, Yarur 2002], [Q., Souplet 2004]), but in the case of (5), known $L^1(\Omega,\delta)$-estimates require serious restrictions (e.g. $r, s < 1$ in [Li 2010]).

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Ideas of the proof of $L^1(\Omega, \delta)$-estimates

Consider one of the difficult cases: $r, s > 1$.
Subcriticality conditions (7) $\Rightarrow$ w.l.o.g. $p + r < p^*$.

Lemma 1

There exists a nonempty interval $A = A(p, q, r, s) \subset (0, 1)$ with the following properties: if $a \in A$ then there exists $\kappa > 1$ such that

$$-\Delta(u^a v^{1-a}) \geq c(u^a v^{1-a})^\kappa.$$

Proof:

$$-\Delta(u^a v^{1-a}) = au^{a-1}v^{1-a}(-\Delta u) + (1-a)u^a v^{-a}(-\Delta v) + a(1-a)\underbrace{\left[u^{a-2}v^{1-a}|\nabla u|^2 + u^a v^{-a-1}|\nabla v|^2 - 2u^{a-1}v^{-a}\nabla u \cdot \nabla v\right]}_{\geq 0}$$

$$\geq au^{r+a-1}v^{p+1-a} + (1-a)u^{q+a}v^{s-a} \geq c(u^a v^{1-a})^\kappa$$

Corollary: Estimates of $u^a v^{1-a}$ and $u^{r+a-1}v^{p+1-a}$ in $L^1(\Omega, \delta)$
(and Liouville-type nonexistence result if $\kappa \leq n/(n-2)$).

Lemma 2

Let $\|u\|_k := \left(\int_\Omega u^k \varphi_1 \, dx\right)^{1/k}$ (norm in $L^k(\Omega,\delta)$), $\|u\|_\infty = \|u\|_{L^\infty(\Omega)}$. Then

$$\|u\|_\infty \le C\|u\|_1.$$

Proof (for $p > 0$): Assume $\|u\|_k \le C\|u\|_1$ for some $k \ge 1$.
Since $p + r < p^*$ and $r > 1$, we have

$$a := \frac{r-1}{p+r-1} \in A, \qquad m := \frac{k(p+1-a)}{p+1-a+pk} \ge 1, \qquad K := \frac{(n+1)m}{(n+1-2m)_+} > k.$$

If $\tilde{k} < K$ then

$$\begin{aligned} \|u\|_{\tilde{k}} &\le C\|u^r v^p\|_m = C\left(\int_\Omega (u^r v^p)^m \varphi_1 \, dx\right)^{1/m} \\ &\le C\underbrace{\left(\int_\Omega u^{r+a-1} v^{p+1-a} \varphi_1 \, dx\right)}_{\le C \quad \text{(Lemma 1)}}^{p/(p+1-a)} \left(\int_\Omega u^k \varphi_1 \, dx\right)^{1/k} \\ &\le C\|u\|_k \le C\|u\|_1. \end{aligned}$$

Lemma 3

Let $\|u\| := \|u\|_1 = \int_\Omega u\varphi_1\, dx$. Then

$$c \le \|u\|^{r-1}\|v\|^p \le C,$$
$$c \le \|u\|^q\|v\|^{s-1} \le C.$$

Proof: Denote $f = u^r v^p$, $g = u^q v^s$. Then

$$\|u\| = \int_\Omega u\varphi_1\, dx = \frac{1}{\lambda_1}\int_\Omega u^r v^p \varphi_1\, dx \le C\|u\|_\infty^r \overbrace{\int_\Omega v^p\varphi_1\, dx \le C\|u\|^r\|v\|^p}^{\text{Lemma 2, } p<p^*}.$$

Using $u(x) = \int_\Omega G(x,y)f(y)\, dy$ and $G(x,y) \ge c\delta(x)\delta(y)$ one obtains

$u(x) \ge c\|f\|\varphi_1$ (and similarly $v(x) \ge c\|g\|\varphi_1$), hence

$$\|u\| = \int_\Omega u\varphi_1\, dx = \frac{1}{\lambda_1}\int_\Omega u^r v^p \varphi_1\, dx \ge c\|f\|^r\|g\|^p \ge c\|u\|^r\|v\|^p.$$

Corollary. $\|u\| + \|v\| \le C$ (use $pq \ne (1-r)(1-s)$).

Existence

Homotopies on large balls:

$$\begin{aligned} -\Delta u &= u^r v^p + F, && x \in \Omega, \\ -\Delta v &= u^q v^s + G, && x \in \Omega, \\ u = v &= 0, && x \in \partial\Omega. \end{aligned}$$

If $pq > (1-r)(1-s)$:

$$F = (tv)^\omega, \quad G = t(v+1), \quad t \in [0, \lambda_1 + 1],$$

where $\omega \in (0,1]$, $\omega < 2/(n-1)$.

If $pq < (1-r)(1-s)$:

$$\begin{aligned} F &= 0, \quad G = tv^s, && t \in [0,1], \\ F &= 0, \quad G = v^s - tu^q v^s, && t \in [0,1], \end{aligned}$$

$\Rightarrow$ w.l.o.g $q = 0$; exchanging the role of u, v: $p = 0$.

Typical noncooperative system

$$\left.\begin{aligned} -\Delta u &= u^r v^{p-1}(v-\lambda u), && x\in\Omega,\\ -\Delta v &= u^{p-1} v^r(u-\lambda v), && x\in\Omega,\\ u &= v = 0, && x\in\partial\Omega. \end{aligned}\right\} \tag{8}$$

Theorem 2

Assume

$$p\ge 1,\quad 1<p+r<p^*,\quad \lambda<1.$$

If $r\in[0,p]$ then there exists $C>0$ such that $\|u\|_\infty+\|v\|_\infty\le C$ for any positive classical solution of (8). In addition, there exists a positive classical solution of (8).

Remark: [Montaru, Souplet, Sirakov 2014] required $r\le 1$ and used the special structure of (8) (less restrictive subcriticality conditions).

Theorem 3

Assume $\lambda, \mu \geq 0$, $\lambda + \mu < 2$.

(i) If $n \leq 3$ then the system

$$\left.\begin{aligned} -\Delta u &\geq uv^{3/2} - \lambda u^2 v, && x \in \mathbb{R}^n, \\ -\Delta v &\geq u^3 v - \mu u^2 v^{3/2}, && x \in \mathbb{R}^n, \end{aligned}\right\} \tag{9}$$

does not admit positive solutions.

(ii) If $n \leq 2$ then the system

$$\left.\begin{aligned} -\Delta u &\geq uv^3 - \lambda u^3 v^2, && x \in \mathbb{R}^n, \\ -\Delta v &\geq u^6 v - \mu u^4 v^2, && x \in \mathbb{R}^n, \end{aligned}\right\} \tag{10}$$

does not admit positive solutions.

The arguments using auxiliary functions $u^a v^{1-a}$ also guarantee (optimal) Fujita-type results. In the case of

$$\left.\begin{aligned} u_t - \Delta u &\geq u^r v^p, \\ v_t - \Delta v &\geq u^q v^s, \end{aligned}\right\} \qquad x \in \mathbb{R}^n,\ t > 0, \tag{11}$$

optimal Fujita-type results were obtained by [Escobedo, Levine 1995]: their proof is long, technical and requires cooperativity.
Our arguments are very simple and can also be used for noncooperative systems, for example

$$\left.\begin{aligned} u_t - \Delta u &\geq uv^{3/2} - \lambda u^2 v, \\ v_t - \Delta v &\geq u^3 v - \mu u^2 v^{3/2}, \end{aligned}\right\} \qquad x \in \mathbb{R}^1,\ t > 0. \tag{12}$$

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