Comparison Results for Semilinear Elliptic Equations in Equimeasurable Domains

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INTRODUCTION

Semilinear elliptic equation

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + H(x,u,\nabla u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Bounded domain $\Omega \subset \mathbb{R}^N$ of class C^2 (dimension $N \geq 2$)

Goal: to compare u with a (radially symmetric) solution v of

$$\begin{cases} -\mathrm{div}(\widehat{A}(x)\nabla v) + \widehat{H}(x,v,\nabla v) = 0 & \text{in } \Omega^* \\ v = 0 & \text{on } \partial\Omega^* \end{cases}$$

where Ω^* is the ball with the same measure as Ω ($|\Omega^*| = |\Omega|$)

and $\widehat{A}$ and $\widehat{H}$ satisfy the same type of constraints as A and H

General assumptions

- $A:\Omega\to\mathcal{S}_n(\mathbb{R})$ is of class $W^{1,\infty}(\Omega)$ and uniformly elliptic:

$$A(x)\geq\Lambda(x)\ \mathrm{Id}$$

with $\Lambda\in L^\infty(\Omega)$ and $\Lambda(x)\geq\lambda>0$ a.e. in Ω

- $H:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is continuous and there exist

$$1\leq q\leq 2$$

and three continuous functions a, b and $f:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ s.t.

$$\begin{cases} H(x,s,p)\geq -a(x,s,p)\,|p|^q+b(x,s,p)\,s-f(x,s,p)\\ b(x,s,p)\geq 0\end{cases}$$

No bound on H from above

- The cases $q=1$ and $1<q\leq 2$ will be treated separately

Existence and uniqueness results are different

Notions of solutions u

- Weak solutions: $u \in H_0^1(\Omega)$, $H(\cdot, u(\cdot), \nabla u(\cdot)) \in L^1(\Omega)$ and

$$\int_\Omega A(x)\nabla u \cdot \nabla \varphi + \int_\Omega H(x, u, \nabla u)\, \varphi = 0$$

for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$

If $H(\cdot, u(\cdot), \nabla u(\cdot)) \in L^2(\Omega)$, then test functions $\varphi \in H_0^1(\Omega)$

- Strong solutions

$$u \in W(\Omega) = \bigcap_{1 \le p < +\infty} W^{2,p}(\Omega)$$

Distribution functions

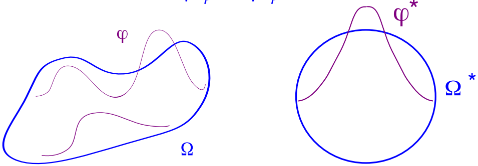
$$\mu_\varphi(t) = \Big|\big\{x \in \Omega,\ \varphi(x) > t\big\}\Big|$$

Schwarz symmetric decreasing rearrangement of φ:

$$\varphi^*(x) = \min\big\{t \in \mathbb{R},\ \mu_\varphi(t) \le \alpha_n\,|x|^n\big\},\quad x \in \Omega^*$$

where $\alpha_n = |B_1|$

$$\mu_\varphi = \mu_{\varphi^*}$$



(a) Function φ (b) Schwarz spherical rearrangement

Goal: given a solution u in Ω, show that

$$u^* \le v \quad \text{in } \Omega^*$$

where v solves a comparable problem in Ω^*

LINEAR GROWTH IN THE GRADIENT

Theorem 1

Assume $q = 1$: $H(x,s,p) \geq -a(x,s,p)\,|p| + b(x,s,p)\,s - f(x,s,p)$

Let $u \in W(\Omega)$ be a solution such that $u > 0$ in Ω and $|\nabla u| \neq 0$ on $\partial\Omega$

Then

$$u^* \leq v \ \text{ a.e. in } \Omega^*$$

where $v \in H_0^1(\Omega^*) \cap C(\overline{\Omega^*})$ is the unique weak solution of

$$-\mathrm{div}(\widehat{\Lambda}(x)\nabla v) - \widehat{a}(x)\,|\nabla v| - f_u^*(x) = 0 \text{ in } \Omega^*, \quad v = 0 \text{ on } \partial\Omega^*$$

with some radially symmetric functions $\widehat{\Lambda}$, $\widehat{a} \in L^\infty(\Omega^*)$ such that

$$\begin{cases} 0 < \operatorname*{ess\,inf}_{\Omega} \Lambda \leq \widehat{\Lambda}(x) \leq \operatorname*{ess\,sup}_{\Omega} \Lambda, & \|\widehat{\Lambda}^{-1}\|_{L^1(\Omega^*)} = \|\Lambda^{-1}\|_{L^1(\Omega)} \\ 0 \leq \inf\limits_{\Omega\times\mathbb{R}\times\mathbb{R}^n} a^+ \leq \widehat{a}(x) \leq \sup\limits_{\Omega\times\mathbb{R}\times\mathbb{R}^n} a^+ & \end{cases}$$

and $f_u^*(x)$ is the Schwarz rearrangement of $f_u(y) = f(y, u(y), \nabla u(y))$

Corollary

For all functions $\overline{a}$ and $\overline{f}$ in $L^\infty(\Omega^*)$ such that $\widehat{a} \leq \overline{a}$ and $f_u^* \leq \overline{f}$:

$$u^* \leq \overline{v} \quad \text{a.e. in } \Omega^*$$

where $\overline{v} \in H_0^1(\Omega^*) \cap C(\overline{\Omega^*})$ is the unique weak solution of

$$-\mathrm{div}(\widehat{\Lambda}(x)\nabla\overline{v}) - \overline{a}(x)\,|\nabla\overline{v}| - \overline{f}(x) = 0 \ \text{ in } \Omega^*, \quad v = 0 \text{ on } \partial\Omega^*$$

Proof.

$$\begin{aligned} -\mathrm{div}(\widehat{\Lambda}(x)\nabla v) - \overline{a}(x)\,|\nabla v| - \overline{f}(x) &= (\widehat{a}(x) - \overline{a}(x))|\nabla v| + f_u^*(x) - \overline{f}(x) \\ &\leq 0 \end{aligned}$$

in the weak $H_0^1(\Omega^*)$ sense.

Then, $v \leq \overline{v}$ a.e. in Ω^* (weak maximum principle [Porretta]).

Finally, $u^* \leq \overline{v}$ a.e. in Ω^*.

Existence and uniqueness for the problem

$$\begin{cases} -\mathrm{div}(\widehat{\Lambda}(x)\nabla v) - \widehat{a}(x)\,|\nabla v| - f_u^*(x) & = & 0 & \text{in } \Omega^* \\ v & = & 0 & \text{on } \partial\Omega^* \end{cases}$$

[Porretta]

Remark: uniqueness $\Longrightarrow$ v is radially symmetric

Existence and uniqueness for the problem

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + H(x,u,\nabla u) & = & 0 \ \text{ in } \Omega \\ u & = & 0 \ \text{ on } \partial\Omega \end{cases}$$

- If
$$\begin{cases} \omega^{-1}(s-s') \le H(x,s,p) - H(x,s',p) \le \omega\,(s-s'), \quad \omega > 0 \\ |H(x,s,p) - H(x,s,p')| \le \alpha(x)\,(1+|s|^{2/N})\,|p-p'| \\ \alpha \in L^r(\Omega), \quad r = N^2/2 \text{ for } N \ge 3,\ r > 2 \text{ for } N = 2,\ r = 2 \text{ for } N = 1 \end{cases}$$
$\Longrightarrow$ uniqueness

- If, additionally, $|H(x,s,p)| \le \beta(x)\,(1+|s|+|p|)$, with $\beta \in L^t(\Omega)$, $t = N$ for $N \ge 3$, $t > 2$ for $N = 2$, $t = 2$ for $N = 1$
$\Longrightarrow$ existence of a unique solution $u \in H_0^1(\Omega)$

- If, additionally, $\beta \in L^\infty(\Omega)$, then $u \in W(\Omega)$

- If, additionally, $H(x,0,p) \le \sigma\,|p|$ and $H(\cdot,0,0) \not\equiv 0$,
$\Longrightarrow$ $u > 0$ in Ω and $|\nabla u| \ne 0$ on $\partial\Omega$

Some references in the literature

- Talenti (1976): $A \geq \mathrm{Id}$, $\Lambda = 1$, $b \geq 0$, $f \in L^\infty(\Omega)$,

$$-\mathrm{div}(A(x)\nabla u) + b(x)u = f(x) \text{ in } \Omega$$

Then $|u|^* \leq v$ where

$$-\Delta v = |f|^* \text{ in } \Omega^*$$

- Talenti (1985): $A \geq \mathrm{Id}$, $\Lambda = 1$, $\alpha \in L^\infty(\Omega, \mathbb{R}^N)$, $b \geq 0$, $f \in L^\infty(\Omega)$,

$$-\mathrm{div}(A(x)\nabla u) + \alpha(x) \cdot \nabla u + b(x)u = f(x) \text{ in } \Omega$$

Then $|u|^* \leq v$ where

$$-\Delta v + \widetilde{\alpha}\, e_r \cdot \nabla v = |f|^* \text{ in } \Omega^*$$

and $\widetilde{\alpha} = \| \, |\alpha| \, \|_{L^\infty(\Omega)}$, $e_r(x) = x/|x|$

- $\Longleftarrow$ our first theorem

- Further results: pointwise or integral comparisons between u^* and v in Ω^* or subdomains

  [Alvino, Trombetti, Lions, Matarasso], [Bandle, Marcus], [Cianchi], [Ferone, Messano], [Messano], [Trombetti, Vazquez]

- In most references: Λ is chosen as a constant λ and the second-order terms in Ω^* are $-\lambda\,\Delta v$
- If one takes $\Lambda = \lambda$ constant, then $\widehat{\Lambda} = \lambda$

  But, Λ can be chosen non-constant and, in general, $\widehat{\Lambda}$ is not constant either

Quantitatively improved inequalities when Ω is not a ball

Theorem 2

Same assumptions and notations as in Theorem 1, and Ω is not a ball

There is $\eta_u > 0$ such that

$$(1+\eta_u)\,u^* \le v \quad \text{a.e. in } \Omega^*$$

Furthermore, if

$$\begin{cases} \|A\|_{W^{1,\infty}(\Omega)} + \|\Lambda^{-1}\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^N)} + \|f\|_{L^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^N)} \le M \\ |H(x,s,p) - H(x,0,0)| \le M\,(|s|+|p|) \\ -M \le H(x,0,0) \le 0, \quad \displaystyle\int_\Omega H(x,0,0)\,dx \le -M^{-1} < 0 \end{cases}$$

then there is $\eta = \eta(\Omega, N, M) > 0$ independent of u, such that

$$(1+\eta)\,u^* \le v \quad \text{a.e. in } \Omega^*$$

AT MOST QUADRATIC GROWTH IN THE GRADIENT

Theorem 3

Assume $H(x,s,p) \geq -a(x,s,p)\,|p|^q + b(x,s,p)\,s - f(x,s,p)$ with $1 < q \leq 2$ and $\inf_{\Omega\times\mathbb{R}\times\mathbb{R}^N} b > 0$.

Let $u \in W(\Omega)$ be a solution s.t. $u > 0$ in Ω and $|\nabla u| \neq 0$ on $\partial\Omega$. Then

$$u^* \leq v \ \text{ a.e. in } \Omega^*$$

where $v \in H^1_0(\Omega^*) \cap C(\overline{\Omega^*})$ is the unique weak solution of

$$-\mathrm{div}(\widehat{\Lambda}(x)\nabla v) - \widehat{a}(x)\,|\nabla v|^q + \widehat{\delta}\,v - \widehat{f}(x) = 0 \text{ in } \Omega^*, \quad v = 0 \text{ on } \partial\Omega^*$$

with $\widehat{\delta} > 0$ and some radially symmetric functions $\widehat{\Lambda}$, $\widehat{a}$, $\widehat{f} \in L^\infty(\Omega^*)$ s.t.

$$\begin{cases} 0 < \mathrm{ess\,inf}\,\Lambda \leq \widehat{\Lambda}(x) \leq \mathrm{ess\,sup}\,\Lambda, \quad \|\widehat{\Lambda}^{-1}\|_{L^1(\Omega^*)} = \|\Lambda^{-1}\|_{L^1(\Omega)} \\ 0 \leq \inf a^+ \times \left(\dfrac{\mathrm{ess\,inf}\,\Lambda}{\mathrm{ess\,sup}\,\Lambda}\right)^{q-1} \leq \widehat{a}(x) \leq \sup a^+ \times \left(\dfrac{\mathrm{ess\,sup}\,\Lambda}{\mathrm{ess\,inf}\,\Lambda}\right)^{2q-2} \\ \inf f \leq \widehat{f}(x) \leq \sup f, \quad \displaystyle\int_{\Omega^*} \widehat{f} = \int_\Omega f_u \end{cases}$$

Furthermore, for every $\varepsilon > 0$, there exists a radially symmetric function $\widehat{f}_\varepsilon \in L^\infty(\Omega^*)$ such that

$$\mu_{\widehat{f}_\varepsilon} = \mu_{f_u}$$

and

$$\|(u^* - v_\varepsilon)^+\|_{L^{2^*}(\Omega^*)} \leq \varepsilon$$

where $v_\varepsilon \in H^1_0(\Omega^*) \cap C(\overline{\Omega^*})$ is the unique weak solution of

$$-\mathrm{div}(\widehat{\Lambda}(x)\nabla v_\varepsilon) - \widehat{a}(x)\,|\nabla v_\varepsilon|^q + \widehat{\delta} v_\varepsilon - \widehat{f}_\varepsilon(x) = 0 \text{ in } \Omega^*, \quad v_\varepsilon = 0 \text{ on } \partial\Omega^*$$

Corollary

For all functions $\overline{a} \geq \widehat{a}$ *and* $\overline{f} \geq \widehat{f}$ *and for all* $0 < \overline{\delta} \leq \widehat{\delta}$*:*

$$u^* \leq \overline{v} \quad \textit{a.e. in } \Omega^*$$

where $\overline{v} \in H_0^1(\Omega^*) \cap C(\overline{\Omega^*})$ *is the unique weak solution of*

$$-\mathrm{div}(\widehat{\Lambda}(x)\nabla\overline{v}) - \overline{a}(x)\,|\nabla\overline{v}|^q + \overline{\delta}\,\overline{v} - \overline{f}(x) = 0 \ \ \textit{in } \Omega^*, \quad \overline{v} = 0 \ \textit{on } \partial\Omega^*$$

Proof.

$$\begin{aligned} -\mathrm{div}(\widehat{\Lambda}(x)\nabla v) - \overline{a}(x)\,|\nabla v|^q + \overline{\delta}\,v - \overline{f}(x) \\ &= (\widehat{a}(x) - \overline{a}(x))|\nabla v|^q + (\overline{\delta} - \widehat{\delta})v + \widehat{f}(x) - \overline{f}(x) \\ &\leq 0 \end{aligned}$$

in the weak $H_0^1(\Omega^*) \cap L^\infty(\Omega^*)$ sense.

Then, $v \leq \overline{v}$ a.e. in Ω^* (weak maximum principle [Barles, Murat]).

Finally, $u^* \leq \overline{v}$ a.e. in Ω^*.

Existence and uniqueness for the problem

$$\begin{cases} -\mathrm{div}(\widehat{\Lambda}(x)\nabla v) - \widehat{a}(x)\,|\nabla v|^q + \widehat{\delta}\, v - \widehat{f}(x) & = & 0 & \text{in } \Omega^* \\ v & = & 0 & \text{on } \partial\Omega^* \end{cases}$$

with $v \in H_0^1(\Omega^*) \cap L^\infty(\Omega^*)$

Existence: [Boccardo, Murat, Puel]

Uniqueness: [Barles, Murat]

Remark: uniqueness $\Longrightarrow$ v is radially symmetric, and continuous in $\overline{\Omega^*}$

Existence and uniqueness for the problem

$$\left\{\begin{array}{rcl} -\mathrm{div}(A(x)\nabla u)+H(x,u,\nabla u) & = & 0 \text{ in } \Omega \\ u & = & 0 \text{ on } \partial\Omega \end{array}\right.$$

- If

$$\left\{\begin{array}{l} H(x,s,p)=\beta(x,s,p)+\widetilde{H}(x,s,p) \\ \beta(x,s,p)s\geq \nu\, s^2, \quad |\beta(x,s,p)|\leq \kappa\,(\gamma(x)+|s|+|p|), \\ |\widetilde{H}(x,s,p)|\leq \rho+\varrho(|s|)\,|p|^2 \end{array}\right.$$

  with ν, κ, $\rho>0$, $0\leq\gamma\in L^2(\Omega)$ and $\varrho:\mathbb{R}_+\to\mathbb{R}_+$ is increasing

  $\Longrightarrow$ existence of a solution $u\in H^1_0(\Omega)\cap L^\infty(\Omega)$

  [Boccardo, Murat, Puel]

- If

$$|H(x,s,p)|\leq M(1+|s|^r+|p|^q)$$

  with $q<1+2/N$, $M\geq 0$, $r\geq 0$, $r(N-2)<N+2$

  $\Longrightarrow u\in W(\Omega)$

- If $\partial_s H$ and $\partial_p H$ exist and

$$\begin{cases} \partial_s H(x,s,p) \geq \sigma > 0, \quad |\partial_p H(x,s,p)| \leq \theta(|s|)\,(1+|p|) \\ |H(x,s,0)| \leq \vartheta(|s|) \end{cases}$$

  for some continuous nonnegative functions θ and ϑ

  $\Longrightarrow$ uniqueness of $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$, and $u \geq 0$ if $H(x,0,0) \leq 0$ [Barles, Murat]

- Further existence results, for the equation satisfied by v, when $\widehat{\delta} = 0$ or even $\widehat{\delta} < 0$, and *when $\widehat{f}$ is small in some spaces*

  [Abdellaoui, Dall'Aglio, Peral], [Ferone, Murat], [Ferone, Posteraro, Rakotoson], [Jeanjean, Sirakov], [Sirakov]

- Existence or uniqueness do not always hold if $\widehat{\delta} \leq 0$ for general $\widehat{f}$

  [Abdellaoui, Dall'Aglio, Peral], [Hamid, Bidaut-Véron], [Jeanjean, Sirakov], [Porretta], [Sirakov]

  Example [Porretta]: consider

  $$-\Delta v - |\nabla v|^2 - f(x) = 0$$

  Set $w = e^v - 1$. Then

  $$-\Delta w - f(x)(1 + w) = 0$$

  Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet b.c.

  If $f(x) \geq \lambda_1$ (> 0), then $v \geq 0$, whence $w \geq 0$ and

  $$-\Delta w = f(x)(1 + w) > \lambda_1 w$$

  which is impossible.

Some comparison results in the literature

- [Alvino, Lions, Trombetti] $H = H(x,p)$ and

$$A \geq \mathrm{Id}, \quad \Lambda = 1, \quad |H(x,p)| \leq f(x) + \kappa\,|p|^q$$

with $\kappa > 0$, $0 \leq f \in L^\infty(\Omega)$

Then $u^* \leq v$ in Ω^*, where $v \in H^1_0(\Omega^*) \cap L^\infty(\Omega^*)$ is any solution of

$$-\Delta v - \kappa\,|\nabla v|^q - f^*(x) = 0 \ \text{ in } \Omega^*$$

provided such a solution exists (it does if $\|f\|_{L^\infty(\Omega)}$ is small enough)

Further results: [Maderna, Pagani, Salsa], [Messano], [Pašić]

- [Ferone, Posteraro]

$A \geq \mathrm{Id}$, $\Lambda = 1$, $H(x,s,p) = \mathrm{div}(F) + \widetilde{H}(x,s,p)$, $|\widetilde{H}(x,s,p)| \leq f(x) + |p|^2$

with $F \in (L^r(\Omega))^N$, $f \in L^{r/2}(\Omega)$ and $r > N$

Then $u^* \leq v$ in Ω^*, where $v \in H^1_0(\Omega^*) \cap L^\infty(\Omega^*)$ is any solution of

$$-\Delta v - |\nabla v|^2 - \mathrm{div}(\widehat{F} e_r) - f^*(x) = 0 \ \text{ in } \Omega^*$$

provided such a solution exists (it does if f and F are small enough)

- [Tian, Li]

$$|H(x,s,p)| \leq f(x) + \kappa\,\Lambda(x)^{2/q}|p|^q$$

with $\kappa > 0$, $0 \leq f \in L^r(\Omega)$ and $r > N^2/(3N-2)$

Then $u^* \leq v$ in Ω^*, where $v \in H_0^1(\Omega^*) \cap L^\infty(\Omega^*)$ is any solution of

$$-\mathrm{div}(\widetilde{\Lambda}(x)\nabla v) - \kappa\,\widetilde{\Lambda}(x)^{2/q}|\nabla v|^q - f^*(x) = 0 \quad \text{in } \Omega^*$$

provided such a solution v exists (it does if f is small enough)

Different assumptions

- Bounds on $|H|$ vs. lower bound on H
- Assumption on the existence of v ($\simeq$ small coefficients)

  vs. assumption $\inf_{\Omega\times\mathbb{R}\times\mathbb{R}^N} b > 0$ and property $\widehat{\delta} > 0$

  (existence is automatically guaranteed, no smallness assumption)
- Problem with $\widehat{H}(x, \nabla v)$ and f^* in Ω^*

  vs. $\widehat{H}(x, v, \nabla v)$ and $\widehat{f}$, $\widehat{f}_\varepsilon$ in Ω^*

Particular case of our results

Choose $\Lambda = \lambda$ constant, $a = \alpha$ constant and $f = \gamma$ constant

Then $u^* \leq v$ in Ω^*, where v is the unique $H_0^1(\Omega^*) \cap C(\overline{\Omega^*})$ solution of

$$\left\{ \begin{array}{rcll} -\lambda \Delta v - \alpha \, |\nabla v|^q + \widehat{\delta} \, v & = & \gamma & \text{in } \Omega^* \\ v & = & 0 & \text{on } \partial\Omega^* \end{array} \right.$$

Even this particular case is new.

Theorem 4 (Quantitatively improved inequalities when Ω is not a ball)

Same assumptions and notations as in Theorem 3, and Ω is not a ball

There is $\eta_u > 0$ such that

$$(1+\eta_u)\,u^* \le v \ \text{ in } \Omega^* \quad \text{and} \quad \left\|\left((1+\eta_u)\,u^* - v_\varepsilon\right)^+\right\|_{L^{2^*}(\Omega^*)} \le \varepsilon$$

Furthermore, if $q < 1 + 2/N$, $r \ge 0$, $r(N-2) < N+2$ and

$$\begin{cases} \|A\|_{W^{1,\infty}(\Omega)} + \|\Lambda^{-1}\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^N)} + \|f\|_{L^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^N)} \le M \\ b(x,s,p) \ge M^{-1} \\ |H(x,s,p) - H(x,0,0)| \le M\left(|s|^r + |p|^q\right) \\ |H(x,s,p)| \le M(1 + |s|^r + |p|^q) \\ -M \le H(x,0,0) \le 0, \quad \displaystyle\int_\Omega H(x,0,0)\,dx \le -M^{-1} < 0, \end{cases}$$

then there is $\eta = \eta(\Omega, N, q, M, r) > 0$ independent of u, such that

$$(1+\eta)\,u^* \le v \ \text{ in } \Omega^* \quad \text{and} \quad \left\|\left((1+\eta)\,u^* - v_\varepsilon\right)^+\right\|_{L^{2^*}(\Omega^*)} \le \varepsilon$$

SOME ELEMENTS OF THE PROOFS

$$-\mathrm{div}(A\nabla u) = -H(x, u, \nabla u)$$

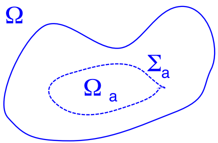
- Assume $A \geq \Lambda\,\mathrm{Id}$, A and Λ are $C^1(\overline{\Omega})$, u is analytic in Ω. Then

  $Z = \left\{a \in \left[0, \max_{\overline{\Omega}} u\right],\ \exists\, y \in \overline{\Omega},\ u(y) = a \text{ and } \nabla u(y) = 0\right\}$ is finite

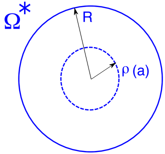
  Denote

  $$\Omega_a = \{y \in \overline{\Omega},\ u(y) > a\}, \quad \Sigma_a = \{y \in \overline{\Omega},\ u(y) = a\}$$

  There holds $|\Sigma_a| = 0$ for all $0 \leq a \leq \max_{\overline{\Omega}} u$



(c)



(d)

- Map $a \mapsto \rho(a) \in [0, R]$ such that $|\Omega_a| = |B_{\rho(a)}|$, is decreasing, continuous, one-to-one and onto.

- Non-critical values: $Y=\big[0,\max_{\overline{\Omega}}u\big]\backslash Z$, $E=\{x\in\overline{\Omega^*},\ |x|\in\rho(Y)\}$
- Symmetrized ellipticity function $\widehat{\Lambda}$, defined for all $x\in E$:

$$\widehat{\Lambda}(x)=\frac{\displaystyle\int_{\Sigma_{\rho^{-1}(|x|)}}|\nabla u(y)|^{-1}d\sigma(y)}{\displaystyle\int_{\Sigma_{\rho^{-1}(|x|)}}\Lambda(y)^{-1}|\nabla u(y)|^{-1}d\sigma(y)}>0$$

  - co-area formula $\Longrightarrow \displaystyle\int_{\Omega^*}\widehat{\Lambda}(x)^{-1}dx=\int_{\Omega}\Lambda(y)^{-1}dy$
  - $\min_{\overline{\Omega}}\Lambda\le\widehat{\Lambda}\le\max_{\overline{\Omega}}\Lambda$ in Ω^*
- Symmetrized function $\widehat{u}(x)=\widetilde{u}(|x|)$, radially symmetric, vanishing on $\partial\Omega^*$ and such that, for $a\notin Z$ and $|x|=\rho(a)$,

$$N\alpha_N\,|x|^{N-1}\,\widehat{\Lambda}(x)\,\widetilde{u}'(|x|)=\int_{B_{\rho(a)}}\mathrm{div}(\widehat{\Lambda}\nabla\widehat{u})(z)dz=\int_{\Omega_a}\mathrm{div}(A\nabla u)(y)dy<0$$

The function $\widehat{u}$ is positive in Ω^*, decreasing in $|x|$, it is of class $W^{1,\infty}(\Omega^*)\cap H^1_0(\Omega^*)$, $C^2(E\cap\Omega^*)$ and $C^1(E\cup\{0\})$

- Denote $a_u(y)=a(y,u(y),\nabla u(y))$

  Symmetrized function $\widehat{a}$ defined for all $x\in E$:

$$\widehat{a}(x)=\begin{cases}\displaystyle\max_{y\in\Sigma_{\rho^{-1}(|x|)}}\big(a_u^+(y)\,\Lambda^{-1}(y)\big)\times\widehat{\Lambda}(x) & \text{if } q=2,\\[2ex] \left(\dfrac{\displaystyle\int_{\Sigma_{\rho^{-1}(|x|)}} a_u^+(y)^{\frac{2}{2-q}}\Lambda(y)^{-\frac{q}{2-q}}|\nabla u(y)|^{-1}\,d\sigma_{\rho^{-1}(|x|)}}{\displaystyle\int_{\Sigma_{\rho^{-1}(|x|)}}|\nabla u(y)|^{-1}\,d\sigma_{\rho^{-1}(|x|)}}\right)^{\frac{2-q}{2}}\times\widehat{\Lambda}(x)^{\frac{q}{2}}\end{cases}$$

  (second case: $1\le q<2$)

- $$\min_{\overline{\Omega}} a_u^+\le\widehat{a}(x)\le\Big(\max_{\overline{\Omega}} a_u^+\Big)\times\left(\frac{\max_{\overline{\Omega}}\Lambda}{\min_{\overline{\Omega}}\Lambda}\right)^{q-1}$$

- Denote $f_u(y) = f(y, u(y), \nabla u(y))$

  Symmetrized function $\widehat{f}$ defined for all $x \in E$:

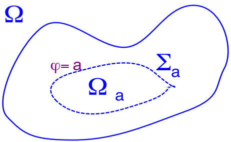
$$\widehat{f}(x) = \frac{\displaystyle\int_{\Sigma_{\rho^{-1}(|x|)}} f_u(y)\,|\nabla u(y)|^{-1} d\sigma_{\rho^{-1}(|x|)}}{\displaystyle\int_{\Sigma_{\rho^{-1}(|x|)}} |\nabla u(y)|^{-1}\, d\sigma_{\rho^{-1}(|x|)}}$$

- $$\min_{\overline{\Omega}} f_u \leq \widehat{f}(x) \leq \max_{\overline{\Omega}} f_u \quad \text{and} \quad \int_{\Omega^*} \widehat{f} = \int_{\Omega} f_u$$

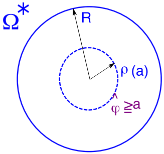
First key inequality: pointwise comparison between u and $\widehat{u}$

For all $x \in \overline{\Omega^*}$ with $|x| = \rho(a)$ and for all $y \in \Sigma_a$,

$$\widehat{u}(x) \geq u(y) = a = \rho^{-1}(|x|) \quad \text{, that is} \quad u^*(x) \leq \widehat{u}(x)$$



(e)



(f)

Improved inequality when Ω is not a ball

$$\widehat{u}(x) \geq (1+\eta)\, u(y) \quad \text{, that is} \quad (1+\eta)\, u^*(x) \leq \widehat{u}(x)$$

where $\eta > 0$ depends on Ω, N and on some bounds of $\|u\|_{C^{1,\alpha}(\overline{\Omega})}$

Second key inequality: partial differential inequality

For all $x \in E \cap \Omega^*$, there is $y \in \Sigma_{\rho^{-1}(|x|)}$ (that is, $u(y) = \rho^{-1}(|x|)$) s.t.

$$\begin{aligned} & -\mathrm{div}(\widehat{\Lambda}\nabla\widehat{u})(x) \;\; - \widehat{a}(x)|\nabla\widehat{u}(x)|^q \;\; - \widehat{f}(x) \\ \leq & -\mathrm{div}(A\nabla u)(y) \;\; - a_u(y)|\nabla u(y)|^q \;\; - f_u(y) \end{aligned}$$

Improved inequality when $m_b = \min_{\overline{\Omega}} b_u > 0$, $b_u = b(\cdot, u(\cdot), \nabla u(\cdot))$

$$\begin{aligned} & -\mathrm{div}(\widehat{\Lambda}\nabla\widehat{u})(x) \;\; - \widehat{a}(x)|\nabla\widehat{u}(x)|^q \;\; + \delta\,\widehat{u}(x) \;\; - \widehat{f}(x) \\ \leq & -\mathrm{div}(A\nabla u)(y) \;\; - a_u(y)|\nabla u(y)|^q \;\; + b_u(y)\,u(y) \;\; - f_u(y) \end{aligned}$$

where $\delta > 0$ depends on m_b, Ω, N and some bounds on some norms of u and the coefficients

Results of independent interest. Different from Schwarz symmetrization.

End of the proof (with $q = 1$)

- $$\begin{aligned} & -\mathrm{div}(\widehat{\Lambda}\nabla\widehat{u})(x) \ -\widehat{a}(x)|\nabla\widehat{u}(x)| - \widehat{f}(x) \\ \leq\ & -\mathrm{div}(A\nabla u)(y) \ - a_u(y)|\nabla u(y)| - f_u(y) \\ \leq\ & -\mathrm{div}(A\nabla u)(y) \ + H(y, u(y), \nabla u(y)) \quad = 0 \end{aligned}$$

  and $\widehat{u}$ is a weak $H_0^1(\Omega^*)$ subsolution of

  $$-\mathrm{div}(\widehat{\Lambda}\nabla\widehat{u}) - \widehat{a}\, e_r \cdot \nabla\widehat{u} - \widehat{f} \leq 0 \ \text{ in } \Omega^*$$

- Let $w \in H_0^1(\Omega^*)$ be the unique solution of

  $$-\mathrm{div}(\widehat{\Lambda}\nabla w) + \widehat{a}\, e_r \cdot \nabla w - \widehat{f} = 0 \ \text{ in } \Omega^*$$

  Maximum principle $\Longrightarrow$ $\widehat{u} \leq w$ in Ω^* $\Longrightarrow$ $u^* \leq \widehat{u} \leq w$ in Ω^*

- Sequence $(g_k)_{k\in\mathbb{N}}$ s.t. $g_k \rightharpoonup \widehat{f}$ in $L^\infty(\Omega^*)$ weak-* and $\mu_{g_k} = \mu_{f_u}$
- Solutions $z_k \in H_0^1(\Omega^*)$ of

$$-\mathrm{div}(\widehat{\Lambda}\nabla z_k) + \widehat{a}\, e_r \cdot \nabla z_k - g_k = 0 \text{ in } \Omega^*$$

  One has: $z_k \to w$ in $H_0^1(\Omega^*)$ weak and $L^2(\Omega^*)$ strong
- Solution $z \in H_0^1(\Omega^*)$ of

$$-\mathrm{div}(\widehat{\Lambda}\nabla z) + \widehat{a}\, e_r \cdot \nabla z - f_u^* = 0 \text{ in } \Omega^*$$

  One has: $z_k \le z$ (in particular: use of Hardy-Littlewood inequality)
- Solution $v \in H_0^1(\Omega^*)$ of

$$-\mathrm{div}(\widehat{\Lambda}\nabla v) - \widehat{a}\,|\nabla v| - f_u^* = 0 \text{ in } \Omega^*$$

  Maximum principle $\Longrightarrow z \le v$
- $\Longrightarrow u^* \le w \simeq z_k \le z \le v \Longrightarrow$ conclusion: $u^* \le v$
- General case: data and eigenfunctions are not smooth enough
  Smooth approximations, uniform estimates,...: many technicalities
- Improved inequalities when Ω is not a ball