Mean field equations, Hyper-elliptic curves and Modular forms

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Abstract

In this talk, we survey some recent joint works with C.L. Chai (U.Penn) and C.L. Wang (NTU), in which we have developed a theory to connect the mean field equation, Green function and the classic Lamé equation. In this theory, we have constructed a family of hyper-elliptic curves (depending on the genus n and moduli parameter $\tau \in \mathbb{H}$: the upper half plane) and a premodular form of degree $\frac{1}{2}n(n+1)$. We prove that the nonlinear PDE on E_τ has a solution iff τ is a zero of this premodular form.

[1] C.L. Chai, C.S. Lin and C.L. Wang: Mean Field Equations, Hyper-elliptic curves and Modular forms, part 1.

[2] C.S. Lin and C.L. Wang: Mean Field Equations, Hyper-elliptic curves and Modular forms, part 2.

§1. Motivation

Let $E = \mathbb{C}/\Lambda_\tau$ be a flat torus, where $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, $\omega_1 = 1$, $\omega_2 = \tau$ and $\operatorname{Im}\tau > 0$. We consider the following equation:

$$\Delta u + e^u = \rho\delta_0 \text{ in } E \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$, a positive constant ρ and δ_0: the Dirac measure at the lattice point 0. In this talk, we mainly consider

$$\rho = 4\pi N,\ N = 2n,\ n \in \mathbb{N}.$$

Equation (1.1) is originated from the conformal geometry. For any compact Riemann surface (M, g),

$$\Delta_g u + e^u - 2k(x) = 4\pi \sum_{i=1}^{n} \alpha_i \delta_{Q_i} \tag{1.2}$$

where $k(x)$ is the Gaussian curvature, $Q_i \in M$ and $\alpha_i > -1$.

For any solution $u(x)$ to (1.2), the Gaussian curvature of the new metric $\tilde{g} = e^{2v} g$ $(2v = u - \log 2)$ is equal to 1 outside those Q_i. Since $u(x)$ has singularities at Q_i, the conformal metric $\tilde{g}$ degenerates at Q_i and is called a metric on M with conic singularities at Q_i.
There is another application of (1.1) to the d-dimensional complex Monge-Ampere equation:

$$\det\left(\frac{\partial^2 w}{\partial z_i \partial \bar{z}_j}\right) = e^{-w} \text{ on } \left(E \backslash \{0\}\right)^d, \tag{1.3}$$

the d-th Cartesian product of $E\backslash\{0\}$. For any solution $u(z)$ of (1.1),

$$w(z_1, \cdots, z_d) = -\sum_{i=1}^{d} u(z_i) + d \log 4$$

satisfies (1.3) with a logarithmic singularity along the normal crossing divisor $D = E^d \backslash \left(E\backslash\{0\}\right)^d$. Hence bubbling solutions will give examples of bubbling solutions to the degenerate complex Monge-Ampere equation.

Clearly, (1.1) is equivalent to

$$\Delta u + \rho \left(\frac{e^u}{\int_E e^u dx} - \frac{1}{|E|} \right) = \rho \left(\delta_0 - \frac{1}{|E|} \right)$$

which is a special case of the mean field equation:

$$\Delta u + \rho \left(\frac{he^u}{\int_M he^u dx} - \frac{1}{|M|} \right) = 4\pi \sum_{i=1}^{N} \alpha_i \left(\delta_{Q_i} - \frac{1}{|M|} \right) \tag{1.4}$$

on a compact Riemann surface (M, g).
Equation (1.4) has been extensively studied for more than 3 decades. Suppose $\alpha_i \in \mathbb{N} \cup \{0\}$. Starting from the work of Brezis-Merle, it was proved that if $\rho \notin 8\pi\mathbb{N}$, then there exists an apriori bound for any solution $u(x)$, away from the singular points Q_i.

This result was obstained by Y.Y. Li and Shafri for $\alpha_j = 0 \ \forall j$ and by Bartolucci-Tarantello for general $\alpha_j \in \mathbb{N}$. Thus, the Leray-Schauder degree is well-defined for each $\rho \notin 8\pi\mathbb{N}$, and is denoted by d_ρ. This d_ρ satisfies

- If $d_\rho \neq 0$, then (1.4) has a solution.
- d_ρ is a constant for $\rho \in (8\pi m, 8\pi(m+1))$.
- d_ρ is a constant independent of h as long as $h \in C^1$ and positive.

The formula for counting the degree d_ρ was obtained in a series of papers by Chen and Lin. Particularly, they showed that if $\alpha_j \in \mathbb{N}$ and the Euler characteristic number $\chi(M) \leq 0$, then $d_\rho > 0 \ \forall \rho \notin 8\pi\mathbb{N}$. Indeed they could write down the generating function for d_ρ explicitly, no matter $\alpha_j \in \mathbb{N}$ or not.

Hereafter, we shall study (1.4) with $\rho \in 8\pi\mathbb{N}$, and equation (1.1) is the simplest (but non-trivial) case.
For any fixed $\rho \notin 8\pi\mathbb{N}$, all the solutions of (1.1) away from 0 have a uniform bound. But if $\rho_k \to 8\pi n$, then there might have a sequence of blow-up solutions u_k with the blowup set $\{p_1, \cdots, p_n\}$. By the Pohozaev identity (a balance condition), those p_i satisfies

$$n\nabla G(p_i) = \sum_{j\neq i}^{n} \nabla G(p_i - p_j) \ \ \forall 1 \leq i \leq n \tag{1.5}$$

where $G(x)$ is the Green function.
Question: How many solutions of (1.5)?

§2. Green function and the Lame equation

How to study the geometry of a flat torus E_τ?

From the analytic point of view, there are two simplest ways to do it:

(a) The Green function $G(z)$:

$$\begin{cases} -\Delta G = \delta_0 - \frac{1}{|E|} \text{ on } E, \\ \int_E G = 0. \end{cases} \tag{2.1}$$

(b) The classic Lame equation:

$$L_{\eta,B}(y) := y'' - (\eta(\eta+1)\wp(z) + B)\, y, \tag{2.2}$$

where $\eta \geq -\frac{1}{2}$, $B \in \mathbb{C}$, $y'(z) = \frac{dy}{dz}$ and $\wp(z)$ is the Weierstrass elliptic function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \; z \in \mathbb{C}$$

- $\wp(z)$ is doubly periodic:

$$\wp(z+\omega_1) = \wp(z) \text{ and } \wp(z+\omega_2) = \wp(z)$$

- $\wp(z)$ is even. Hence $\wp'\left(\frac{\omega_i}{2}\right) = 0$, $1 \le i \le 3$. Denote $e_i = \wp\left(\frac{\omega_i}{2}\right)$.
- $\wp(z)$ satisfies the ODE:

$$\begin{aligned} \wp'(z)^2 &= 4\wp^3(z) - g_2\wp(z) - g_3 \\ &= 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3) \end{aligned}$$

  where $e_1 + e_2 + e_3 = 0$, $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3)$ and $g_3 = 4e_1e_2e_3$.
- In the Lame equation, η is called the index and B is called the accessory parameter. In the literature, people are interested in what are B such that (2.2) has only algebraic solutions or equivalently, has a finite monodromy group.

- It is well-known that the Green function is closely related to the **concentration** phenomenon of some nonlinear PDE. For example, suppose u_k is a sequence of blowup solutions to (1.1):

$$\Delta u_k + e^{u_k} = \rho_k \delta_0 \text{ in } E_\tau,$$

where $\rho_k \to 8\pi n$. Then u_k has n blowup points $\{p_1, \cdots, p_n\}$. $(0 \notin \{p_1, \cdots, p_n\})$, and the locations of p_i satisfies (1.5), i.e.

$$n\nabla G(p_i) = \sum_{j\neq i}^{n} \nabla G(p_i - p_j) \ \ \forall 1 \le i \le n. \tag{2.3}$$

For example, $n = 1$ (i.e. $\rho_k \to 8\pi$), the blowup point p satisfies $\nabla G(p) = 0$, i.e. p is a critical point of G.
For general n, we want to understand how many solutions $\mathbf{p} = (p_1, \cdots, p_n)$ equation (2.3) might have. For $n = 1$, this problem is simply to ask how many critical points of G might have.

How to connect (2.3) with the Lame equation (2.2) ? To answer this question, we have to write down explicitly the Green function in terms of elliptic functions. There are many ways to write it in terms of theta function (the odd one), or the following:

$$G(z) = \frac{1}{|E|} \int_E \log |\wp(z) - \wp(\xi)| \, d\xi_1 d\xi_2 + C \tag{2.4}$$

where C is a constant to match the condition $\int_E G = 0$. From (2.4), we could derive

$$-4\pi G_z(z) = \zeta(z) - \eta_1 z - \frac{2\pi i x_2}{b} \tag{2.5}$$

where $G_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) G(z)$, $z = x_1 + ix_2$, $b = \mathrm{Im}\,\tau = \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right)$, and $\zeta(z)$ is the Weierstrass zeta function:

$$\zeta^{'}(z) = -\wp(z), \; \zeta(z + \omega_i) = \zeta(z) + \eta_i, \; i = 1, 2 \tag{2.6}$$

By the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, (2.5) is equivalent to

$$-4\pi G_z(z) = \zeta(z) - \eta_1 t - \eta_2 s, \tag{2.7}$$

if $z = t\omega_1 + s\omega_2$. From (2.5) or (2.7), $G_z(z)$ is not a holomorphic function in z. Hence (2.3) is not a system of algebraic equations.

- By using the anti-symmetry of ∇G, we have

$$n\sum_{i=1}^{n}\nabla G(p_i) = \sum_{i=1}^{n}\sum_{j\neq i}^{n}\nabla G(p_i - p_j) = 0. \tag{2.8}$$

  Write $p_i = t_i\omega_1 + s_i\omega_2$, by (2.7), (2.3) and (2.8) becomes (after multiplying -4π),

$$\begin{aligned} & \sum_{j\neq i}^{n}\zeta(p_i - p_j) - \eta_1\sum_{j\neq i}^{n}(t_i - t_j) - \eta_2\sum_{j\neq i}^{n}(s_i - s_j) \qquad (2.9) \\ = \quad & n\left[\zeta(p_i) - \eta_1 t_i - \eta_2 s_i\right] \end{aligned}$$

and

$$\sum_{j=1}^{n} \zeta(p_j) - \left(\eta_1 \sum_{j=1}^{n} t_j + \eta_2 \sum_{j=1}^{n} s_j \right) = 0.$$

We have

$$\begin{aligned} & \sum_{j \neq i}^{n} (\zeta(p_i - p_j) + \zeta(p_j)) + \zeta(p_i) - n(\eta_1 t_i + \eta_2 s_i) \\ = \ & n\zeta(p_i) - n(\eta_1 t_i + \eta_2 s_i) \end{aligned}$$

i.e.

$$\sum_{j \neq i}^{n} (\zeta(p_i - p_j) + \zeta(p_j) - \zeta(p_i)) = 0, \ \forall 1 \leq i \leq n. \tag{2.10}$$

Therefore (2.3) is equivalent to (2.8) and (2.10). Note that (2.10) is an algebraic equation in $\mathbf{p} = (p_1, \cdots, p_n)$.

The equation (2.10) appears in the literature of 19th century.

- Recall the Lame equation:

$$y'' = \left(n\left(n+1\right)\wp'\left(z\right)+B\right)y \tag{2.11}$$

Classically, there is an ansatz for solutions to (2.11):

$$w_p\left(z\right) = e^{z\sum_{j=1}^{n}\zeta(p_j)}\prod_{j=1}^{n}\frac{\sigma\left(z-p_j\right)}{\sigma\left(z\right)\sigma\left(p_j\right)}$$

is a solution of (2.11) for some B iff $\mathbf{p} = \left(p_1, \cdots, p_n\right)$ satisfies

$$\sum_{j\neq i}\left(\zeta\left(p_i-p_j\right)+\zeta\left(p_j\right)-\zeta\left(p_i\right)\right)=0,\ \forall 1\leq i\leq n$$

i.e. (2.10) holds. Moreover, $B = \left(2n-1\right)\sum_{i=1}^{n}\wp\left(p_i\right)$. See Whittaker-Watson: A course of modern Analysis, or [1].

§3. Hyper-elliptic curves

Set

$$Y_n = \left\{ (p_1, \cdots, p_n) \;\middle|\; \begin{array}{l} p_i \in E \backslash \{0\}, p_i \neq p_j \text{ for } i \neq j, \text{ and} \\ \sum_{j \neq i} (\zeta(p_i - p_j) + \zeta(p_j) - \zeta(p_i)) = 0, \ \forall 1 \leq i \leq n \end{array} \right\}$$

- Is Y_n a smooth complex manifold?
- Is the map $B : Y_n \to \mathbb{C}$ by $B(\mathbf{p}) = (2n-1) \sum_{i=1}^{n} \wp(p_i)$ onto?

It is not easy to trace the proof of the first question in the literature. The second question is yes if we re-interprete some results in the literature. In any case, we prove in [1]:

Theorem 3.1

Let $\bar{Y}_n$ be the closure of Y_n in $E \times \cdots \times E$. Then
(i) $\bar{Y}_n = Y_n \cup \{0\}$
(ii) There exist $\{B_1, \cdots, B_{2n+1}\} \in \mathbb{C}$ (counted with multiplicities) such that the equation

$$\sum_{i=1}^{n} \wp(p_i) = \frac{B}{2n-1}$$

has exactly two solutions in Y_n for $B \notin \{B_1, \cdots, B_{2n+1}\}$. Indeed, solutions appear in pair $\pm \mathbf{p} = \pm(p_1, \cdots, p_n)$. For each B_k,

$$\sum_{i=1}^{n} \wp(p_i) = \frac{B_k}{2n+1}$$

has exactly only one solution.
(iii) Set $X_n = B^{-1}(\mathbb{C} \setminus \{B_1, \cdots, B_{2n+1}\})$. Then X_n is a smooth 1-dimensional complex manifold. Let $\mathbf{p} = (p_1, \cdots, p_n) \in Y_n$. Moreover,

$$\mathbf{p} \in X_n \text{ iff } \{p_1, \cdots, p_n\} \cap \{-p_1, \cdots, -p_n\} = \emptyset,$$

i.e.

$$Y_n \backslash X_n = \{\mathbf{p} | \mathbf{p} \in Y_n, \ \{p_1, \cdots, p_n\} = \{-p_1, \cdots, -p_n\}\}$$

(iv) *Y_n can be parametrized as :*

$$B = \frac{1}{2n-1} \sum_{i=1}^{n} \wp(p_i) \text{ and } C = \wp'(p_i) \prod_{j \neq i}^{n} (\wp(p_i) - \wp(p_j)) \ \forall 1 \leq i \leq n.$$

Then $Y_n = \left\{(B, C) \,|\, C^2 = \ell_n(B)\right\}$ where $\ell_n(B)$ is a monic polynomial of degree $2n+1$. Y_n is an hyper-elliptic curve with arithmetic genus g.

Remark $\mathbf{p} = (p_1, \cdots, p_n) \in Y_n \backslash X_n$ *is called a branch point of the hyper-elliptic curve Y_n. By (iii) of Theorem 3.1, it is characterized by*

$$\{p_1, \cdots, p_n\} = \{-p_1, \cdots, -p_n\}.$$

Either $p_i = -p_i$ for some i or $\forall i, \exists j \neq i$ such that $p_j = -p_i$.
If the former holds then $C = 0$ because $\wp'(p_i) = 0$
If the later holds then $C = 0$ because $\wp(p_i) - \wp(p_j) = 0$.

Hence the branch points of Y_n are $\{(B,0)\,|\ell_n(B)=0\}$.
Remark *If $\ell_n(B)=0$ has $(2n+1)$ distinct zeros, then $Y_n \cup \{\infty\}$ is a compact Riemann surface of genus n. Y_n is called the Lame curve.*

- Recall (2.3):

$$n\nabla G(p_i) = \sum_{j\neq i}^{n} \nabla G(p_i - p_j) \ \ \forall 1 \leq i \leq n.$$

which is equivalent to

$$\sum_{j\neq i} \left(\zeta(p_i - p_j) + \zeta(p_j) - \zeta(p_i)\right) = 0, \ \forall 1 \leq i \leq n$$

and

$$\sum_{j=1}^{n} \nabla G(p_j) = 0$$

Corollary 3.2

Any solution $\mathbf{p}$ *of (2.3) is in* Y_n*. Furthermore, if* $\mathbf{p}$ *is a branch point, then (2.8) holds automatically, that is, any branch point is a solution of (2.3).*

Now, we return to equation (1.1). The geometry of branch points deeply reflect the structure of solutions of (1.1). For example, suppose u_k is a blowup solution of

$$\Delta u_k + e^{u_k} = \rho_k \delta_0 \text{ with } \rho_k \to 8\pi n$$

Theorem 3.3

Let $\{p_1, \cdots, p_n\}$ *be the blowup set of* u_k*.*
(i) If $\rho_k \neq 8\pi n$ *for large* k*, then* $\{p_1, \cdots, p_n\} = \{-p_1, \cdots, -p_n\}$*, i.e.* $\mathbf{p} = (p_1, \cdots, p_n)$ *is a branch point of* Y_n*.*
(ii) If $\rho_k = 8\pi n$ *for large* k*, then* $\{p_1, \cdots, p_n\} \cap \{-p_1, \cdots, -p_n\} = \emptyset$*, i.e.* $\mathbf{p}$ *is not a branch point.*

Definition 3.4

$\mathbf{p} = (p_1, \cdots, p_n)$ is called a non-trival solution of (2.3) if $\mathbf{p}$ is not a branch point. Otherwise, it is called a trival solution.

If Y_n is smooth, then (2.3) has exactly $(2n+1)$ trival solutions.
The following theorem tell the connection of (1.1) and non-trival solutions.

Theorem 3.5

Equation (1.1) with $N = 2n$ has a solution if and only if (2.3) has non-trivial solutions.

How many non-trivial solutions? The answer should depend on the torus itself. Hence it is natural to study this question via the deformation of τ. So in the next section, we consider $n = 1$ and τ deforms in the moduli space $\mathbb{H}$.

§4. Deformation in τ and modular form

In this section, we discuss solutions **p** of (2.3) with $n = 1$, i.e.

$$\nabla G(p) = 0.$$

- p is a trivial solution iff p is half peroid.

$$p = -p \pmod{\Lambda_\tau}$$

(Notice that $G(z)$ is even. Hence $\nabla G\left(\frac{\omega_i}{2}\right) = 0$ for any half period $\frac{\omega_i}{2}$)
p is a nontrivial solution iff p is a non-half-period critical point of G.

Theorem 4.1

For any torus, the Green function G has at most one pair $\pm p$ of non-trivial solutions. If it exists, then $\pm p$ is the (global) minimal point of G.

Note that $\nabla G(p) = 0$ iff $p = s\omega_1 + t\omega_2$ satisfies

$$\zeta(s\omega_1 + t\omega_2) - (s\eta_1 + t\eta_2) = 0. \qquad (4.1)$$

Equation (4.1) is not an algebraic equation. At the first sight, (4.1) looks elementary. But its proof is not so easy. Actually, Theorem 4.1 is proved by applying some uniqueness result of (1.1). (Nonlinear PDE $\Longrightarrow$ the linear result.)

Example 4.2

If $\tau = ib$, $b > 0$, i.e. E_τ is a rectangle, then G has three critical points only, $\frac{\omega_3}{2}$ is the minimal point and the other two are saddle points. All the three are non-degenerates critical points of G. ($\det D^2 G \neq 0$)

Example 4.3

If $\tau = e^{\frac{i\pi}{3}}$ is the rhombus of sixty degree, then G has two non-trivial solutions at $\pm p = \pm\frac{\omega_3}{3}$. All five critical points are non-degenerates.

Remark *It is an elementary fact that $E_\tau = E_{\tau'}$ iff $\tau' = \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$,*
$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{Z}$ $\det\gamma = 1$. *(But then $\omega_1, \omega_2, \omega_3$ will be permuted.)*

- Let $\Omega_5 = \{\tau | G \text{ has five critical points}\}$.

$$\Omega_5 \diagup SL_2(\mathbb{Z}) \subseteq \mathbb{H} \cup \{\infty\} \diagup SL_2(\mathbb{Z}) \cong \mathbb{S}^2.$$

We might abuse the notation and denote $\Omega_5 \diagup SL_2(\mathbb{Z})$ by Ω_5 if no confusion is caused. Similarly, we define Ω_3. It is important to study the geometry of Ω_5 and Ω_5 in the moduli space $\mathbb{S}^2$ of tori.

Theorem 4.4

(i) Ω_5 is open and $\Omega_3 = \mathbb{S}^2 \backslash \Omega_5$ is a closed set.
(ii) Both Ω_5 and $\Omega_3^0 =$ the interior of Ω_3 are simply connected, and $\partial\Omega_5 = \partial\Omega_3$ is a smooth S^1.
(iii) $\partial\Omega_5 = \partial\Omega_3$ consists of those tori such that one of three half-periods are degenerate. Moreover, for any tori, at most one half period is degenerate (i.e. $\det D^2 G = 0$)
(iv) $\det D^2 G\left(\frac{\omega_i}{2}\right) \neq 0$ except when $\tau \in \partial\Omega_3$ where $\frac{\omega_i}{2}$ is the only degenerate critical points. In particular, $D^2 G(p) < 0$ if p is a saddle point of G.

- Proof of (i) follows from the proof of Theorem 4.1 (the uniqueness of nonlinear PDE)
- The proof of (ii) is interesting. So, we will sketch it. For the full proof, we refer [2].

Definition 4.5 (Modular form)

Let $f(\tau)$ be a holomorphic function in $\mathbb{H}$, and also at ∞. Suppose f satisfies

$$f(\gamma \cdot \tau) = (c\tau + d)^{-k} f(\tau), \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Then $f(\tau)$ is called a modular form of weight k w.r.t. $SL_2(\mathbb{Z})$.

- We will use a formula of counting zero for a modular form f of weight k. (See Serre's book: A course in Arithmetic)

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{p \neq \infty, i, \rho} v_p(f) = \frac{k}{12} \tag{4.2}$$

where $v_p(f)$= the order of zero of f at p, $i = \sqrt{-1}$ and $\rho = e^{i\pi/3}$. We will use the case $k = 8$. The LHS$=\frac{2}{3}$, which implies

$$v_\infty(f) = v_i(f) = \sum_{p \neq \infty, i, \rho} v_p(f) = 0 \text{ and } v_\rho(f) = 2$$

i.e. any modular form (w.r.t. $SL_2(\mathbb{Z})$) of weight 8 could have zero of order 2 at $\tau = \rho$.

- We need another elementary lemma, which could be proved by addition theorem.

Lemma 4.6

For any $\tau \in \mathbb{H}$,
(i) $\zeta\left(\frac{3}{4}\omega_1 + \frac{1}{4}\omega_2\right) \neq \frac{3}{4}\eta_1 + \frac{1}{4}\eta_2$
(ii) $\zeta\left(\frac{1}{6}\omega_1 + \frac{1}{6}\omega_2\right) \neq \frac{1}{6}\eta_1 + \frac{1}{6}\eta_2$
(iii) $\zeta\left(\frac{2}{6}\omega_1 + \frac{3}{6}\omega_2\right) \neq \frac{2}{6}\eta_1 + \frac{3}{6}\eta_2$

- Set up of the proof of Theorem 4.4 (ii)
  We consider the domain F defined by

$$F = \left\{ \tau \in \mathbb{H} \middle| 0 \leq \mathrm{Re}\,\tau \leq 1, \left|\tau - \frac{1}{2}\right| \geq \frac{1}{2} \right\}$$

  and denote $\Omega_5 \cap F$ by Ω_5.
  Let

$$\Gamma_0(2) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \mod 2 \right\}.$$

  Then F is a fundamental domain of $\Gamma_0(2)$ and three times of the fundamental domain of $SL_2(\mathbb{Z})$.

- By Theorem 4.1, for any $\tau \in \Omega_5$, there exists an unique pair $\pm p(\tau)$ of non-half-period critical point of G. Without loss of generality, we write $p = t\omega_1 + s\omega_2$, $0 \le t < 1$ and $0 \le s \le \frac{1}{2}$, but $(t,s) \notin \left\{(0,0), \left(\frac{1}{2},0\right), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right)\right\}$. Thus , we define a map form Ω_5 to $\square = \left\{(t,s)\,|0 \le t \le 1,\, 0 \le s \le \frac{1}{2}\right\}$ by

$$\pi : \tau \in \Omega_5 \longmapsto p(\tau) = t\omega_1 + s\omega_2 = (t,s).$$

- Recall the equation for critical points of G:

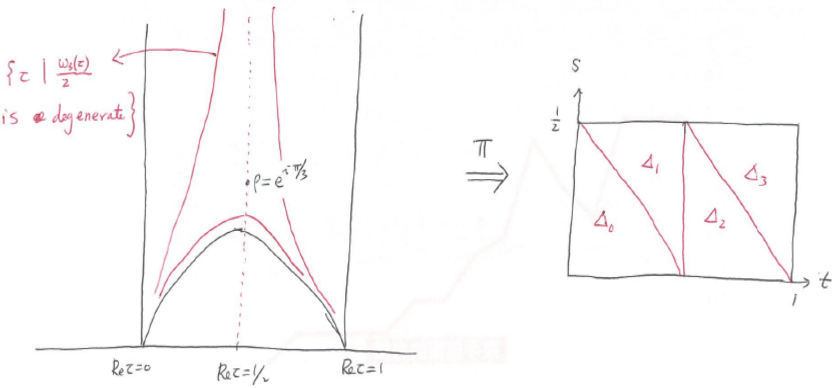
$$Z(t,s;\tau) = Z_{(t,s)}(\tau) = \zeta(t\omega_1 + s\omega_2;\tau) - (t\eta_1(\tau) + s\eta_2(\tau)).$$

Suppose $(t,s) \notin \left\{(0,0), \left(\frac{1}{2},0\right), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right)\right\}$. Then for any fixed (t,s),

$$Z_{(t,s)}(\tau) = 0 \text{ iff } \Delta u + e^u = 8\pi\delta_0 \text{ in } E_\tau \text{ has a solution.}$$

When (t,s) is a N-torsion point, $Z_{(t,s)}(\tau)$ is a modular form of weight 1 w.r.t. $\Gamma(N)$ (Hecke, 1926)

- We claim: $\pi : \Omega_5 \xrightarrow{onto} \Delta_1 = \left\{(t,s) \,|0 < t, s < \frac{1}{2}, t+s > \frac{1}{2}\right\} \subseteq \square$, or equivalently, for any $(t,s) \in \square$, $Z_{(t,s)}(\tau)$ has a solution in F iff $\tau \in \Delta_1$.



F is bounded by three black curves,

and $\{\infty, 0, 1\}$ are three cusp for $\Gamma_0(2)$.

Step 1: $\forall (t, s) \in \square$, we have

$$Z_{(t,s)}(\tau) \neq 0 \ \forall \tau \in \partial F.$$

(PDE's non-existence of $\Delta u + e^u = 8\pi\delta_0$ in E_τ if $\tau \in \partial F$)
Step 2: Consider

$$f(\tau) = \prod_{(t,s)} Z_{(t,s)}(\tau)$$

where $(t, s) = \left(\frac{m}{3}, \frac{n}{3}\right)$, $0 \leq m, n \leq 3$, and $(3, m, n)$ has no common divisor. Then $f(\tau)$ is a modular form of weight 8. Then we can use the counting zero formulas (4.2) to conclude that

$$Z_{\left(\frac{m}{3}, \frac{n}{3}\right)}(\tau) = 0$$

for $\left(\frac{m}{3}, \frac{n}{3}\right) \in \square$ iff $(m, n) = (1, 1)$ or $(2, 2)$. Note that $\left(\frac{1}{3}, \frac{1}{3}\right) \in \Delta$, because $\frac{1}{3} + \frac{1}{3} > \frac{1}{2}$. Furthermore, $Z_{\left(\frac{1}{3}, \frac{1}{3}\right)}(\tau) = 0$ has noly a zero at $\tau = e^{i\pi/3}$ which is also simple.

Step 3: the limit of $Z_{(t,s)}(\tau) \neq 0$ as $\tau \to \infty, i, 1+i$ (cusps of F) if $t \neq \frac{1}{2}$, $s \neq \frac{1}{2}$ and $t+s \neq \frac{1}{2}$. Hence $\forall (t,s) \in \Delta$, $\exists 1$ τ such that

$$Z_{(t,s)}(\tau) = 0.$$

Step 4: $\forall (t,s) \in \Delta_0 \cup \Delta_2 \cup \Delta_3$, there are no τ such that $Z_{(t,s)}(\tau) = 0$, because by Lemma 4.6,

$$\left(\frac{1}{6}, \frac{1}{6}\right) \in \Delta_0 \text{ and } \left(\frac{3}{4}, \frac{1}{4}\right) \in \Delta_1 \cup \Delta_2.$$

Step 5: $Z_{(t,s)}(\tau)$ has no solutions if $(t,s) \in \partial\Delta_0$.

In conclusion, for $n = 1$, we list some important results which will be generalized to $n \geq 2$:

(a) While deforming in τ, critical points of $G(z;\tau)$ could be bifurcated only at half periods. (i.e. trivial solutions)

(b) The equation

$$\Delta u + e^u = 8\pi\delta_0 \text{ in } E_\tau$$

has a solution iff there is

$$(t,s) \notin \left\{(0,0), \left(\frac{1}{2},0\right), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

such that τ is a zero of $Z_{(t,s)}(\tau)$.

§5. Generalization to $n \geq 2$

Let

$$G_k\left(p_1, \cdots, p_n\right) = \sum_{i<j} G\left(p_i - p_j\right) - n \sum G\left(p_i\right), \; p_i \in E.$$

Then any critical point of G_k is a solution to (2.3)

$$n \nabla G\left(p_i\right) = \sum_{j \neq i}^{n} \nabla G\left(p_i - p_j\right) \;\; \forall 1 \leq i \leq n. \tag{2.3}$$

where it is equivalent to (2.8) and (2.10)

$$n \sum_{i=1}^{n} \nabla G\left(p_i\right) = 0. \tag{2.8}$$

$$\sum_{j \neq i}^{n} \left(\zeta\left(p_i - p_j\right) + \zeta\left(p_j\right) - \zeta\left(p_i\right)\right) = 0, \; \forall 1 \leq i \leq n. \tag{2.10}$$

Recall that the Lame curve Y_n :

$$Y_n = \left\{ (p_1, \cdots, p_n) \left| \sum_{j \neq i} (\zeta(p_i - p_j) + \zeta(p_j) - \zeta(p_i)) = 0 \ \forall 1 \leq i \leq n \right. \right\}.$$

Then we can associate a "premodular form" $Z_n(\sigma; \tau)$, $\sigma \in E_\tau$, $\tau \in \mathbb{H}$ such that $Z_n(\sigma; \tau)$ is a modular form of weight $\frac{n(n+1)}{2}$ (if σ is a N-torsion point) w.r.t. the modular group $\Gamma(N)$.

Theorem 5.1

The equation

$$\Delta u + e^u = 8\pi\delta_0 \text{ in } E_\tau$$

has a solution iff there is some $\sigma \notin \left\{ (0,0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$ *such that*

$$Z_n(\sigma; \tau) = 0.$$

Conjecture 5.2 *For* $\sigma \notin \left\{ (0,0), \left(\frac{1}{2},0\right), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right) \right\}$, $Z_n(\sigma;\tau)$, *as a holomorphic function of* τ, *could have simple zeros only.*

The conjecture 5.2 can be applied to prove the following generation of (a) at $n = 1$.

Critical points (non-trivial solution of (2.3)) of G_k *could be bifurcated only at trivial solutions.*

The main purpose of our study is to generalize Theorem 4.1 for $n \geq 2$, i.e., to understand the solution structure of (2.3) in the moduli space of E_τ.

Remark *The Lame curve turns out to be the same as the spectral curve if the kdv when the potential is* $n(n+1)\wp(z)$.

Thank You Very Much!