Steady states with unbounded mass of the Keller-Segel system

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”Geometric Properties of Nonlinear Elliptic and Parabolic Equations”,

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References

A. Pistoia, G. Vaira.
Steady states with unbounded mass of the Keller-Segel system.
Proceedings Royal Society Edinburgh (to appear)

M. del Pino, A. Pistoia, G. Vaira.
Large mass boundary condensation patterns in the stationary Keller-Segel system
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The Keller-Segel model in chemotaxis
is a system of PDE's modelling
a mutual attraction of amoebae
caused by releasing a chemical substance

$$(KS) \qquad \begin{cases} v_t = \Delta v - \nabla(v \nabla u) & \text{in } \Omega, \\ \tau u_t = \Delta u - u + v & \text{in } \Omega, \\ v, u > 0 & \text{in } \Omega, \\ v(x,0) = v_0(x),\ u(x,0) = u_0(x) & \text{in } \Omega, \\ \partial_\nu v = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

- v is the concentration of amoebae
- u is the chemical released
- Ω is an open bounded domain in $\mathbb{R}^n$, $n \geq 2$
- $\tau > 0$
- ν is the outward normal at $\partial\Omega$

Stationary states

(u, v) is a stationary state of the system,

i.e. (u, v) solves

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$$(E)_\lambda \qquad \begin{cases} -\Delta u + u = \lambda e^u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \qquad (\text{we use the } 2^{nd} \text{ equation})$$

Known results

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- Senba & Suzuki (2000), Wang & Wei (2002) $\Rightarrow$ if $\Omega \subset \mathbb{R}^2$ there exist solutions with mass different from some critical values, i.e.

  if $\mu \in (0, \mu_0)$, $\mu \neq 4\pi, 8\pi, 12\pi, \ldots$ there exists a solution u_λ of $(E)_\lambda$ such that

$$\int_\Omega \lambda e^{u_\lambda(x)} dx = \mu.$$

Theorem (Del Pino & Wei (2006))

Let $\Omega \subset \mathbb{R}^2$.

For any integer k there exists λ_k such that if $\lambda \in (0, \lambda_k)$

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If $k = l + 2h$, then u_λ has

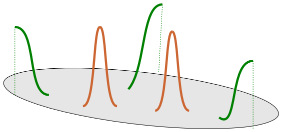
l boundary peaks $\eta_1, \dots, \eta_l \in \partial\Omega$

h interior peaks $\xi_1, \dots, \xi_h \in \Omega$

and

$$u_\lambda(x) \to \sum_{i=1}^{l} 4\pi G(x, \eta_i) + \sum_{i=1}^{h} 8\pi G(x, \xi_i)$$

uniformly on compact subsets of $\overline{\Omega} \setminus \cup\{\eta_i, \xi_j\}$.



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$G(x, y)$ denotes the Green's function

$$-\Delta_x G + G = \delta_y \text{ in } \Omega, \quad \partial_\nu G = 0 \text{ on } \partial\Omega.$$

Theorem (Pistoia-Vaira (2012))

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be the unit ball

There exists λ_0 such that for any $\lambda \in (0, \lambda_0)$

there exists a radial solution u_λ of $(E)_\lambda$ such that

$$\lim_{\lambda \to 0} \frac{1}{|\ln \lambda|} \lambda e^{u_\lambda(x)} = c\delta_{\partial\Omega},$$

where c is a positive constant and

$\delta_{\partial\Omega}$ is the Dirac measure on the boundary $\partial\Omega$.

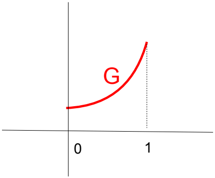
More precisely

There exists $\epsilon_\lambda \to 0$ as $\lambda \to 0$ such that

$$\epsilon_\lambda u_\lambda(r) \to \frac{\sqrt{2}}{G'(1)} G(r) \quad \text{uniformly in } [0,1]$$

where G solves

$$-G'' - \frac{n-1}{r} G' + G = 0 \text{ in } (0,1), \quad G(1) = 1.$$



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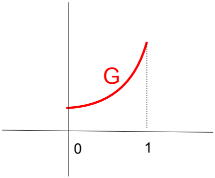


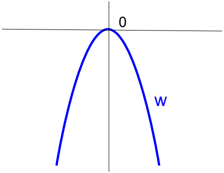
Moreover,

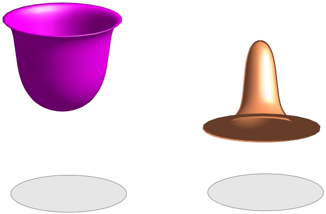
$$\tilde{u}_\lambda(r) := u_\lambda(\epsilon_\lambda r + 1) - \ln \lambda \to w(r)$$

C^1−uniformly in compact sets of $\mathbb{R}$,

where $w(r) := \ln \frac{4e^{\sqrt{2}r}}{(1+e^{\sqrt{2}r})^2}$ solves

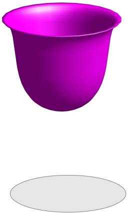
$$-w'' = e^w \text{ in } \mathbb{R}, \; w(0) = w'(0) = 0.$$





The new radial solution

The known radial solution



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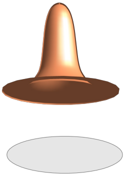

The known radial solution

Question

Does there exist such a solution in any domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$!

Theorem (Del Pino-Pistoia-Vaira (2014))

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain.

There exist $\lambda_0 > 0$ and a sequence $\lambda_i^* \to 0$ such that for any $\lambda \in (0, \lambda_0) \setminus \{\lambda_i^*\}$

there exists a solution u_λ of $(E)_\lambda$ such that

$$\lim_{\lambda \to 0} \frac{1}{|\ln \lambda|} \lambda e^{u_\lambda(x)} = c\delta_{\partial\Omega},$$

where c is a positive constant and

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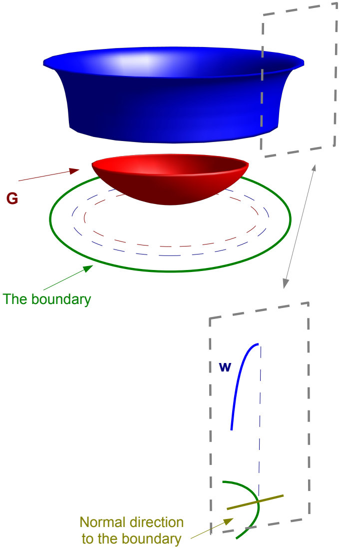
- close to the boundary
  u_λ looks like w

  $$-w'' = e^w \text{ in } \mathbb{R}$$

- far away from the boundary
  u_λ looks like G

  $$\begin{cases} -\Delta G + G = 0 & \text{in } \Omega, \\ G = 1 & \text{on } \partial\Omega \end{cases}$$



The proof

I will give the proof in the radial case.

In the general case the idea is the same, but it requires a lot of technicalities.

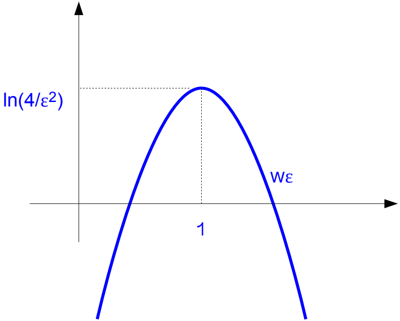
An idea of the proof

Let

$$w_\epsilon(r) := \ln \frac{4}{\epsilon^2} \frac{e^{\sqrt{2}\frac{r-1}{\epsilon}}}{\left(1 + e^{\sqrt{2}\frac{r-1}{\epsilon}}\right)^2} \quad r \in \mathbb{R},\ \epsilon > 0.$$

be the solution of

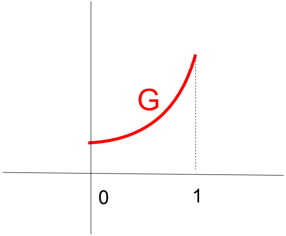
$$-w'' = e^w \text{ in } \mathbb{R}$$



The approximation far away from the boundary

Let G be the solution of

$$-G'' - \frac{n-1}{r} G' + G = 0 \text{ in } (0,1), \quad G(1) = 1.$$



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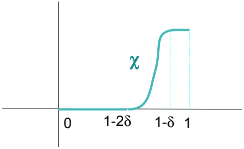

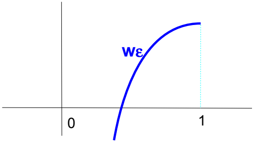

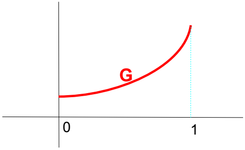

A contraction mapping argument

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$$\underbrace{-\phi'' - \frac{n-1}{r}\phi' + \phi - \lambda e^{U_\lambda}\phi}_{\text{the linear map } L_\lambda(\phi)} = \underbrace{\lambda\left[e^{U_\lambda + \phi} - e^{U_\lambda} - e^{U_\lambda}\phi\right]}_{\text{the second order term } N_\lambda(\phi)}$$

$$+ \underbrace{U_\lambda'' + \frac{n-1}{r}U_\lambda' - U_\lambda + \lambda e^{U_\lambda}}_{\text{the error term } R_\lambda}$$

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We prove that T_λ is a contraction map, since if $\lambda \to 0$

- the error term goes to zero, i.e. $\|R_\lambda(\phi)\|_{\mathrm{L}^1([0,1])} = o(\epsilon)$
- the linear map is invertible, i.e. $\left\|L_\lambda^{-1}\right\|_{\mathcal{L}(\mathrm{L}^1([0,1]),\mathrm{L}^\infty([0,1]))} = O(1)$

The study of the linear theory is quite involved!

Theorem (Grossi (2008))

The solutions to the linear problem

$$-\phi'' = \frac{4e^{\sqrt{2}r}}{(1+e^{\sqrt{2}r})^2}\phi \quad \text{in } \mathbb{R}$$

are linear combination of

$$\underbrace{\phi_1(r) := \frac{e^{\sqrt{2}r}-1}{e^{\sqrt{2}r}+1}}_{\text{this does not decay at infinity!}} \quad \text{and} \quad \underbrace{\phi_2(y) := -2 + 2r\frac{e^{\sqrt{2}r}-1}{e^{\sqrt{2}r}+1}}_{\text{this is unbounded!}}$$

Estimate of the error term close to the boundary

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$$= \cancel{e^{w_\epsilon(r)}} + \frac{n-1}{r} \underbrace{w_\epsilon'(r)}_{\sim \frac{n-1}{r}\frac{\sqrt{2}}{\epsilon}} - \underbrace{w_\epsilon(r)}_{\sim \ln\frac{4}{\varepsilon^2} + \frac{\sqrt{2}}{\epsilon}(r-1)} + \underbrace{\ln\lambda}_{= \ln\frac{4}{\varepsilon^2} + \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon} + \cancel{\lambda e^{w_\epsilon(r) - \ln\lambda}}$$

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$$\sim \underbrace{\frac{n-1}{r}\frac{\sqrt{2}}{\epsilon} - \frac{\sqrt{2}}{\epsilon}(r-1) + \frac{\omega_1}{\varepsilon}}_{\text{has to be killed!}} = O\left(\frac{1}{\epsilon}\right)$$

Estimate of the error term close to the boundary

$$R_\lambda(r) = w_\epsilon''(r) + \frac{n-1}{r} w_\epsilon'(r) - (w_\epsilon(r) - \ln\lambda) + \lambda e^{w_\epsilon(r) - \ln\lambda}$$

$$= \cancel{e^{w_\epsilon(r)}} + \frac{n-1}{r} \underbrace{w_\epsilon'(r)}_{\sim \frac{n-1}{r}\frac{\sqrt{2}}{\epsilon}} - \underbrace{w_\epsilon(r)}_{\sim \ln\frac{4}{\varepsilon^2} + \frac{\sqrt{2}}{\epsilon}(r-1)} + \underbrace{\ln\lambda}_{= \ln\frac{4}{\varepsilon^2} + \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon} + \cancel{\lambda e^{w_\epsilon(r) - \ln\lambda}}$$

$$\sim \underbrace{\frac{n-1}{r}\frac{\sqrt{2}}{\epsilon} - \frac{\sqrt{2}}{\epsilon}(r-1) + \frac{\omega_1}{\varepsilon}}_{\text{has to be killed!}} = O\left(\frac{1}{\epsilon}\right)$$

$$\Downarrow$$

we have to add the local term α_ϵ

$$-\alpha_\epsilon''(r) - \frac{n-1}{r}\alpha_\epsilon'(r) = \frac{n-1}{r}\frac{\sqrt{2}}{\epsilon} - \frac{\sqrt{2}}{\epsilon}(r-1) + \frac{\omega_1}{\varepsilon} \text{ in } [0,1],\ \alpha_\epsilon(1) = \alpha_\epsilon'(1) = 0$$

$$R_\lambda(r) = w_\epsilon''(r) + \frac{n-1}{r} w_\epsilon'(r) - (w_\epsilon(r) - \ln\lambda) + \lambda e^{w_\epsilon(r) - \ln\lambda}$$

$$= \cancel{e^{w_\epsilon(r)}} + \frac{n-1}{r} \underbrace{w_\epsilon'(r)}_{\sim \frac{n-1}{r}\frac{\sqrt{2}}{\epsilon}} - \underbrace{w_\epsilon(r)}_{\sim \ln\frac{4}{\varepsilon^2} + \frac{\sqrt{2}}{\epsilon}(r-1)} + \underbrace{\ln\lambda}_{= \ln\frac{4}{\varepsilon^2} + \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon} \cancel{+\lambda e^{w_\epsilon(r) - \ln\lambda}}$$

$$\sim \underbrace{\frac{n-1}{r}\frac{\sqrt{2}}{\epsilon} - \frac{\sqrt{2}}{\epsilon}(r-1) + \frac{\omega_1}{\varepsilon}}_{\text{has to be killed!}} = O\left(\frac{1}{\epsilon}\right)$$

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$$\Downarrow$$

$$R_\lambda(r) = (w_\epsilon + \alpha_\epsilon)''(r) + \frac{n-1}{r}(w_\epsilon + \alpha_\epsilon)'(r) - (w_\epsilon(r) - \ln\lambda + \alpha_\epsilon(r)) + \lambda e^{w_\epsilon(r) - \ln\lambda + \alpha_\epsilon(r)}$$

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$$\sim \frac{4}{\varepsilon^2}\frac{e^{\sqrt{2}s}}{\left(1+e^{\sqrt{2}s}\right)^2}\underbrace{\alpha_\epsilon(\varepsilon s + 1)}_{\sim \varepsilon\alpha_0(s)}$$

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$$\Downarrow$$

we have to add the global term $v_\epsilon(r) := \epsilon v\left(\frac{r-1}{\varepsilon}\right)$

$$-v'' - 4\frac{e^{\sqrt{2}s}}{\left(1+e^{\sqrt{2}s}\right)^2}v = 4\frac{e^{\sqrt{2}s}}{\left(1+e^{\sqrt{2}s}\right)^2}\alpha_0(s) \quad \text{in } \mathbb{R}.$$

$$R_\lambda(r) = (w_\epsilon + \alpha_\epsilon)''(r) + \frac{n-1}{r}(w_\epsilon + \alpha_\epsilon)'(r) - (w_\epsilon(r) - \ln\lambda + \alpha_\epsilon(r)) + \lambda e^{w_\epsilon(r) - \ln\lambda + \alpha_\epsilon(r)}$$

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$$\Downarrow$$

$$R_\lambda(r) \sim O(1)$$

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$$\Downarrow$$

$$R_\lambda(r) \sim O(1)$$

$$\Downarrow$$

we have to add a local term β_ε and a global term z_ε till we have

$$R_\lambda(r) = o(\varepsilon)$$

Estimate of the error term far away from the boundary

$$R_\lambda(r) = \underbrace{(\gamma_\epsilon G(r))'' + \frac{n-1}{r}(\gamma_\epsilon G(r))' - \gamma_\epsilon G(r)}_{=0} + \lambda e^{\gamma_\epsilon G(r)}$$

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$$= \lambda e^{\gamma_\epsilon \overbrace{G(r)}^{\leq G(1)}}$$

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$$= o(\epsilon) \text{ if } \omega_1 > \gamma_1$$

$$R_\lambda(r) = \underbrace{(\gamma_\epsilon G(r))'' + \frac{n-1}{r}(\gamma_\epsilon G(r))' - \gamma_\epsilon G(r)}_{=0} + \lambda e^{\gamma_\epsilon G(r)}$$

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Recall that

Estimate of the error term far away from the boundary

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Recall that

- $\lambda = \frac{4}{\epsilon^2} e^{-\left(\frac{\omega_1}{\epsilon} + \omega_2 + \omega_3 \epsilon\right)}$

Estimate of the error term far away from the boundary

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- $\gamma_\varepsilon = \frac{\gamma_1}{\epsilon} + \gamma_2 + \gamma_3 \epsilon$

Estimate of the error term far away from the boundary

$$R_\lambda(r) = \underbrace{(\gamma_\epsilon G(r))'' + \frac{n-1}{r}(\gamma_\epsilon G(r))' - \gamma_\epsilon G(r)}_{=0} + \lambda e^{\gamma_\epsilon G(r)}$$

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Recall that

- $\lambda = \frac{4}{\epsilon^2} e^{-\left(\frac{\omega_1}{\epsilon} + \omega_2 + \omega_3 \epsilon\right)}$
- $\gamma_\varepsilon = \frac{\gamma_1}{\epsilon} + \gamma_2 + \gamma_3 \epsilon$
- $G \nearrow$

Estimate of the error term in the interspace

We have to glue U_λ^{bd} and U_λ^{in} up to the third order:

$$\left|U_\lambda^{bd}(r) - U_\lambda^{in}(r)\right| = O\left(\epsilon^2\right), \quad \left|\left(U_\lambda^{bd}(r)\right)' - \left(U_\lambda^{in}(r)\right)'\right| = O(\epsilon)$$

$$\Downarrow$$

We have to glue U_λ^{bd} and U_λ^{in} up to the third order:

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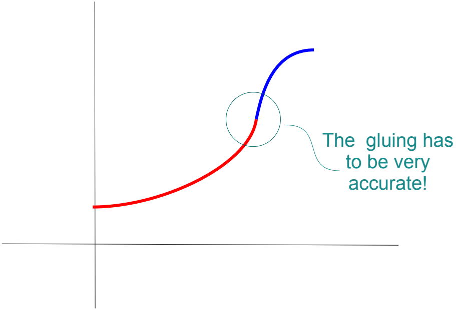
$$\Downarrow$$

We have to choose $\omega_1, \omega_2, \omega_3$ and $\gamma_1, \gamma_2, \gamma_3$!



- $U_\lambda^{bd}(r) = \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon + \underbrace{\ln \frac{e^{\sqrt{2}\frac{r-1}{\epsilon}}}{\left(1 + e^{\sqrt{2}\frac{r-1}{\epsilon}}\right)^2} + \alpha_\epsilon(r)}_{=O\left(\frac{1}{\epsilon}\right)} + \underbrace{v_\epsilon(r) + \beta_\varepsilon(r)}_{=O(1)} + \underbrace{z_\varepsilon(r)}_{=O(\epsilon)}$

- $U_\lambda^{bd}(r) = \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon + \underbrace{\ln \frac{e^{\sqrt{2}\frac{r-1}{\epsilon}}}{\left(1+e^{\sqrt{2}\frac{r-1}{\epsilon}}\right)^2} + \alpha_\epsilon(r)}_{=O(\frac{1}{\epsilon})} + \underbrace{v_\epsilon(r) + \beta_\varepsilon(r)}_{=O(1)} + \underbrace{z_\varepsilon(r)}_{=O(\epsilon)}$

- $U_\lambda^{in}(r) = \left(\frac{\gamma_1}{\epsilon} + \gamma_2 + \gamma_3\epsilon\right) G(r),$

- $U_\lambda^{bd}(r) = \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon + \underbrace{\ln \frac{e^{\sqrt{2}\frac{r-1}{\epsilon}}}{\left(1+e^{\sqrt{2}\frac{r-1}{\epsilon}}\right)^2} + \alpha_\epsilon(r)}_{=O(\frac{1}{\epsilon})} + \underbrace{v_\epsilon(r) + \beta_\varepsilon(r)}_{=O(1)} + \underbrace{z_\varepsilon(r)}_{=O(\epsilon)}$
- $U_\lambda^{in}(r) = \left(\frac{\gamma_1}{\epsilon} + \gamma_2 + \gamma_3\epsilon\right) G(r),$

$$U_\lambda^{bd}(r) - U_\lambda^{in}(r)$$

- $U_\lambda^{bd}(r) = \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon + \underbrace{\ln \frac{e^{\sqrt{2}\frac{r-1}{\epsilon}}}{\left(1 + e^{\sqrt{2}\frac{r-1}{\epsilon}}\right)^2} + \alpha_\epsilon(r)}_{=O(\frac{1}{\epsilon})} + \underbrace{v_\epsilon(r) + \beta_\varepsilon(r)}_{=O(1)} + \underbrace{z_\varepsilon(r)}_{=O(\epsilon)}$
- $U_\lambda^{in}(r) = \left(\frac{\gamma_1}{\epsilon} + \gamma_2 + \gamma_3\epsilon\right) G(r),$

$$U_\lambda^{bd}(r) - U_\lambda^{in}(r) = \overbrace{\frac{\omega_1}{\epsilon} + \frac{\sqrt{2}}{\epsilon}(r-1)}^{U_\lambda^{bd}(r)} - \overbrace{\frac{\gamma_1}{\epsilon}\left[G(1) + G'(1)(r-1)\right]}^{U_\lambda^{in}(r)} + O(1)$$

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$= O(1)$ if we choose $\boxed{\omega_1 = \gamma_1 G(1) \quad \& \quad \sqrt{2} = \gamma_1 G'(1)}$

$$\Downarrow$$

the term of order $\frac{1}{\varepsilon}$ is ok!

- $U_\lambda^{bd}(r) = \frac{\omega_1}{\varepsilon} + \omega_2 + \omega_3\epsilon + \underbrace{\ln \frac{e^{\sqrt{2}\frac{r-1}{\epsilon}}}{\left(1+e^{\sqrt{2}\frac{r-1}{\epsilon}}\right)^2} + \alpha_\epsilon(r)}_{=O(\frac{1}{\epsilon})} + \underbrace{v_\epsilon(r) + \beta_\varepsilon(r)}_{=O(1)} + \underbrace{z_\varepsilon(r)}_{=O(\epsilon)}$
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$\Downarrow$

the term of order $\frac{1}{\varepsilon}$ is ok!

...and so on!

Conjecture on the Dirichlet problem

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain s.t.

there exists a curve $\gamma \subset \Omega$ such that the problem

$$\begin{cases} \Delta G = 0 \text{ in } \Omega \setminus \gamma, \ G = 0 \text{ on } \partial\Omega \\ G = 1 \text{ on } \gamma, \ \partial_{\nu_+} G = -\partial_{\nu_-} G \text{ on } \gamma \end{cases}$$

has a *non degenerate* solution.

Then there exist $\lambda_0 > 0$ and a sequence $\lambda_i^* \to 0$ such that for any $\lambda \in (0, \lambda_0) \setminus \{\lambda_i^*\}$

there exists a solution u_λ of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$\lim_{\lambda \to 0} \frac{1}{|\ln \lambda|} \lambda e^{u_\lambda(x)} = c\delta_\gamma,$$

where c is a positive constant depending on γ and δ_γ is the Dirac measure on the curve γ.

A first question!

When does there exist such a curve? For example, if Ω is a hole ...

More precisely, the profile looks like ...

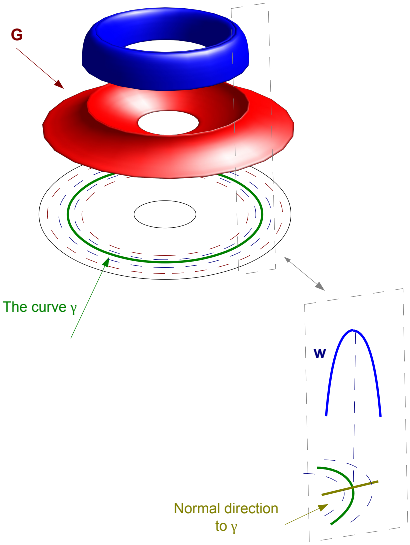
- close to γ
  u_λ looks like w

  $$-w'' = e^w \text{ in } \mathbb{R}$$

- far away from γ
  u_λ looks like G

  $$\begin{cases} \Delta G = 0 \text{ in } \Omega \setminus \gamma, \ G = 0 \text{ on } \partial\Omega \\ G = 1 \text{ on } \gamma, \ \partial_{\nu_+} G = -\partial_{\nu_-} G \text{ on} \end{cases}$$

- THE MAIN DIFFICULTY:
  to make the gluing
  of the two profiles!



THANK YOU FOR YOUR ATTENTION!