On the Chern-Simons Equations on a Torus

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(joint work with Chang-Shou Lin)

Consider the following Chern-Simons-Higgs equation:

(1)
$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^u(1 - e^u) = 4\pi \sum_{j=1}^{N} \delta_{p_j}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

where Ω is a parallelogram in $\mathbb{R}^2$, $p_j \in \Omega$, $j = 1, \cdots, N$, and some of the p_i may coincide.

Introduce

$$u_0(x) = -4\pi \sum_{j=1}^{N} G(x, p_j),$$

where $G(x, p_j)$ is the Green function:

$$\Delta G(x, p_j) = -\delta_{p_j} + \frac{1}{|\Omega|},$$

$$\int_\Omega G(x, p_j)\, dx = 0.$$

Note that $G(x, p_j)$ is not harmonic. Near each vortex point p_j,

$$u_0(x) = 2m \ln |x - p_j| + O(1),$$

where m is the number of p_i satisfying $p_i = p_j$.

We can use the function u_0 to remove the singularities from (1).

Replace u by $u + u_0$ in (1), then u satisfies

$$
(2) \qquad \begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^{u+u_0}(1 - e^{u+u_0}) = \frac{4\pi N}{|\Omega|}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases}
$$

Note that u_0 has a singularity at p_j. But near p_j,

$$
e^{u_0} \sim |x - p_j|^{2m}
$$

and e^{u_0} is a smooth function.

A Result for (2)**:**

Theorem A(K.Choe and N.Kim, 2008) For $\varepsilon_n \to 0$, then one of the following is true

(a) $u_n + u_0 \to 0$ uniformly in any compact subset of $\Omega \setminus \{p_1, \cdots, p_N\}$;

(b) $u_n + \ln \frac{1}{\varepsilon_n^2}$ is bounded;

(c) there are $x_{1,n}, \cdots, x_{k,n} \in \Omega$, such that as $n \to +\infty$, $x_{j,n} \to q_j$,

$$u_n(x_{j,n}) + \ln \frac{1}{\varepsilon_n^2} \to +\infty, \quad \forall\, j = 1, \cdots, k,$$

and

$$u_n(x) + \ln \frac{1}{\varepsilon_n^2} \to -\infty, \quad \text{uniformly on any compact subset of } \Omega \backslash \{q_1, \cdots, q_k\} \subset \Omega.$$

Moreover,

$$\frac{1}{\varepsilon_n^2} e^{u_n + u_0}(1 - e^{u_n + u_0}) \to \sum_{j=1}^{k} M_j \delta_{q_j}, \quad M_j \geq 8\pi.$$

Solution satisfying (a) is called a topological solution. Solution satisfying (b) or (c) is called a non-topological solution. Solution satisfying (c) is called a bubbling solution.

Integrate the equation, we find

$$4\pi N = \frac{1}{\varepsilon_n^2} \int_\Omega e^{u_n+u_0}(1 - e^{u_n+u_0}).$$

So,

$$\sum_{j=1}^{k} M_j = 4\pi N.$$

From $M_j \geq 8\pi$, we see $k \leq \frac{N}{2}$. If $k = \frac{N}{2}$, then $M_j = 8\pi$.

Questions Concerning the Bubbling Solutions:

(1) Can different type of blow-up points appear in a sequence of bubbling solutions?

(2) Necessary and sufficient conditions for the existence.

(3) Local uniqueness of bubbling solutions: Suppose $u_{n,i}$, $i = 1, 2$, are two sequence of blow-up solutions and they blow up at the same points $\{q_1, \cdots, q_k\}$. Is $u_{n,1} = u_{n,2}$ for large n?

(4) Can we count the exact number of the solutions for (2)?

(1) Non-coexistence of Bubbles

Define the local strength of a bubbling solution u_ε at q_i as follows:

(3) $$M_{\varepsilon,i} = \frac{1}{\varepsilon^2} \int_{B_\delta(x_{\varepsilon,i})} e^{u_\varepsilon + u_0}(1 - e^{u_\varepsilon + u_0}), \quad i = 1, \cdots, k,$$

where $x_{\varepsilon,i} \in B_\delta(q_i)$ is a point such that $u_\varepsilon(x_{\varepsilon,i}) = \max_{y \in B_\delta(q_i)} u_\varepsilon(y)$.

The limit problems:

Assume the blow-up point $q_j \notin \{p_1, \cdots, p_N\}$. After a suitable scaling, the solutions converge to an entire solution u to either

(4) $$\Delta u + e^u = 0, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u = 8\pi = \lim_{\varepsilon \to 0} M_{\varepsilon,j},$$

or

(5) $$\Delta u + e^u(1 - e^u) = 0, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u(1 - e^u) = M_j := \lim_{\varepsilon \to 0} M_{\varepsilon,j}.$$

The type of the blow-up point q_i is determined by the local strength $M_{\varepsilon,i}$.

A blow-up point q_j is called the mean field type, or MF type, if the limit equation is (4), while it is called Chern-Simons type, or CS type, if the limit equation is (5).

Question 1. Do the MF type blow-up point and the CS type blow-up point, or different CS type blow-up points, co-exist in a sequence of bubbling solutions?

Answer: No.

Theorem 0.1. *Suppose that* u_ε *is a sequence of bubbling solutions for* (2)*, whose blow-up set is* $\{q_1, \cdots, q_k\}$ *as* $\varepsilon \to 0$*. If* $q_i \notin \{p_1, \cdots, p_N\}$*,* $i = 1, \cdots, k$*, then* $M_{\varepsilon,i} = \frac{4\pi N}{k} + o(1)$*,* $i = 1, \cdots, k$*. Moreover, all the* q_i *are of mean field type if* $N = 2k$*, while all the* q_j *are of Chern-Simon type if* $N > 2k$*.*

Outline of the Proof of Theorem 0.1

(6) $$u_\varepsilon(x) = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon + O(1), \quad \text{in } C^1(\Omega \setminus \cup_{i=1}^k B_\theta(x_{\varepsilon,i})),$$

where $u_\varepsilon(x_{\varepsilon,i}) = \max_{B_\theta(q_i)} u_\varepsilon(y)$.

Simple blow-up (this means that in $B_\theta(x_{\varepsilon,i})$, u_ε is controlled by one bubble) implies :

(7) $$u_\varepsilon(x) = M_{\varepsilon,i} \ln \varepsilon + O(1), \quad \text{in } \partial B_\theta(x_{\varepsilon,i}).$$

Combining (6) and (7) yields

$$M_{\varepsilon,i} \ln \varepsilon = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon + O(1).$$

Existence

Using a Pohozaev identity and Theorem 0.1, we can prove that if $\{q_1, \cdots, q_k\}$ is a blow-up set of a sequence of bubbling solutions, $q_j \notin \{p_1, \cdots, p_N\}$, then $\mathbf{q}$ must satisfy

(8)
$$D_h u_0(q_j) + \frac{4\pi N}{k} \sum_{i \neq j,\ 1 \leq i \leq k} D_{q_{jh}} G(q_i, q_j) = 0$$

which implies $\mathbf{q}$ is a critical point of the function

(9)
$$G_k(\mathbf{x}) = \frac{2\pi N}{k} \sum_{i \neq j, 1 \leq i,\, j \leq k} G(x_i, x_j) + \sum_{j=1}^{k} u_0(x_j), \quad \mathbf{x} = (x_1, \cdots, x_k),\ x_j \in \Omega.$$

Question 2. If $\mathbf{q}$ is a non-degenerate critical point of $G_k(\mathbf{x})$, is there a sequence of bubbling solutions, whose blow-up set is $\{q_1, \cdots, q_k\}$?

Answer Yes in the Chern-Simons case ($2k < N$). No in the MF case ($N = 2k$).

In the MF case, there is an extra necessary condition in addition to $DG_k(\mathbf{q}) = 0$.

Extra necessary condition in the MF case.

Define the quantity $D(\mathbf{q})$ as follows.

(10) $$D(\mathbf{q}) = \sum_{i=1}^{k} \rho_i \Big(\lim_{\theta \to 0} \int_{\Omega_i \setminus B_\theta(q_i)} \frac{e^{f_{\mathbf{q},i}} - 1}{|y - q_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - q_i|^4} \Big),$$

where Ω_i is any open set satisfying with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, $\cup_{i=1}^{k} \bar{\Omega}_i = \bar{\Omega}$, $B_\delta(q_i) \subset\subset \Omega_i, i = 1, \cdots, k$,

$$
\begin{aligned}
f_{\mathbf{q},i}(y) =& 8\pi\Big(\gamma(y,q_i) - \gamma(q_i,q_i) + \sum_{j\neq i}\big(G(y,q_j) - G(q_i,q_j)\big)\Big) \\
& + u_0(y) - u_0(q_i),
\end{aligned}
$$

and

$$
\rho_i = e^{8\pi\left(\gamma(q_i,q_i)+\sum_{j\neq i} G(q_i,q_j)\right)+u_0(q_i)}.
$$

Remark. The quantity $D(\mathbf{q})$ is independent of the decomposition. It is only defined for $\mathbf{q}$ satisfying $DG_k(\mathbf{q}) = 0$.

Theorem 0.2. *Suppose $q_j \notin \{p_1, \cdots, p_N\}$. If u_n is a MF type bubbling solution to* (2)*, then the blow-up set $\mathbf{q}$ satisfies $D(\mathbf{q}) \leq \mathbf{0}$.*

To prove Theorem 0.2, we need to find the equation that links ε with the local maximum points of the bubbling solution u_ε.

We have the following existence result:

Theorem 0.3. *Suppose that $\mathbf{q}$ satisfies $q_j \notin \{p_1, \cdots, p_N\}$, $DG_k(\mathbf{q}) = 0$ and $det(D^2 G_k(\mathbf{q})) \neq 0$. Then for $\varepsilon > 0$ small,*

(i) (2) *has a CS type bubbling solution, whose blow set is $\{q_1, \cdots, q_k\}$ if $N > 2k$;*

(ii) (2) *has a MF type bubbling solution, whose blow set is $\{q_1, \cdots, q_k\}$ if $D(\mathbf{q}) < 0$ and $N = 2k$.*

Local Uniqueness

We have the following result.

Theorem 0.4. *Suppose that* $u_{\varepsilon,1}$ *and* $u_{\varepsilon,2}$ *are two solutions of* (2) *which blow up at* $\mathbf{q}$ *satisfying* $DG_k(\mathbf{x})) = 0$, $q_j \notin \{p_1, \cdots, p_N\}$, $j = 1, \cdots, k$ *and* $det(D^2 G_k(\mathbf{x})) \neq 0$. *Then*

(i) $u_{\varepsilon,1} = u_{\varepsilon,2}$ *if* $N = 2k$ *and* $D(\mathbf{q}) < \mathbf{0}$*;*

(ii) $u_{\varepsilon,1} = u_{\varepsilon,2}$ *if* $N > 2k$.

Main difficulty to prove Theorem 0.4

Suppose that (2) has two different solutions. Consider the normalization of the difference of $u_{n,i}$: $\xi_n = (u_{n,1} - u_{n,2})/\|u_{n,1} - u_{n,2}\|_{L^\infty(\Omega)}$

- Heuristically, ξ_n attains its maximum in a neighborhood of some q_j.
- After a suitable scaling, ξ_n will converge to $\xi(x)$, satisfying a linearized equation in $\mathbb{R}^2$, whose kernel is three dimensional in the MF blow-up, or two dimensional in the Chern-Simons blow-up.
- The difficult part is to kill the non-trivial kernel, because we do not know a priori that ξ is orthogonal to the kernel of the linear operator (in the proof of existence, this can be achieved a priori). Various kind of Pohozaev identities and sharp estimates for the local bubbling behavior are used to kill the non-trivial kernel.

Exact Number of Solutions

We count the number of solutions according to the classification in Theorem A.

Topological Solution

- Existence: Caffarelli and Y.Yang, Spruck and Y. Yang.
- Uniqueness. Tarantello.

So,

Theorem B. *For any configuration* $p_1, \cdots, p_N$, (2) *has a unique topological solution provided that* $\varepsilon > 0$ *is small enough.*

Solutions satisfying (b) in Theorem A.

If u_ε is a solution satisfying (b) in Theorem A. Then $u_\varepsilon \to u$ which satisfies

(11)
$$\begin{cases} \Delta u + e^{u+u_0} = \frac{4\pi N}{|\Omega|}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

If (11) has no solution, then (2) has no solution satisfying (b) in Theorem A.

Existence or non-existence of solutions satisfying (b) in Theorem A is a very difficult problem. We consider that all the vortex points collapse into one point, say 0, and Ω is a rectangle. Then, $u_0 = -4\pi N G(x, 0)$.

Theorem C. (C.-S. Lin and C.-L. Wang) *If $N = 2, 4$,* (11) *has no solution. In particular,* (2) *has no solution satisfying (b) in Theorem A.*

On the other hand, we are able to prove

Theorem 0.5. *Let $N = 2m - 1$, $m = 1, 2, 3$ and Ω is a rectangle. Then the number of solutions for* (11) *is m, all of which are non-degenerate. In particular, the number of solutions to (2) satisfying (b) of Theorem A is equal to m if ε is small.*

Bubbling Solutions

To count the number of the bubbling solutions, we need to know

- the number of critical points of G_k, whether they are all non-degenerate;
- in the case $N = 2k$, whether it always holds $D \neq 0$ at any critical point of G_k, and the number of critical points of G_k at which $D < 0$.

Similar to the discussion of case (b) solutions, we assume $p_1 = \cdots = p_N = 0$ and Ω is a rectangle. So, we have $u_0 = -4\pi N G(x)$, $G(x) = G(x, 0)$. The critical point $(q_1, \cdots, q_N)$ of G_k satisfies

$$-kD_h G(q_j) + \sum_{i \neq j,\ 1 \leq i \leq k} D_{q_{jh}} G(q_i, q_j) = 0 \tag{12}$$

To count the number of solutions for (12), we will use the theories for elliptic functions (the Weierstrass $\wp(z)$ functions). This was discussed in C.S.Lin's talk.

The quantity D

Recall

(13) $$D(\mathbf{q}) = \sum_{i=1}^{k} \rho_i \Big(\lim_{\theta \to 0} \int_{\Omega_i \setminus B_\theta(q_i)} \frac{e^{f_{\mathbf{q},i}} - 1}{|y - q_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - q_i|^4} \Big),$$

Rewrite D as

(14)
$$D(\mathbf{q}) = \sum_{i=1}^{k} \rho_i \lim_{\theta \to 0} \Big(\int_{\Omega_i \setminus B_\theta(q_i)} e^{4 \ln \frac{1}{|y - q_i|^4} + f_{\mathbf{q},i}} - \int_{\Omega_i \setminus B_\theta(q_i)} \frac{1}{|y - q_i|^4} \Big),$$

Note that

$$4\ln\frac{1}{|y-q_i|^4}+f_{\mathbf{q},i}(y)=8\pi\sum_{i=1}^{k}\big(G(y,q_j)-G(q_i,q_j)\big)-4N\pi\big(G(x)-G(q_i)\big)$$

is a doubly periodic harmonic function with singularities 0, q_j. The behavior of this function at each singularity is clear. So we can write

$$4\ln\frac{1}{|y-q_i|^4}+f_{\mathbf{q},i}(y)=\ln|f(z)|+C$$

where $f(z)$ is an elliptic function with singularities at 0, q_j, and C is a constant. Now the integral $\int_\Omega |f(z)|$ is computable.

Theorem D. (C.-C.Chen, C.-S. Lin and G. Wang) *If $k = 1$ ($N = 2$), $G_1 = -4\pi N G(x, 0)$ has exactly three critical points (the three half periods); all of them are all non-degenerate; and $D < 0$ only at the maximum point of G_1.*

- $G(x, 0)$ is symmetric with respect to the two coordinate axis. So $G(x, 0)$ only have the three half periods as its critical points.

- Using the maximum principle, $G_{x_1 x_1} \neq 0$, $G_{x_2 x_2} \neq 0$ and $G_{x_1 x_2} = 0$ at the critical point. This implies the non-degeneracy of the critical points.

- The method used in the paper by C.-C.Chen, C.-S. Lin and G. Wang to check the sign of D does not reveal the relation between the sign of D and the Hessian of $G(x, 0)$.

We can prove the following relation

$$D(q) = -c(q)det\big(D^2G(q,0)\big), \tag{15}$$

where q is a critical point, and $c(q) > 0$ is a constant.

From (15), $D > 0$ at the two saddle points, while $D < 0$ at the minimum point.

Theorem E. (C.-S. Lin and C.-L. Wang) *If $k = 2$ ($N = 4$), G_2 has exactly five critical points.*

For the properties of the five critical points of G_2, we have

Theorem 0.6. *All the critical points of G_2 are non-degenerate. Moreover, $det\big(D^2G_2(\mathbf{q})\big)$ is negative at two of the critical points of G_2, while it is positive at the other three.*

We have the following connection between D and D^2G_2.

Theorem 0.7. *For each critical points $\mathbf{q}$ of G_2, there is a constant $c(\mathbf{q}) > 0$, such that*

$$D(\mathbf{q}) = c(\mathbf{q})det\big(D^2G_2(\mathbf{q})\big). \tag{16}$$

In particular, $D \neq 0$ at each critical points $\mathbf{q}$ of G_2.

Theorem 0.8. *The quantity D is negative at two of the critical points of G_2, while it is positive at the other three.*

Conclusion

If $N = 1$, (2) has exactly 2 solutions: one topological solution and another solution satisfying (b) of Theorem A.

If $N = 2$, (2) has exactly 2 solutions: one topological solution and one MF type bubbling solution with one blow-up point.

If $N = 3$, (2) has exactly 6 solutions: one topological solution, two solutions satisfying (b) of Theorem A, and three CS type bubbling solutions with one blow-up point.

If $N = 4$, (2) has exactly 6 solutions: one topological solution, three CS type bubbling solutions with one blow-up point and two MF type bubbling solutions with two blow-up points.

If $N = 5$, (2) has exactly 12 solutions: one topological solution, three solutions satisfying (b) of Theorem A, three CS type bubbling solutions with one blowup point and five CS type bubbling solutions with two blowup points.

Theorem 0.9. *Suppose* $p_j = 0$, Ω *is a rectangle and* $1 \leq N \leq 5$. *Then (2) has exactly* $m(m+1)$ *solutions, where* $m = \frac{N+1}{2}$ *if* N *is odd, while* $m = \frac{N}{2}$ *if* N *is even.*

We propose the following conjecture.

Conjecture: Assume that $p_j = 0$, Ω is a rectangle. Suppose $N = 2m - 1$ or $2m$, $m \in \mathbb{N}$. Then equation (2) for small $\varepsilon > 0$ has exactly $m(m+1)$ solutions.