Harish-Chandra Series in Unitary Groups and Crystal Graphs

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Contents

This is a joint project with many contributors, in particular

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LEVI SUBGROUPS OF FINITE GROUPS OF LIE TYPE

Let G be a finite group of Lie type; then G is a finite group with a split BN-pair, of characteristic p, say.

There is a distinguished class of subgroups of G, the parabolic subgroups.

A parabolic subgroup P has a Levi decomposition $P = LU$, where U is the unipotent radical, L a Levi complement of P.

Levi subgroups of G resemble G; in particular, they are again groups of Lie type.

Later in this talk I will concentrate on the unitary groups.

THE GENERAL UNITARY GROUPS

Let $G = \mathrm{GU}_n(q) = \{A \in \mathrm{GL}_n(q^2) \mid A^{tr} J \overline{A} = J\}$, with

$J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{n \times n}$, and $\overline{[a_{ij}]} := [a_{ij}^q]$.

Let $r, m \in \mathbb{N}$ with $r + 2m = n$. Then

$$\left\{ \begin{bmatrix} A & & \\ & B & \\ & & A^\dagger \end{bmatrix} \mid A \in \mathrm{GL}_m(q^2), B \in \mathrm{GU}_r(q) \right\}$$

$$\cong \mathrm{GU}_r(q) \times GL_m(q^2)$$

is a Levi subgroup of G (where $A^\dagger = J\overline{A}^{-tr} J$).

Choosing all possible r, m with $r + 2m = n$, and in $\mathrm{GL}_m(q^2)$ all Levi subgroups, we obtain all Levi subgroups of G.

HARISH-CHANDRA INDUCTION

Let k be an algebraically closed field with $\mathrm{char}(k) = \ell \neq p$ (this includes the case $\ell = 0$).

Let $L \leq G$ be a Levi subgroup, and $P \leq G$ parabolic with Levi complement L.

The functor

$$R_L^G : kL\text{-mod} \to kG\text{-mod}$$

$$Y \mapsto R_L^G(Y) := \mathrm{Ind}_P^G(\mathrm{Infl}_L^P(Y))$$

is called Harish-Chandra induction (or parabolic induction).

It is independent of the choice of P with Levi complement L (Dipper-Du ('93), Howlett-Lehrer ('94)).

For $Y \in kL$-mod, we put $H(L, Y) := \mathrm{End}_{kG}(R_L^G(Y))$ for the Hecke algebra of the pair (L, Y).

HARISH-CHANDRA CLASSIFICATION

A simple $X \in kG$-mod is called cuspidal, if $X \not\leq R_L^G(Y)$ for all **proper** Levi subgroups $L \leq G$ and all $Y \in kL$-mod.

A cuspidal pair (L, Y) consists of a Levi subgroup $L \leq G$ and a cuspidal $Y \in kL$-mod.

Harish-Chandra theory yields the following classification.

THEOREM (HARISH-CHANDRA ('70), GECK-H.-MALLE ('96))

$$\{X \mid X \in kG\text{-mod } \textit{simple}\}\,/\textit{iso.}$$
$$\updownarrow$$
$$\left\{(L, Y, \theta) \;\Big|\; \begin{array}{c} (L, Y) \textit{ a cuspidal pair} \\ \theta \in H(L, Y)\text{-mod } \textit{simple} \end{array}\right\}/\textit{conj.}$$

Let (L, Y) be a cuspidal pair.
$\mathcal{E}(G; L, Y) := \{X \leftrightarrow (L, Y, \theta) \mid \theta \in H(L, Y)\text{-mod simple}\}/\text{iso.}$
is the Harish-Chandra series corresponding to (L, Y).

UNIPOTENT MODULES

From now on assume that $G = \mathrm{GU}_n(q)$.

Restrict to unipotent kG-modules.

Distinguished set of simple kG-modules.

1. $\mathrm{char}(k) = 0$
   - labelled by partitions of n (Lusztig-Srinivasan, '77)
   - write Y_λ for the unipotent kG-module labelled by $\lambda \vdash n$
2. $\mathrm{char}(k) = \ell > 0$
   - ℓ-dec. matrix of the Y_λs lower unitriangular (Geck, '91)
   - $\rightsquigarrow$ labelling of unipotent kG-modules by partitions of n
   - write X_λ for the unipotent kG-module labelled by $\lambda \vdash n$
   - $\{Y_\lambda \mid \lambda \vdash n\}$, $\{X_\lambda \mid \lambda \vdash n\}$ unions of Harish-Chandra series

A NATURAL QUESTION

Given $\lambda \vdash n$, determine Harish-Chandra series of Y_λ and X_λ.

EXAMPLE (LUSZTIG ('77), FONG-SRINIVASAN, ('90))

1. *Y_λ is cuspidal if and only if $\lambda = \Delta_t := (t, t-1, \ldots, 1)$ for some $0 \leq t \leq n$ with $|\Delta_t| \equiv n \pmod 2$.*
2. *Harish-Chandra series of Y_λ determined by $\lambda_{(2)}$ (2-core).*
3. *Given t with $|\Delta_t| \equiv n \pmod 2$, let $r := |\Delta_t|$, $m = (n-r)/2$, put $(L, Y) = (\mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m, Y_{\Delta_t})$.*
   Then $H(L, Y) \cong \mathcal{H}_{k,q^{2t+1},q^2}(B_m)$ (Iwahori-Hecke algebra).
   Simple $H(L, Y)$-modules labelled by bipartitions of m.
4. *The bijection*

$$\mathcal{E}(G; L, Y) \leftrightarrow \{\theta \in H(L, Y)\text{-mod } simple\ \}/iso.$$

is given by the 2-quotient of a partition.

THE MODULAR CASE

Assume now that $\ell > 0$ and put

$$e := \min\{0 \neq i \in \mathbb{N} \mid \ell \text{ divides } (-q)^i - 1\}.$$

If e is **even**, division of $\{X_\lambda \mid \lambda \vdash n\}$ into Harish-Chandra series is known (Dipper-James ('86), Fong-Srinivasan ('89), Geck-H.-Malle ('94), Gruber-H. ('97)).

THEOREM (GECK-H.-MALLE, '96)

Suppose that $e = 1$, $\ell > n$ and let $\lambda, \mu \vdash n$. Then

1. *X_λ is cuspidal if and only if λ' is 2-regular.*
2. *Write $\lambda = \lambda_1 + 2\lambda_2$ such that λ_1' is 2-regular, similarly for μ. Then X_λ and X_μ are in the same Harish-Chandra series, if and only if $\lambda_1 = \mu_1$.*

Want: Similar combinatorial description of Harish-Chandra series for **odd** $e > 1$.

EXAMPLE: $\mathrm{GU}_7(q)$ (DUDAS-MALLE, '13)

7	1														
61	.	1													
52	1	.	1												
51^2	.	.	.	1											
43	.	.	.	.	1										
421	1	.	.	.	.	1									
41^3	.	.	.	.	1	.	1								
3^21	.	.	.	1	.	.	.	1							
32^2	.	.	.	.	.	.	.	.	1						
321^2	.	.	1	.	.	.	.	.	.	1					
31^4	.	.	.	.	.	.	.	.	1	.	1				
2^31	.	.	.	.	.	.	1	.	.	.	.	1			
2^21^3	1	.	1	.	.	1	.	.	.	.	.	.	1		
21^5	.	.	.	.	.	.	.	.	.	.	.	.	.	1	
1^7	.	.	1	.	.	.	.	.	.	1	.	2	1	.	1
	ps	21	ps	ps	21	1^3	21	1^3	ps	B	1^3	c	1^3	21	c

PURE LEVI SUGROUPS

The Dynkin diagram of G equals

DEFINITION

A Levi subgroup of G is called pure, if it corresponds to a left connected subset of the Dynkin diagram of G.

A pure Levi subgroup L satisfies $L \cong \mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m$ for some $r, m \leq n$ with $r + 2m = n$.

REMARK

Let $L, M \leq G$ be pure Levi subgroups and let $x \in N$. Then

$$L^x \cap M$$

is a pure Levi subgroup.

HC-CLASSIFICATION WITH PURE LEVI SUBGROUPS

A simple $X \in kG$-mod is called weakly cuspidal, if $X \not\leq R_L^G(Y)$ for all **proper pure** Levi subgroups $L \leq G$ and all $Y \in kL$-mod.

A weak cuspidal pair (L, Y) consists of a pure Levi subgroup $L \leq G$ and a weakly cuspidal $Y \in kL$-mod.

Harish-Chandra theory yields the following classification.

THEOREM (VARIOUS AUTHORS)

$$\{X \mid X \in kG\text{-mod } \textit{simple}\} \,/\, \textit{iso.}$$
$$\updownarrow$$
$$\left\{(L, Y, \theta) \mid \begin{array}{c} (L, Y) \text{ a weak cuspidal pair} \\ \theta \in H(L, Y)\text{-mod simple} \end{array}\right\} \,/\, \textit{conj.}$$

Let (L, Y) be a weak cuspidal pair.
$\mathcal{E}(G; L, Y) := \{X \leftrightarrow (L, Y, \theta) \mid \theta \in H(L, Y)\text{-mod simple}\}/\text{iso.}$
is the weak Harish-Chandra series corresponding to (L, Y).

Example: $\mathrm{GU}_7(q)$ (Dudas-Malle, '13)

7	1														
61	.	1													
52	1	.	1												
51^2	.	.	.	1											
43	.	.	.	.	1										
421	1	.	.	.	.	1									
41^3	.	.	.	.	1	.	1								
3^21	.	.	.	1	.	.	.	1							
32^2	.	.	.	.	.	.	.	.	1						
321^2	.	.	1	.	.	.	.	.	.	1					
31^4	.	.	.	.	.	.	.	.	1	.	1				
2^31	.	.	.	.	.	.	1	.	.	.	.	1			
2^21^3	1	.	1	.	.	1	.	.	.	.	.	.	1		
21^5	.	.	.	.	.	.	.	.	.	.	.	.	.	1	
1^7	.	.	1	.	.	.	.	.	.	1	.	2	1	.	1
	ps	21	ps	ps	21	1^3	21	1^3	ps	B	1^3	c	1^3	21	c

THE HARISH-CHANDRA BRANCHING GRAPH

Let $\iota \in \{0, 1\}$.
The Harish-Chandra branching graph $\mathcal{B}_{\iota,\ell}$ has vertices

$$\{\lambda \vdash n \mid n \equiv \iota \pmod 2\}.$$

There is a directed edge $\lambda \to \mu$ if and only if $\lambda \vdash n$ and $\mu \vdash n + 2$ for some $n \in \mathbb{N}$, and

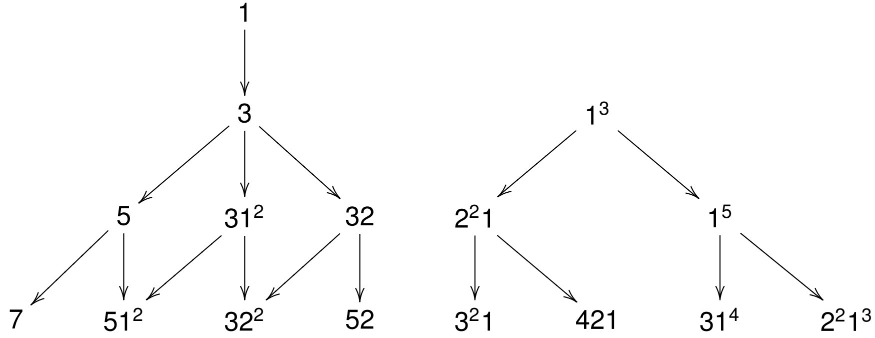
$$X_\mu \leq R_{\mathrm{GU}_n(q)}^{\mathrm{GU}_{n+2}(q)}(X_\lambda).$$

PROPOSITION

1. *The root vertices of $\mathcal{B}_{\iota,\ell}$ correspond to the weak cuspidal pairs.*
2. *Let κ be a root vertex in $\mathcal{B}_{\iota,\ell}$ and let λ be any vertex in $\mathcal{B}_{\iota,\ell}$. Then X_λ lies in the weak Harish-Chandra series of κ, if and only if there is a path from κ to λ in $\mathcal{B}_{\iota,\ell}$.*

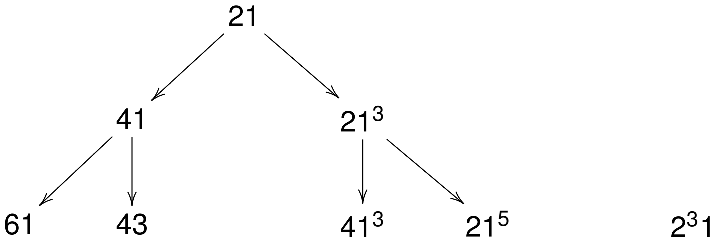
A Truncated Harish-Chandra Branching Graph

Let $\iota = 1$, $\ell \mid q^2 - q + 1$ $(e = 3)$, $n \leq 7$.



Two further root vertices: 1^7, 321^2

A Truncated Harish-Chandra Branching Graph, Continued



THE FOCK SPACE (OF LEVEL 2)

Fix $\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2$ and $2 \leq e \in \mathbb{Z}$.

The Fock space (of level 2) and charge $\mathbf{c}$ is the $\mathbb{Q}(v)$-vector space

$$\mathcal{F}_{\mathbf{c},e} := \bigoplus_{m\in\mathbb{N}} \bigoplus_{\mu\vdash_2 m} \mathbb{Q}(v)|\mu, \mathbf{c}\rangle.$$

There is an action of the quantum group $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$ on $\mathcal{F}_{\mathbf{c},e}$, with:

1. $\mathcal{F}_{\mathbf{c},e}$ is an integrable $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-module;
2. $|\mu, \mathbf{c}\rangle$ is a weight vector for every $m \in \mathbb{N}$ and $\mu \vdash_2 m$;
3. $|\emptyset, \mathbf{c}\rangle$ is a highest weight vector and

$$\mathcal{U}'_v(\widehat{\mathfrak{sl}_e}).|\emptyset, \mathbf{c}\rangle \cong V(\Lambda(\mathbf{c})),$$

the simple highest weight module with weight $\Lambda(\mathbf{c})$ (computable from $\mathbf{c}$).

THE CRYSTAL GRAPH

There is a crystal graph $\mathcal{G}_{\mathbf{c},e}$, describing the canonical basis of the integrable $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-module $\mathcal{F}_{\mathbf{c},e}$.

The vertices of $\mathcal{G}_{\mathbf{c},e}$ are the highest weight vectors $|\mu, \mathbf{c}\rangle$, $m \in \mathbb{N}$, $\mu \vdash_2 m$.

There is a directed edge $|\mu, \mathbf{c}\rangle \to |\nu, \mathbf{c}$ if and only if ν is obtained from μ by adding a good node.

The notion of good addable node depends on e and $\mathbf{c}$.

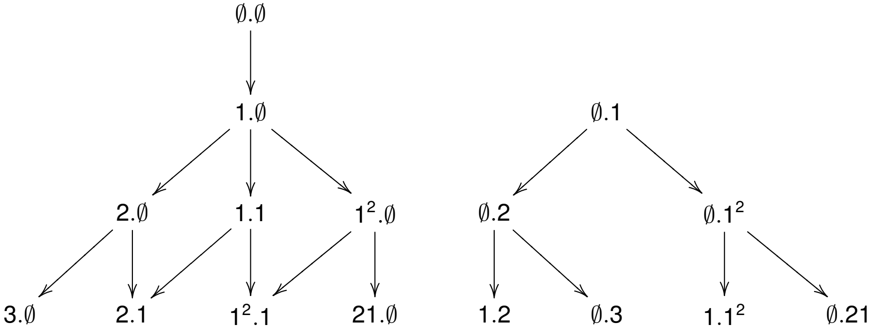
The good nodes of $|\mu, \mathbf{c}\rangle$ can be computed algorithmically.

Each connected component of $\mathcal{G}_{\mathbf{c},e}$ spans a simple highest weight module of $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$, whose highest weight vector is the unique root vertex of the component.

(Jimbo, Misra, Miwa, Okada ('91); Uglov ('99))

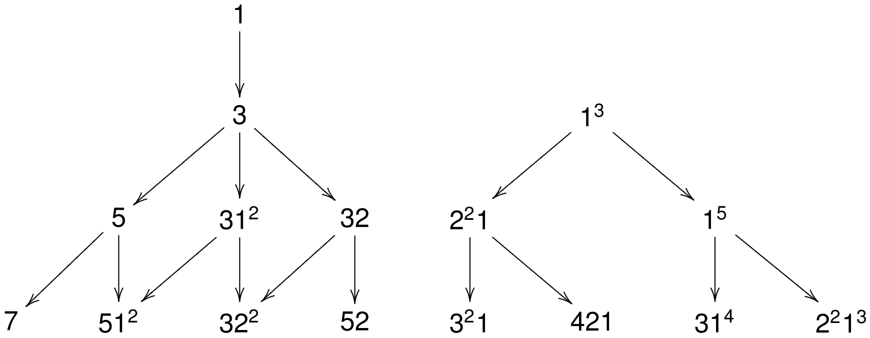
A Truncated Crystal Graph

Let $e = 3$, $\mathbf{c} = (0, 0)$.



Two further root vertices: $\emptyset.1^3$, $1^3.\emptyset$

A TRUNCATED HARISH-CHANDRA BRANCHING GRAPH



Two further root vertices: 1^7, 321^2

CONJECTURE I

Suppose that e is odd.

CONJECTURE I

Let $\lambda, \mu \vdash n$. If X_λ, X_μ lie in the same weak Harish-Chandra series, then λ and μ have the same 2-core.

In other words, Y_λ and Y_μ lie in the same (ordinary) Harish-Chandra series.

If Conjecture I is true, the weak Harish-Chandra series form a refinement of the ordinary Harish-Chandra series.

Let $t \in \mathbb{N}$ and let $\iota = |\Delta_t| \bmod 2 \in \{0, 1\}$ and let $\mathcal{B}_{\iota,\ell,t}$ denote the induced subgraph of $\mathcal{B}_{\iota,\ell}$ with vertices λ such that $\lambda_{(2)} = \Delta_t$.

If Conjecture I is true, $\mathcal{B}_{\iota,\ell,t}$ is a union of connected components of $\mathcal{B}_{\iota,\ell}$.

CONJECTURE II

Let $t \in \mathbb{N}$ and let $\iota = |\Delta_t| \bmod 2 \in \{0, 1\}$.

Let $\mathcal{B}'_{\iota,\ell,t}$ be obtained from $\mathcal{B}_{\iota,\ell,t}$ by replacing the lables λ by their 2-quotients.

CONJECTURE II

With the above notation, $\mathcal{B}'_{\iota,\ell,t}$ agrees with the crystal graph $\mathcal{G}_{\mathbf{c},e}$ with $\mathbf{c} = (t + (1-e)/2, 0)$ below rank ℓ, i.e. for vertices that correspond to partitions of $n < \ell$.

In particular, the root vertices of $\mathcal{G}_{\mathbf{c},e}$ correspond to the weakly cuspidal $k\mathrm{GU}_n(q)$-modules, if $n < \ell$,

and the vertices at distance m from a root vertex of $\mathcal{G}_{\mathbf{c},e}$ label the modules in the weak Harish-Chandra series in $\mathrm{GU}_n(q)$ corresponding to this root vertex for $n = |\Delta_t| + 2m < \ell$.

Perhaps Conjecture II is true without the restrictions on n.

THE EVIDENCE

1. The conjectures are true for $n \leq 10$ and $e \in 3, 5$ and $\ell > n$.
2. The conjectures are true for $n = 12$, $e = 3$ and $\ell \geq 13$ (as far as I could compute the Harish-Chandra series).
3. The "In particular" part of Conjecture II is true for the connected component of $\mathcal{B}'_{\iota,\ell,t}$ containing Δ_t, provided $\ell >> 0$ (Geck, Geck-Jacon, '06),

   i.e. the vertices at distance m from the root vertex are the same in both graphs.

   Either of these sets of vertices labels the simple modules of $\mathcal{H}_{k,q^{2t+1},q^2}(B_m)$ for $m \in \mathbb{N}$.
4. Some consequences of the conjectures for $\mathcal{G}_{\mathbf{c},e}$ hold (e.g., X_{1^n} is cuspidal, if and only if $e \mid n$ or $e \mid n-1$).

CONJECTURE III

CONJECTURE III

Let $\lambda \vdash n$ such that X_λ is weakly cuspidal.

Then the e-core of λ is a 2-core.

Again, Conjecture III is true for $n \leq 10$.
It is also true for $\lambda \vdash n$ of e-weight 1.

PROPOSITION

Let $\lambda \vdash r$ such that X_λ is weakly cuspidal und such that the e-core of λ equals Δ_s for some $s \in \mathbb{N}$.

Let $n = r + 2m$, $G = \mathrm{GU}_n(q)$ and $L = \mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m$.

Then $H(L, X_\lambda) \cong \mathcal{H}_{k,q^{2s+1},q^2}(B_m)$.

This gives a parametrization of the (L, X_λ) Harish-Chandra series by the simple modules of $\mathcal{H}_{k,q^{2s+1},q^2}(B_m)$, i.e. by the connected component of $\mathcal{B}'_{\iota,\ell,s}$ containing Δ_s.