Complex Monge-Ampère equations on Compact Kähler manifolds

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06.04.2014-11.04.2014

1 Overview of the Field

The workshop focused on analytic methods in complex algebraic and Kähler geometry, with a special emphasis on complex Monge-Ampère equations. From a PDE point of view the latter are fully non-linear and possibly degenerate second order elliptic equations, and are quite ubiquitous in complex geometry and analysis, most strikingly in the context of Kähler geometry.

Indeed on the one hand the complex Monge-Ampère operator is closely related to intersection theory since it can be seen as the top degree self-intersection operator on closed positive $(1, 1)$-currents. On the other hand the Ricci curvature of a Kähler metric is expressed in terms of the complex Monge-Ampère operator of the Kähler potential, which explains why the existence problem for Kähler-Einstein metrics and the study of the Kähler-Ricci flow boil down to the study of a complex Monge-Ampère equation and the associated parabolic evolution equation.

An impressive number of works have been devoted to the existence, uniqueness and regularity of solutions to complex Monge-Ampère equations, both on compact manifolds and on domains. These problems were settled for Kähler-Einstein equations of negative and zero curvature by Yau's resolution of the Calabi conjecture in the late 70's [Yau78]. Getting a geometric understanding of the existence of Kähler-Einstein metrics on higher dimensional Fano manifolds and more generally of constant scalar curvature Kähler metrics on polarized manifolds is a more delicate problem and a very active research area.

Roughly at the same time as Yau's result, Bedford and Taylor's fundamental work on degenerate Monge-Ampère equations in domains [BT82] opened new research directions in several complex variables and pluripotential theory. Let us mention notably the work of Kolodziej [Kol05].

It is interesting to note that until very recently the differential-geometric and potential-theoretic sides have developed rather independently, mostly because of a lack of common vocabulary and interest.

Complex geometry allows to build a bridge between complex analysis and pluripotential theory on the one hand, and complex differential and algebraic geometry on the other hand. This was well illustrated by [Dem93], [DP04] where Monge-Ampère equations were used to obtain the first general results in the direction of Fujita's conjecture and to get a numerical characterization of the Kähler cone respectively.

2 Recent Developments and Open Problems

One of the most famous problems in Kähler geometry is that of finding when Kähler-Einstein metrics exist on Fano manifolds. Since Kähler-Einstein metrics can also be viewed as the stationary points of the Kähler-Ricci flow, this problem is the same as the one of the convergence of this flow. Its relation with the Monge-Ampère equation is particularly strong, since the Kähler-Ricci flow is just a parabolic version of the Monge-Ampère equation.

Perelman's work [SeT08] has created new tools for the study of the Kähler-Ricci flow [TZ07, PSSW08, ST09], while the recent breakthrough of Birkar, Cascini, Hacon and McKernan [BCHM10] in the Minimal Model Program has motivated the study of Kähler-Einstein metrics on singular varieties [EGZ09, BEGZ10, ST12].

Campana's birational classification scheme [Camp11], which aims at understanding the hyperbolicity properties of complex varieties, also calls for Kähler-Einstein metrics on geometric orbifolds. Birational geometry of higher dimensional varieties more generally leads to consider complex Monge-Ampère equations in more degenerate situations:

-the cohomology classes involved are no longer Kaehler,

-the measures to be considered are no longer volume forms,

-the solutions are merely weak (non-smooth, possibly unbounded).

These additional complications require the use of fine tools from complex analysis, pluripotential theory and algebraic geometry.

It thus appears that the study of complex Monge-Ampere equations in a context general enough to fit with the birational classification schemes calls for a whole range of very diversified techniques.

The last decade has witnessed an explosive growth in the subject, which has opened up entire new venues for investigation. Let us stress important progress on

-developing new methods for solving Monge-Ampère equations (algebraic approximation [PS06], variational methods [BerBo10, BBGZ13])

-regularity properties of degenerate solutions [DZ10, EGZ11],

-extensions of Bando-Mabuchi uniqueness theorem [CT08, Ber09, Ber11],

-conical Kähler-Einstein metrics [Don11, CGP11, JMR11],

-limits of Kähler-Einstein manifolds [DS12, BBEGZ11, CDS12a, CDS12b, CDS13, Tian12].

A strong motivation for all these works comes from the Yau-Tian-Donaldson conjecture (see [PS10]) which states that the existence of constant scalar curvature Kähler metrics in a Hodge class is equivalent to a suitably modified version of GIT stability of the underlying polarized manifold. This influential conjecture has recently attracted great interest among algebraic geometers (see e.g. [Oda12]).

Kähler geometry has recently known dramatic progress through the resolution, in an important special case (the "anticanonically polarized" case), of this conjecture. The diversity of techniques involved, extending across algebraic geometry, complex analysis, fully non-linear partial differential equations, pluripotential theory, and the Riemannian geometry of possibly singular spaces, makes it a very challenging task to understand in detail all aspects of the proof. At the same time it is clear that the new methods introduced are bound to play a major role in the forthcoming developments of Kähler geometry, as was well illustrated by the talks and informal discussions during our workshop.

3 Presentation Highlights

3.1 The Yau-Tian-Donaldson conjecture

The study of special Kähler metrics on compact Kähler manifolds, pioneered by Calabi in the 1950's, has been a guiding question in the field ever since, which led to an impressive number of remarkable developments, among which the solution by Yau of the Calabi conjecture in the late 1970's [Yau78] was one of the crowning achievements. The latter settled the existence and uniqueness problem for Kähler-Einstein metrics of negative or zero curvature, which amounts to the resolution of certain complex Monge-Ampère equations.

The case of Kähler-Einstein metrics of positive curvature turned out to be of a very different nature. Besides being non-unique in general (in fact, precisely in the presence of holomorphic vector fields according to a result of Bando and Mabuchi [BM87]), obstructions to their existence were also constructed, first in terms of holomorphic vector fields (Futaki invariants), before being generalized by Tian through the introduction of the notion of K-stability in the late 90's [Tia97]. The latter was given an algebro-geometric interpretation by Donaldson around 2000, which led to the precise formulation of the Yau-Tian-Donaldson conjecture: the equivalence between the existence of positively curved Kähler-Einstein metrics (or, more generally, of constant scalar curvature Kähler metrics) and K-(poly)stability. One remarkable feature of this conjecture is the possibility to encode in a purely algebro-geometric condition the exact obstruction to solving certain fully non-linear elliptic PDE's.

A proof of this fundamental conjecture has recently been announced by Chen-Donaldson-Sun and by Tian, independently [CDS12a, CDS12b, CDS13, Tian12]. The method relies on a continuity method where the parameter is the cone angle of a Kähler metric with singularities along a smooth complex hypersurface. A crucial ingredient is Gromov's compactness theorem for Riemannian manifolds with Ricci curvature bounded below, which allows to consider Gromov-Hausdorff limits of such manifolds. Such limits are typically very singular compact metric spaces, and understanding the structure of these limit spaces is the subject of the subtle Cheeger-Colding theory. A particularly remarkable step in the proof of the Yau-Tian-Donaldson establishes that Gromov-Hausdorff limits of positively curved Kähler-Einstein manifolds of fixed volume are actually projective algebraic varieties with fairly mild singularities (to be described within the Minimal Model Program in birational geometry), endowed with a singular Kähler-Einstein metric.

Several talks of the workshop were directly connected to this major breakthrough. *M.Paun* surveyed recent developments around conical Kähler-Einstein metrics, *S.Paul* explained how the notion of (semi)stable pair is equivalent to the existence of Kähler-Einstein metrics on Fano manifolds with finite automorphism group, *G.Szekelyhidi* proved a partial $\mathcal{C}^0$-estimate along the classical continuity method, *X.Wang* showed that any two toric n-manifolds can be joined in Gromov-Hausdorff topology by a continuous path of conical Kähler-Einstein toric manifolds, and *X.Zhu* studied the Gromov-Hausdorff convergence of almost Kähler-Ricci solitons.

3.2 Kähler-Ricci flow and the Minimal Model Program

From a general perspective, the Kähler-Ricci flow is obtained by specializing Hamilton's Ricci flow to Kähler manifolds; however, the corresponding PDE reduces in the Kähler case to a parabolic complex Monge-Ampère equation, and has therefore been studied quite independently from the general Riemannian case.

Along the flow, the cohomology class of the Kähler form evolves linearly towards the canonical class, and its positivity is in fact the only obstruction to the existence of the flow. The existence time can therefore be described by a purely cohomological condition, which turns out to fit exactly the procedure used to set up one step of the Minimal Model Program (MMP) in birational geometry.

Building on this important fact, Song and Tian have developed a program [ST09] viewing the Kähler-Ricci flow as a metric version of the MMP, each step of the MMP corresponding to a surgery that is used to repair a finite time singularity of the flow and start it over again. Moreover, they have been able to analyze the long time behavior of the flow at the final stage of the MMP, where the variety has become minimal.

However, most of the convergence results obtained so far stay away from the singularities themselves, proving C^∞ convergence on compact sets away from the singularities. Understanding the global behavior of the flow in the Gromov-Hausdorff topology is a fundamental and very challenging problem. The recent work of Chen-Donaldson-Sun and Tian has considerably improved our knowledge of Gromov-Hausdorff limit of algebraic Kähler-Einstein manifolds. Recently, in [CD11] Chen and Donaldson have revisited and improved some of the results of Cheeger-Colding-Tian. Moreover Colding, Cheeger and Naber have also given new information about non collapsed Gromov-Hausdorff limit of manifolds with a lower bound on the Ricci curvature [CN13, CN12].

Several lectures of the workshop concerned this interplay between the MMP and the Kähler-Ricci flow. *F.Campana* explained how the solution of the Abundance conjecture in dimension 3 can be adapted to the Kähler setting. *T.Collins* showed that finite time singularities of the Kähler-Ricci flow always form along analytic subvarieties. *J.Song* studied Calabi-Yau varieties with crepant singularities, identifying the metric completion of the Kähler-Einstein metric space constructed by Eyssidieux-Guedj-Zeriahi. *A.Zeriahi* developed the first steps of a viscosity theory in order to study the Kähler-Ricci flow on mildly singular varieties.

3.3 Degenerations of Kähler-Einstein manifolds

In algebraic geometry, degenerations are used to study points at infinity in moduli spaces. In the special case of K3 surfaces and, more generally, Calabi-Yau manifolds, the choice of a polarization (ample line bundle or Kähler cohomology class) uniquely determines a Ricci-flat Kähler metric, and it is a central problem to investigate the metric behavior of polarized Calabi-Yau manifolds approaching the boundary of the moduli space.

While the case of arbitrary degenerations seems completely out of reach at the moment, the case of maximal degenerations, also known as 'large complex structure limits', has attracted a lot of attention recently. Indeed, a remarkable conjecture of Kontsevich-Soibelman [KS01] and Gross-Siebert predicts that the Gromov-Hausdorff limit should be found as the dual complex of a relative minimal model of the degeneration, equipped with a metric of Monge-Ampère type, a real analogue of a complex Ricci flat metric. This conjecture originates from the Strominger-Yau-Zaslow picture of mirror symmetry, involving special Lagrangian fibrations that are only expected to exist in such large complex structure limits.

The Kontsevich-Soibelman version of the picture actually suggests to look for the limit metric using non-Archimedean geometry, building on the fact that the dual complex embeds in a natural way in the Berkovich space attached to the degenerations. Recent progress has been accomplished [MN12, NX13] on the algebro-geometric aspects of this conjecture, which connect in an exciting way the Minimal Model Program and non-Archimedean geometry.

An apparently more managable problem consists in analyzing the metric behavior as the cohomology class of the metric approaches the boundary of the Kähler cone of a fixed Calabi-Yau manifold. As shown by Tosatti and Zhang, an important feature of this situation is the existence of an a priori bound on the diameter. Combined with the fundamental Gromov compactness theorem, this guarantees the existence, up to a subsequence, of a limit in the Gromov-Hausdorff topology.

In the non-collapsed case, the limit class has positive volume and is the pull-back of a Kähler class by a birational contraction to a Calabi-Yau variety with canonical singularities. Combining [RoZh13, RuZh11] with the fondamental result of Donaldson-Sun [DS12], the limit is known to be given by the corresponding Ricci-flat metric on this singular Calabi-Yau variety.

In the collapsed case, the limit class is the pull-back of a class on the base of a fibration, and the limit metric is conjecturally the unique metric in this class with Ricci curvature equal the Weil-Petersson metric. This was pioneered by Gross and Wilson in the case of K3 surfaces [GW00]. In higher dimensions, [Tos10] proves $C^{1,\alpha}$ convergence away from the singular fibers; C^{∞} convergence away from the singular fibers is established in [GTZ13a] for abelian fibrations, and Gromov-Hausdorff convergence is obtained in [GTZ13b] when the base is further assumed to be one-dimensional.

The lectures by *V.Tosatti* surveyed his recent works with Gross and Zhang on this theme. *H.Guenancia* explained his construction (joint work with Berman) of Kähler-Einstein metrics on stable varieties, an important step towards understanding the compactification of the moduli space of Kähler-Einstein manifolds of negative curvature.

Working with pairs impose to consider similar questions on quasi-projective manifolds. *H.Auvray* reviewed the construction of Poincaré type metrics in this context, *M.Haskins* studied complete Ricci flat Kähler manifolds that are asymptotic to cylinders at infinity, *H.C.Lu* solved degenerate Calabi-type equations of quasi-projective varieties.

3.4 Other talks

C.Arezzo explained how to construct constant scalar curvature Kähler metrics on blow-ups and desingularization of orbifolds with isolated quotient singularities.

S.Dinew reviewed the local regularity theory of complex Monge-Ampère equations, while *C.Li* studied the critical solvability exponents of the latter.

J.Ross gave detailed asymptotics of the partial Bergman Kernels, *B.Weinkove* developed Calabi-Yau type theorems for Gauduchon and balanced metrics, *D.Witt-Nyström* established local regularity results for geodesic rays in the space of Kähler potentials.

4 Outcome of the Meeting

The objective of the workshop was to bring together an international group of leading experts with complementary backgrounds from each above mentioned field, to report on and discuss recent progress and open problems in the area and thus foster interaction and collaboration between researchers in diverse subfields.

The workshop was timely, since the bulk of the progress described above took place in the last three or four years. It is remarkable that major contributions came from researchers from all over the world. Thus the workshop has also provided a unique opportunity for interaction between different groups who would normally reside in several distinct continents.

We had thirty-eight participants, including some of the main researchers in the fields of interest, coming from eleven different countries (Canada, China, France, Germany, Italy, Japan, Korea, Poland, Sweden, UK, USA). It was important for us that the workshop provide an opportunity for postdocs and graduate students to interact with experts. We have been quite successful in that respect, having five graduate students attending (from the Universities of Columbia, Grenoble, Paris, Purdue and Roma) and eight postdocs (from the Ecole Normale Supérieure and the Universities of Cambridge, Göteborg, Leibniz, McMaster, Stony Brook and Waterloo).

The workshop stimulated a number of fruitful discussions and collaborations. S.Dinew and H.C.Lu took the opportunity to finish their paper on mixed Hessian inequalities (arXiv April 2014), so did P.Eyssidieux, V.Guedj and A.Zeriahi with their study of weak Kähler-Ricci flows (arXiv June 2014). The discussions following Paun's lecture lead him to provide another proof of his main result (which he is presently writing up). S.Boucksom, T.Hisamoto and M.Jonsson took advantage of the free wednesday afternoon to have a lengthy discussion and start a joint project.

The participants were very enthusiastic about the scientific content of the workshop. The warm hospitality and professionalism of the staff were very much appreciated, the beautiful scenery and the great facilities helped make the workshop a success.

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