Oscillating singularities of Lévy processes

...and alpha-stable processes more particularly

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Banff - Multifractal analysis workshop - 2014

February 25, 2014

Pointwise Hölder regularity

Definition

A function f belongs to C_t^α, with $\alpha > 0$ and $t \in \mathbf{R}$, if

$$\forall u \in B(t,\rho); \quad |f(u) - P_t(u)| \leq C|t-u|^\alpha,$$

where P_t is a polynomial.
The *pointwise Hölder exponent* of f at t is then defined by

$$\alpha_{f,t} = \sup\{\alpha : f \in C_t^\alpha\}.$$

Briefly, when $\alpha_{f,t} < 1$,

$$\alpha_{f,t} = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t,\rho)} \frac{|f(u)-f(v)|}{\rho^\alpha} < \infty\right\}.$$

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Multifractal analysis

Definition (Multifractal spectrum)

Geometrical description of level sets of the pointwise exponent (iso-Hölder sets),

$$\forall h \in \mathbf{R}_+; \quad E_h = \{t \in \mathbf{R}_+ : \alpha_{f,t} = h\}.$$

The *multifractal spectrum* is defined by

$$\forall h \in \mathbf{R}_+; \quad d_f(h,V) = \dim_{\mathrm{H}}(E_h \cap V).$$

where $V \in \mathscr{O}$ is a non-empty open set.

Local Hölder regularity

Definition

A function f belongs to $\widetilde{C}_t^{\alpha}$, $\alpha \in (0,1)$, if

$$\forall u, v \in B(t,\rho); \quad |f(u) - f(v)| \leq C|u-v|^{\alpha}.$$

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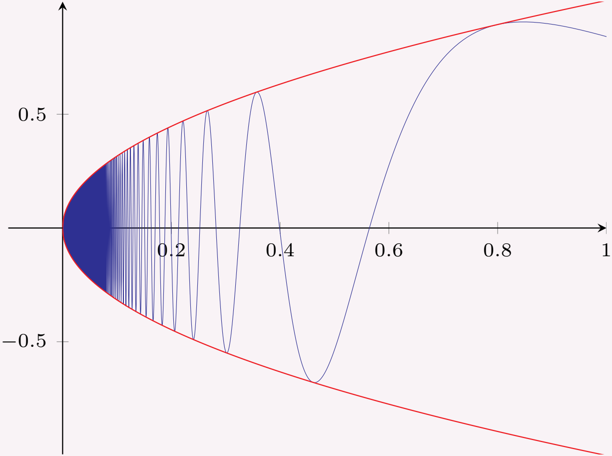
Examples: cusp and chirp functions

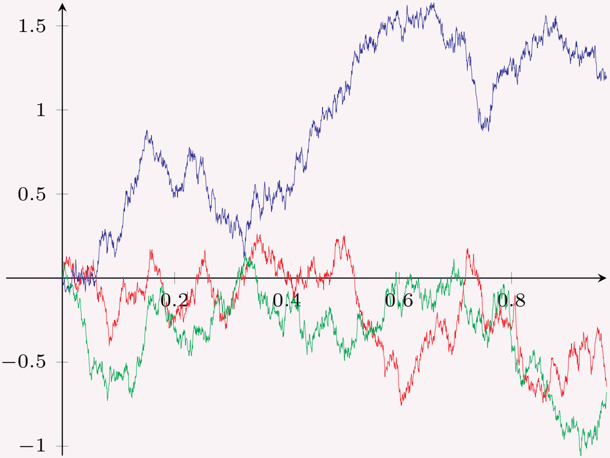
$$f(x) = |x|^\alpha \quad \text{and} \quad h(x) = |x|^\alpha \sin(|x|^{-\beta})$$

$$\alpha_{f,0} = \alpha_{h,0} = \alpha \quad \text{and} \quad \widetilde{\alpha}_{f,0} = \alpha,\ \widetilde{\alpha}_{h,0} = \alpha/(1+\beta).$$



Examples: cusp and chirp functions

$$\forall t \in \mathbf{R}; \quad \alpha_{B,t} = \widetilde{\alpha}_{B,t} = 1/2.$$



Limits of the pointwise Hölder exponent

- Lack of stability under the action of pseudo-differential operators;
- Do not give a complete picture of the local regularity.

Oscillation exponent

Definition

The *oscillation exponent* of f at t is defined by

$$\beta^o_{f,t} = \left.\frac{\mathrm{d}\alpha_{I^\varepsilon_+ f,t}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0_+} - 1,$$

where $I^\varepsilon_+ f$ designates the fractional integral of order ε

$$(I^\varepsilon_+ f)(u) = \frac{1}{\Gamma(\varepsilon)} \int_{\mathbf{R}} (u-s)_+^{\varepsilon-1} f(s)\,\mathrm{d}s.$$

Oscillation exponent

A few basic properties

For any $\varepsilon > 0$,

$$\alpha_{I_+^\varepsilon f,t} \geq \alpha_{f,t} + \varepsilon \quad \Longrightarrow \quad \beta^o_{f,t} \geq 0.$$

If there exists ε_0 such that $\alpha_{I_+^{\varepsilon_0} f,t} = \alpha_{f,t} + \varepsilon_0$, then for any $\varepsilon \in (0, \varepsilon_0)$,

$$\alpha_{I_+^\varepsilon f,t} = \alpha_{f,t} + \varepsilon \quad \Longrightarrow \quad \beta^o_{f,t} = 0.$$

Consequence of the semi-group property $I_+^\alpha I_+^\beta f = I_+^{\alpha+\beta} f$.

- *Cusp singularities:* $\beta^o_{f,t} = 0$;
- *Oscillating singularities:* $\beta^o_{f,t} > 0$.

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- *Cusp singularities*: $\beta^o_{f,t} = 0$;
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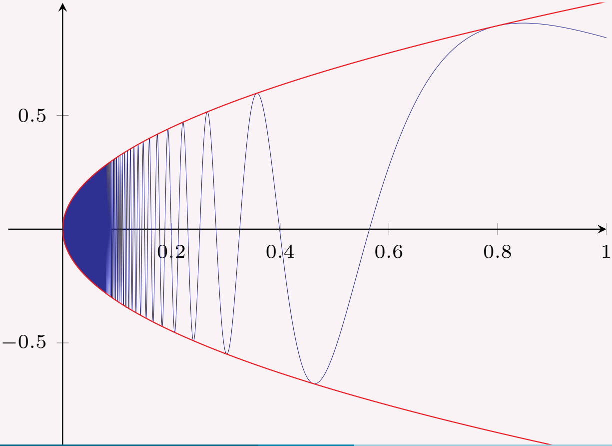
Examples: cusp and chirp functions

$$\alpha_{I_+^\varepsilon f,0} = \alpha + \varepsilon \quad \text{and} \quad \alpha_{I_+^\varepsilon h,0} = \alpha + \varepsilon(1+\beta);$$

$$\beta^o_{f,0} = 0 \quad \text{and} \quad \beta^o_{h,0} = \beta$$



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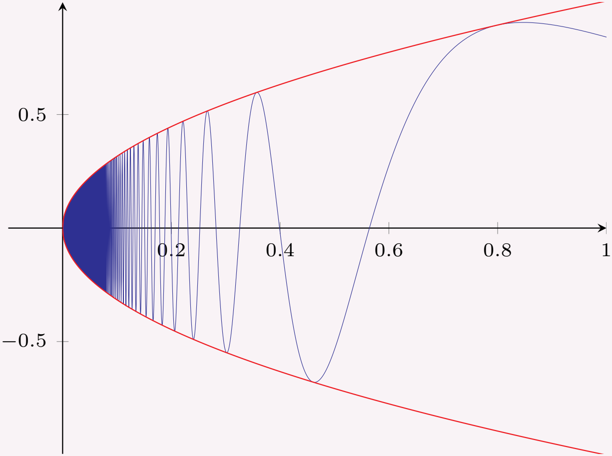

Singularities of stochastic processes

Do common stochastic processes have oscillating singularities?

Example

Suppose B^H is a *fractional Brownian motion*:

$$B^H_t = \int_{\mathbf{R}} \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}B_u$$

With probability one and for all $t \in \mathbf{R}$,

$$\alpha_{B^H,t} = H \quad \text{and} \quad \widetilde{\alpha}_{B^H,t} = H \quad \Longrightarrow \quad \beta^o_{B^H,t} = 0.$$

Possibly Lévy processes, since $\alpha_{B^H,t} > \widetilde{\alpha}_{B^H,t} = 0$.

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Lévy processes

Theorem (Lévy–Khintchine formula)

The law of $(X_t)_{t\in\mathbf{R}}$ *is characterized by* $\mathbb{E}[e^{i\lambda X_t}] = e^{-t\psi(\lambda)}$, *where*

$$\forall \lambda \in \mathbf{R}; \quad \psi(\lambda) = ia\lambda + \frac{\sigma^2\lambda^2}{2} + \int_{\mathbf{R}} \{1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{|x|\leq 1}\} \pi(\mathrm{d}x).$$

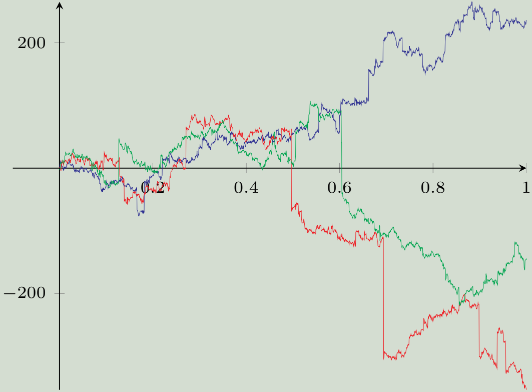
$\pi(\mathrm{d}x)$ *is a Lévy measure, i.e.* $\int_{\mathbf{R}} (1 \wedge x^2)\, \pi(\mathrm{d}x) < +\infty$.

Alpha-stable Lévy processes

Example

Pure jump Lévy process whose Lévy measure π is given by

$$\pi(\mathrm{d}x) = a^-|x|^{-\alpha-1}\mathbf{1}_{\mathbf{R}_-}\,\mathrm{d}x + a^+|x|^{-\alpha-1}\mathbf{1}_{\mathbf{R}_+}\,\mathrm{d}x \quad \text{where } \alpha \in (0,2).$$



Assumption on the Lévy process

We assume that $\pi(\mathbf{R}) = +\infty$ and X has no Brownian and drift components, i.e.

$$a = 0 \quad \text{and} \quad \sigma = 0.$$

Definition (Blumenthal–Getoor exponent)

The Blumenthal–Getoor exponent is defined by

$$\beta = \inf\left\{\gamma \geq 0 : \int_{\mathbf{R}} \left(1 \wedge |x|^{\gamma}\right) \pi(\mathrm{d}x) < \infty\right\} \in [0,2].$$

If X is an alpha-stable Lévy process, then $\beta = \alpha$.

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Multifractal spectrum of Lévy processes

Theorem (Pruitt(1981))

Suppose X is a Lévy process. Then,

$$\forall t \in \mathbf{R}_+; \quad \alpha_{X,t} \overset{\text{a.s.}}{=} \frac{1}{\beta}.$$

Theorem (Jaffard(1999))

Suppose X is a Lévy process with $\beta > 0$. Then, with probability one

$$\forall h \in \mathbf{R}_+ \ \forall V \in \mathcal{O}; \quad d_X(h,V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta] \\ -\infty & \text{otherwise.} \end{cases}$$

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Singularities of Lévy processes

Cusp and oscillating singularities

We distinguish *cusp singularities*

$$\widetilde{E}_h = \left\{ t \in E_h : \beta^o_{X,t} = 0 \right\}$$

from *oscillating singularities*

$$\widehat{E}_h = \left\{ t \in E_h : \beta^o_{X,t} > 0 \right\} = E_h \setminus \widetilde{E}_h.$$

Singularities of Lévy processes

Theorem

Suppose X is a Lévy process with $\beta > 0$. Then, with probability one,

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widetilde{E}_h \cap V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta]; \\ -\infty & \text{if } h \in (1/\beta, +\infty]. \end{cases}$$

Furthermore,

$$\dim_{\mathrm{H}}(\widehat{E}_h) \le \begin{cases} 2\beta h - 1 < \beta h & \text{if } h \in (1/2\beta, 1/\beta); \\ -\infty & \text{if } h \in [0, 1/2\beta] \cup [1/\beta, +\infty]. \end{cases}$$

And $\beta^{o}_{X,t} \le 2\beta h - 1$.

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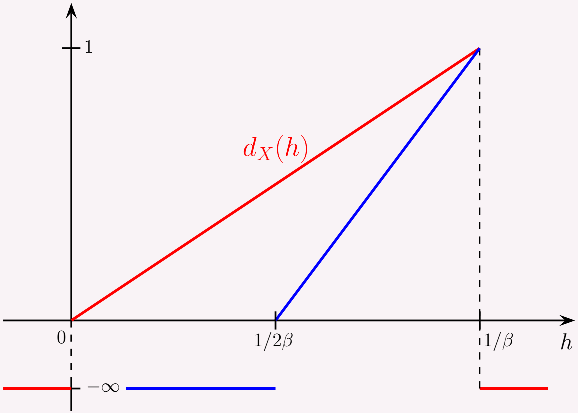

Remark

- For all $h \in [0, 1/\beta]$, $\dim_H(\widehat{E}_h) < \dim_H(\widetilde{E}_h)$. Hence, *cusp singularities* are more common than *chirp oscillations*;

Oscillating singularities

One-side Lévy processes

Suppose π is such that $\pi((-\infty,0)) = 0$. Then, with probability one

$$\forall h \in \mathbf{R}_+; \quad \widehat{E}_h = \emptyset.$$

i.e. for all $t \in \mathbf{R}$, $\beta^o_{X,t} = 0$.

Oscillating singularities

Theorem (Alpha-stable processes)

Suppose X is an alpha-stable Lévy process such that $\alpha \in (0,2)$ and $\beta_\alpha \in (-1,1)$. Then, with probability one

$$\forall V \in \mathcal{O}; \quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) = \begin{cases} 2\alpha h - 1 & \text{if } h \in (1/2\alpha, 1/\alpha); \\ -\infty & \text{otherwise.} \end{cases}$$

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Oscillating singularities

Consider a Lévy measure

$$\pi(\mathrm{d}x) = a_1 \, |x|^{-1-\alpha_1} \, \mathbf{1}_{\mathbf{R}_+} \mathrm{d}x + a_2 \, |x|^{-1-\alpha_2} \, \mathbf{1}_{\mathbf{R}_-} \mathrm{d}x,$$

where $a_1, a_2 > 0$ and $\alpha_1, \alpha_2 \in (0,2)$. Note $\beta = \max(\alpha_1, \alpha_2)$.

Theorem

Suppose X is a Lévy process parametrised by $(0,0,\pi)$. Then, with probability one,

$$\dim_{\mathrm{H}}(\widehat{E}_h \cap V) = \begin{cases} (\alpha_1+\alpha_2)h - 1 & \text{if } h \in \left(1/(\alpha_1+\alpha_2), 1/\beta\right); \\ -\infty & \text{otherwise.} \end{cases}$$

Oscillating singularities, but...

Oscillating singularities of Lévy processes are still quite different from the Chirp function. Recall that for any $\gamma > 0$,

$$\alpha_{I_+^\gamma h,0} = \alpha + \gamma(1+\beta) \quad \text{and} \quad \beta^o_{I_+^\gamma h,0} = \beta.$$

On the other hand, Lévy processes are such that

$$\alpha_{I_+^\gamma X,t} \leq 1/\beta + \gamma \quad \text{and} \quad \beta^o_{I_+^\gamma X,t} \longrightarrow_{\gamma \to 0} 0.$$

In fact, most oscillating singularities disappear after a small fractional integration.

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In fact, most oscillating singularities disappear after a small fractional integration.

Grand-canonical spectrum (ongoing work)

Interested in studying both the pointwise and the oscillation exponents:

$$E_{h,\gamma} = \{t \in \mathbf{R} : \alpha_{X,t} = h \quad \text{and} \quad \beta^{o}_{X,t} = \gamma\}.$$

Note that

$$\widetilde{E}_h = E_{h,0} \quad \text{and} \quad \widehat{E}_h = \bigcup_{\gamma>0} E_{h,\gamma}.$$

Grand-canonical spectrum (ongoing work)

Theorem

Suppose X is an alpha-stable Lévy process with $\beta_\alpha \in (-1,1)$. Then, with probability one

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_{h,0} \cap V) = \begin{cases} \alpha h & \text{if } h \in [0, 1/\alpha]; \\ -\infty & \text{otherwise.} \end{cases}$$

Furthermore,

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_{h,\gamma} \cap V) = \begin{cases} 2\alpha h - 1 - \gamma & \text{if } h \in (1/2\alpha, 1/\alpha) \\ & \text{and } \gamma \in (0, 2\alpha h - 1]; \\ -\infty & \text{otherwise.} \end{cases}$$

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Beyond oscillating singularities of Lévy processes

Cusp and oscillating singularities of Lévy processes induce the knowledge of the multifractal spectrum of the *linear fractional stable motion*.

$$X_t = \int_{\mathbf{R}} \left\{ (t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right\} M_\alpha(\mathrm{d}u),$$

Theorem

Suppose X is a LFSM with $\alpha \in (1,2)$ and $H > 1/\alpha$. Then, its multifractal spectrum is equal to

$$\forall V \in \mathscr{O}; \quad d_X(h,V) = \begin{cases} \alpha(h-H)+1 & \text{if } h \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{otherwise.} \end{cases}$$

A few interesting questions

1. Determine completely the oscillating singularities of Lévy processes;
2. Have a better understanding of their form;
3. Numerically highlight these singularities;
4. Find other stochastic processes with such oscillations.

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