Self-similarity, symmetry and anisotropy in the multivariate and multiparameter settings

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Multifractal Analysis: From Theory to Applications and Back (BIRS)

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Outline

1. introduction
2. harmonizable representations
3. symmetry, exponents, anisotropy
4. wavelet analysis
1. Introduction
why talk about Gaussian monofractals?!
why talk about Gaussian monofractals?!...

... because of multidimensionality!
Introduction

From a technical standpoint...

difficulty = $\alpha_{An}$ Analysis + $\alpha_{Alg}$ Algebra + $\alpha_{Top}$ Topology

• typical probability problem: $\alpha_{An} \approx 1$

• multidimensional problems: $\alpha_{An} \approx \alpha_{Alg} \gg \alpha_{Top} > 0$
why multidimensional? (applications)

• *(range)* data is usually *multivariate*: anomalous diffusion particle position \((X(t)-Y(t))\), climatology, Internet data comes in packets or bytes within bins of certain duration...

• *(domain)* random fields have become a topic of great interest: image processing (bone radiographic imaging), hydrology (aquifers), geostatistics..., especially in the presence of *anisotropy*
why multidimensional? (theory)


- **(domain):** operator scaling measures, scalar s.s. random fields (Samorodnitsky & Taqqu (1994), Meerschaert & Scheffler (2001), Xiao (2009), and many, many others)

current stage of this research: carry out the synthesis between the range and the domain, both in probability and statistics (Li & Xiao (2011))
Introduction

inter-related ideas:

• symmetry and self-similarity/scaling are related in Probability theory ($\Rightarrow$ multidimensionality)

• anisotropy and intrinsic physical symmetries

• (non)identifiability of the parametrization

identifiability (statistical inference): $\theta \mapsto P_\theta$ is injective
isotropy means that...

\[ \{X(Ot)\}_{t \in \mathbb{R}^m} \equiv \{X(t)\}_{t \in \mathbb{R}^m}, \quad O \in O(m) \]

i.e. every \( O \in O(m) \) is a (domain) symmetry of \( X \)

\( m = 1 \): time-reversibility
Harmonizable representations
Harmonizable representations

**def.** (o.s.s. vector random fields): \( \{X(t)\}_{t \in \mathbb{R}^m} \) satisfies

\[
\{X(c^E t)\}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \{c^H X(t)\}_{t \in \mathbb{R}^m}, \quad E \in M(m), H \in M(n)
\]

matrix exponentiation \( c^M := \exp(M \log(c)) = \sum_{k=0}^{\infty} \frac{(M \log(c))^k}{k!} \)

**def.** (operator fractional Brownian field (OFBF)): Gaussian, stationary increment, o.s.s. random field

**def.** (operator fractional Brownian motion (OFBM)): \( m = 1, E = 1 \)
Harmonizable representations

definition: FBM ($H$ characterizes the law up to a constant):

$$EB_H(s)B_H(t) = \frac{EB_H(1)^2}{2} \{ |t|^{2H} + |s|^{2H} - |t - s|^{2H} \}$$

OFBM: $EB_H(t)B_H(s)^* + EB_H(s)B_H(t)^*$

$$= |t|^H \Gamma(1, 1)|t|^{H^*} + |s|^H \Gamma(1, 1)|s|^{H^*} - |t - s|^H \Gamma(1, 1)|t - s|^{H^*},$$

$$\Gamma(1, 1) = EB_H(1)B_H(1)^*$$

It is not true in general that $EB_H(t)B_H(s)^* = EB_H(s)B_H(t)^*$
Harmonizable representations

**goal:** Fourier domain representations for OFBFs

polar coordinates (Biermé, Meerschaert & Scheffler (2007)): given a matrix $E$ ($\min \Re \text{eig}(E) > 0$), there exists a norm $\| \cdot \|_0$ such that

\[
\mathbb{R}^m \ni x = \text{radial}^E * \text{spherical} = \tau(x)^E l(x) \quad \text{homeomorphically}
\]

where $S_0 = \{ x : \| x \|_0 = 1 \} = \{ x : \tau(x) = 1 \}$
Harmonizable representations

**Theorem (Baek, D & Pipiras (2014)):** For an OFBF \( \{X(t)\}_{t \in \mathbb{R}} \) with \( 0 < \min \Re \text{eig}(H) \leq \max \Re \text{eig}(H) < \min \Re \text{eig}(E) \) and density \( f_X(x) \),

\[
\{X(t)\}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \left\{ \int_{\mathbb{R}} (e^{i\langle t, x \rangle} - 1) \tau(x) - H_E f_X^{1/2}(l(x)) \tilde{B}(dx) \right\}_{t \in \mathbb{R}^m},
\]

- \( H_E = H + \text{tr}(E^*)I/2 \)

- \( \tilde{B}(dx) \) := \( \tilde{B}_1(dx) + i\tilde{B}_2(dx) \) (complex-valued multivariate Brownian measure satisfying \( \tilde{B}(-dx) = \overline{\tilde{B}(dx)} \)).
Harmonizable representations

in terms of the spectral measure...

\[ f_X(x) = \tau(x)^{-HE} f_X(l(x)) \tau(x)^{-HE^*} \]

(\( f_X(x) \) is assumed to exist)
Harmonizable representations

in terms of the spectral measure...

\[ f_X(x) = \tau(x)^{-H_E} f_X(l(x)) \tau(x)^{-H_E^*} \]

(\( f_X(x) \) is assumed to exist)

\( m = 1 \):

\[ f_X(x) = \frac{1}{x^2} (x^+ AA^* x^+ D^* + x^- AA^* x^- D^*) \]

(\( f_X(x) \) exists)
Harmonizable representations

\textbf{application} (m = 1):

\textbf{time reversibility:} \{X(t)\}_{t \in \mathbb{R}} \xrightarrow{\mathcal{L}} \{X(-t)\}_{t \in \mathbb{R}}

\[EB_H(s)B_H(t)^* = \frac{1}{2}(|t|^{H} \Gamma(1, 1)|t|^{H^*} + |s|^{H} \Gamma(1, 1)|s|^{H^*} - |t - s|^{H} \Gamma(1, 1)|t - s|^{H^*})\]

\[\iff \{B_H(t)\}_{t \in \mathbb{R}} \text{ is time-reversible}\]

(... \Rightarrow \text{the univariate reasoning is insufficient})
Harmonizable representations

**application** \((m = 1)\):

time reversibility:
\[
\{ X(t) \}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{ X(-t) \}_{t \in \mathbb{R}}
\]

\[
EB_H(s)B_H(t)^* \\
= \frac{1}{2}(|t|^H \Gamma(1, 1)|t|^H^* + |s|^H \Gamma(1, 1)|s|^H^* - |t - s|^H \Gamma(1, 1)|t - s|^H^*)
\]

\[\Leftrightarrow \Im( AA^*) = 0\]

in which case

\[
f_X(x) = \frac{1}{x^2}(x_+^D AA^* x_+^D^* + x_-^D AA^* x_-^D^*) = \frac{1}{x^2} |x|^{-D} AA^* |x|^{-D^*}
\]
Harmonizable representations

\textbf{application} (n = 1):

Biermé, Meerschaert & Scheffler (2007):

- $X_\psi$: OFBF with harmonizable representation based on the filter $\psi$
- $X_\varphi$: OFBF with a MA representation based on the filter $\varphi$

$\psi, \varphi$ are operator-homogeneous: $\varphi(c^E z) = c \varphi(z)$
Harmonizable representations

application ($n = 1$):

Biermé, Meerschaert & Scheffler (2007): under isotropy, moving average and harmonizable representations of $(m, 1)$-OFBFs yield

$$EX\varphi(s)X\varphi(t) = \frac{1}{2}\{||s||^{2H}c_\varphi + ||t||^{2H}c_\varphi - ||s - t||^{2H}c_\varphi\},$$

$$EX\psi(s)X\psi(t) = \frac{1}{2}\{||s||^{2H}c_\psi + ||t||^{2H}c_\psi - ||s - t||^{2H}c_\psi\},$$

$$\Rightarrow X_\varphi \overset{\mathcal{L}}{=} X_\psi \text{ (up to a constant)}$$
Harmonizable representations

application \((n = 1)\):

Biermé, Meerschaert & Scheffler (2007): however, under anisotropy, we have

\[
EX\varphi(s)X\varphi(t) = \frac{1}{2}\left\{\tau(s)^{2H}\sigma_l^2(s) + \tau(t)^{2H}\sigma_l^2(t) - \tau(s - t)^{2H}\sigma_l^2(s - t)\right\},
\]

\[
EX\psi(s)X\psi(t) = \frac{1}{2}\left\{\tau(s)^{2H}\omega_l^2(s) + \tau(t)^{2H}\omega_l^2(t) - \tau(s - t)^{2H}\omega_l^2(s - t)\right\},
\]

open problem: under what conditions on \(\psi\) and \(\varphi\) do we have

\[X\varphi \overset{\mathcal{L}}{=} X\psi?\]
Harmonizable representations

**Theorem** (Baek, D & Pipiras (2014)) under assumptions, there is a function $\varphi : \mathbb{R}^m \to \mathbb{R}$ s.t.

(i) $\varphi(t - \cdot) - \varphi(\cdot) \in L^2(\mathbb{R}^m)$

(ii) $\varphi$ is $E/(1 - \text{tr}(E)/2)$-homogeneous

(iii) $\{B_E(t)\}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \left\{ \int_{\mathbb{R}^m} \varphi(t - u) - \varphi(-u)B(du) \right\}_{t \in \mathbb{R}^m}$

where

$$\varphi(t - u) - \varphi(-u) = \int_{\mathbb{R}^m} e^{-i\langle u, x \rangle}(e^{i\langle t, x \rangle} - 1)\hat{a}(x)dx$$
Symmetry, exponents, anisotropy
non-linear = “not linear”: not very informative…

anisotropic: can we do better than say “not isotropic”?…
Symmetry, exponents, anisotropy

Isotropy:

\[ \{X(Ob)\}_{x \in \mathbb{R}^m} \cong \{X(t)\}_{t \in \mathbb{R}^m}, \quad O \in O(m) \]

i.e. every \( O \in O(m) \) is a (domain) symmetry of \( X \)

Overarching question: in what ways can the isotropy condition be violated?
Symmetry, exponents, anisotropy

For any field $X$, the symmetry sets are...

\[ G_{1}^{\text{ran}} = \{ A \in GL(n, \mathbb{R}); \{ AX(t)\}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \{ X(t)\}_{t \in \mathbb{R}^m} \} \]

\[ G_{1}^{\text{dom}} = \{ A \in GL(n, \mathbb{R}); \{ X(At)\}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \{ X(t)\}_{t \in \mathbb{R}^m} \} \]
Symmetry, exponents, anisotropy

For any field $X$, the symmetry sets are...

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G_{1}^{\text{ran}} = \{ A \in GL(n, \mathbb{R}); \{ AX(t) \}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \{ X(t) \}_{t \in \mathbb{R}^m} \}
\]

\[
G_{1}^{\text{dom}} = \{ A \in GL(n, \mathbb{R}); \{ X(At) \}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \{ X(t) \}_{t \in \mathbb{R}^m} \}
\]

$G_{1}^{\text{dom}}$ and $G_{1}^{\text{ran}}$ are compact subgroups of $GL(n, \mathbb{R})$

\[
\Rightarrow G_{1}^{\bullet} = W \mathcal{O}_0 W^{-1}, \mathcal{O}_0 \subseteq O(n)
\]
Symmetry, exponents, anisotropy

under mild conditions, for any o.s.s. random field $X$...

• Hudson & Mason (1982), Li & Xiao (2011): given $E$, there exists $H$ such that...

• D, Meerschaert & Pipiras (2014): given $H$, there exists $E$ such that...

$$\{X(c^E t)\}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{c^H X(t)\}_{t \in \mathbb{R}}$$
Symmetry, exponents, anisotropy

under mild conditions, for any o.s.s. random field $X$...

Furthermore, it turns out that...

\[
\{X(c^{E+?}t)\}_{t \in \mathbb{R}^m} \overset{\mathcal{L}}{=} \{c^{H+?'}X(t)\}_{t \in \mathbb{R}^m}
\]

question: why should exponents be non-unique in the first place?
Symmetry, exponents, anisotropy

**example:** \( \{ B_H(t) \}_{t \in \mathbb{R}} \) is a vector of uncorrelated FBM\( s \) with the same scalar parameter \( h \ (H = hI) \)

For \( L \in so(n), \ c^L \in SO(n) \) since

\[
c^L(c^L)^* = c^{L+L^*} = I \quad \text{and} \quad \det(c^L) = e^{\text{tr}(L \log(c))} = 1
\]

\[
\{ B_H(ct) \}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{ c^H B_H(t) \}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{ c^H c^L B_H(t) \}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{ c^{H+L} B_H(t) \}_{t \in \mathbb{R}}
\]

\[ \Rightarrow \text{both } H \text{ and } H + L \text{ are exponents for } B_H \]
Symmetry, exponents, anisotropy

**Theorem** (D, Meerschaert & Pipiras (2014)): under mild conditions, for any o.s.s. random field $X$...

- $\mathcal{E}^{\text{ran}}(X) = H + T(G_1^{\text{ran}})$
- $\mathcal{E}^{\text{dom}}(X) = E + T(G_1^{\text{dom}})$

$$T(G_1^{\bullet}) = \left\{ \lim_{k \to \infty} \frac{A_k - I}{d_k}, \text{ some } \{A_k\} \subseteq G_1, \text{ some } 0 \neq d_k \to 0 \right\}$$
**Symmetry, exponents, anisotropy**

**theorem** (D, Meerschaert & Pipiras (2014)): under mild conditions, for any o.s.s. random field $X$...

- $\mathcal{E}^{\text{ran}}(X) = H + T(G_1^{\text{ran}})$
- $\mathcal{E}^{\text{dom}}(X) = E + T(G_1^{\text{dom}})$
Symmetry, exponents, anisotropy

**Theorem** (D, Meerschaert & Pipiras (2014)): under mild conditions, for any o.s.s. random field $X$...

- $\mathcal{E}^{\text{ran}}(X) = H + T(G^{\text{ran}}_1)$
- $\mathcal{E}^{\text{dom}}(X) = E + T(G^{\text{dom}}_1)$

**Example**: OFGN (non-identifiable parametrization):

$$g_{Y_H}(x) \sim x^{-D} AA^* x^{-D^*}, \quad x \to 0$$
Symmetry, exponents, anisotropy

**Theorem** (D, Meerschaert & Pipiras (2014)): under mild conditions, for any o.s.s. random field $X$ . . .

- $\mathcal{E}^{\text{ran}}(X) = H + T(G_1^{\text{ran}})$
- $\mathcal{E}^{\text{dom}}(X) = E + T(G_1^{\text{dom}})$

**Example**: OFGN (non-identifiable parametrization):

$$g_{Y_H}(x) \sim x^{-D(G_1^{\text{ran}})}(AA^*)(G_1^{\text{ran}}, G_1^{\text{dom}})x^{-D(G_1^{\text{ran}})^*}, \quad x \to 0$$
Symmetry, exponents, anisotropy

To recap:

- $G_{1}^{\text{ran}} = W \mathcal{O}_0 W^{-1}$, $\mathcal{E}^{\text{ran}}(X) = H + T(G_{1}^{\text{ran}})$
- $G_{1}^{\text{dom}} = W' \mathcal{O}'_0 W'^{-1}$, $\mathcal{E}^{\text{dom}}(X) = E + T(G_{1}^{\text{dom}})$

**question**: domain and range symmetries look quite analogous. Are they of the same nature, and thus amenable to the same kind of technique?
Symmetry, exponents, anisotropy

To recap:

• \( G^\text{ran}_1 = W \mathcal{O}_0 W^{-1} \), \( \mathcal{E}^\text{ran}(X) = H + T(G^\text{ran}_1) \)

• \( G^\text{dom}_1 = W' \mathcal{O}'_0 W'^{-1} \), \( \mathcal{E}^\text{dom}(X) = E + T(G^\text{dom}_1) \)

**question**: domain and range symmetries look quite analogous. Are they of the same nature, and thus amenable to the same kind of technique?

**answer**: no, they are completely different
Symmetry, exponents, anisotropy

**Fact** (later): $SO(2)$ cannot be a domain symmetry group

**Theorem** (types): For proper Gaussian random fields in $\mathbb{R}^2$,

$G_{\mathrm{ran}} \cong \ldots$

(i) ... $\{\pm I\}$ (*cyclic*)

(ii) ... $\{\pm I, \pm \text{diag}(1, -1)\}$ (*dyhedral*)

(iii) ... $SO(2)$ (*rotational*)

(iv) ... $O(2)$ (*full*)
Symmetry, exponents, anisotropy

**example:** blind source separation

\[ X(t) = (X_1(t), X_2(t))^* : \text{independent, unobservable FGN signals with parameters } d_1, d_2 \]

\[ P \in GL(2, \mathbb{R}) : \text{mixing matrix} \]

\[ \{Y(t)\} = \{PX(t)\} : \text{mixed signal, observable} \]

**goal:** retrieve \( X \), i.e., estimate \( P \)

(hope: \( \hat{P}^{-1}Y(t) \approx X(t) \))
Symmetry, exponents, anisotropy

proposition:

(a) \(d_1 < d_2 \Rightarrow G^\text{ran}_1 = \{\pm I, \pm \text{diag}(1, -1)\}\)

(b) \(d_1 = d_2 \Rightarrow G^\text{ran}_1 = O(2)\)

Then

(a) \{\text{demixing matrices}\} = \{Q = PO, O \in \{\pm I, \pm \text{diag}(1, -1)\}\}\)

(b) \{\text{demixing matrices}\} = \{Q = PO, O \in O(2)\}\)

... \Rightarrow \text{any estimation procedure estimates } P \text{ up to a factor } O \in G^\text{ran}_1\)

question: consequences for asymptotics? \(\hat{P} \overset{P}{\rightarrow} ?\)
Symmetry, exponents, anisotropy

**question**: why are $G_1^{\text{dom}}$, $G_1^{\text{ran}}$ so different?
Symmetry, exponents, anisotropy

range:

$X$: Gaussian, $W = I$, $G_{1}^{\text{ran}} \subseteq O(n)$

\[
O \in G_{1}^{\text{ran}} \iff E[OX(s)X(t)^*O^*] = EX(s)X(t)^*, \quad s, t \in \mathbb{R}^m
\]

\[
\iff OEX(s)X(t)^* = EX(s)X(t)^*O, \quad s, t \in \mathbb{R}
\]

\[
\Rightarrow G_{1}^{\text{ran}} = \text{centralizer of } \{\Gamma(s, t)\}_{s, t \in \mathbb{R}^m}
\]
Symmetry, exponents, anisotropy

**domain (example):** $E = I \in M(2, \mathbb{R})$, $H = \beta - 1 \in \mathbb{R}$

$$\mathbb{R} \ni EX(s)X(t) = \int_{\mathbb{R}^2} (e^{i\langle s,x \rangle} - 1)(e^{-i\langle t,x \rangle} - 1)\|x\|^{-2\beta}_1 dx$$

$$\Rightarrow$$

$$\{X(At)\} \overset{\mathcal{L}}{=} \{X(t)\} \iff \|(A^*)^{-1}x\|_{1}^{-2\beta} = \|x\|_{1}^{-2\beta}, \quad x \in \mathbb{R}^m \setminus \{0\}$$

$$\iff \left\|(A^*)^{-1} \frac{x}{\|x\|_1}\right\|_1 = 1, \quad x \in \mathbb{R}^m \setminus \{0\}$$

$$\Rightarrow G_{\text{dom}}^1 = \text{symmetry group of } S_{\|\cdot\|_1} = \text{symmetry group of a rhombus}$$
Symmetry, exponents, anisotropy

what is the symmetry group of?...
Symmetry, exponents, anisotropy

\[ = G_1^{\text{dom}} \quad (E = I, H = \beta - 1) \]
Symmetry, exponents, anisotropy

- **range:** $A \in G_1^{\text{ran}} \iff AF_X(dx)A^* = F_X(dx)$ (commutativity)
- **domain:** $A \in G_1^{\text{dom}} \iff F_X((A^*)^{-1}dx) = F_X(dx)$ (measure)
Symmetry, exponents, anisotropy

domain: (Meerschaert & Veeh (1995))

eexample: \( SO(2)x = O(2)x, \ x \in \mathbb{R}^m \Rightarrow [SO(2)] = [O(2)] \)

\( \Rightarrow SO(2) \) is not maximal in \([O(2)]\)
Symmetry, exponents, anisotropy

**domain:** (Meerschaert & Veeh (1995))

**example:** \( SO(2)x = O(2)x, \ x \in \mathbb{R}^m \Rightarrow [SO(2)] = [O(2)] \)

\( \Rightarrow \) \( SO(2) \) is not maximal in \([O(2)]\)

**answer to the overarching question:**

**theorem** (D, Meerschaert & Pipiras (2014)): the domain symmetry group \( G_1^{\text{dom}} \) of OFBFs is maximal (this **fully characterizes** all types of anisotropy)

**corollary:** \( SO(2) \) cannot be a domain symmetry group
Symmetry, exponents, anisotropy

can we convey a **parametric** characterization of isotropy?

**question:**

\((m, n)\text{-OFBF: } F_X(dx) = r^{-H} \Lambda(d\theta) r^{-H^*} r^{-1} dr\)

\[
\text{isotropy} \\
\iff \exists \eta > 0 \text{ such that } E_0 = \eta I \in \mathcal{E}^{\text{dom}}(X)
\]
can we convey a **parametric** characterization of **isotropy**?

**Theorem** (D, Meerschaert & Pipiras (2014)):

\[(m, n)\text{-OFBF: } F_X(dx) = r^{-H} \Lambda(d\theta) r^{-H^*} r^{-1} dr\]

**isotropy**

\[\Leftrightarrow\]

**(radial) \( \exists \eta > 0 \text{ such that } E_0 = \eta I \in \mathcal{E}^{\text{dom}}(X)\)**

**(spherical) \( \Lambda(d\theta) = \Lambda(O^*d\theta), \ O \in O(m), \ S_0 = c_0^{-1} S^{m-1} \)**
**Symmetry, exponents, anisotropy**

**issue:** how to incorporate departures from isotropy ("poorer" $G_{1}^{\text{dom}}$) into the analysis of physical systems?
Symmetry, exponents, anisotropy

**Illustration:** $(2, n)$-OFBFs. All possible domain symmetry groups are

(i) $C_n = \{O_{k2\pi/n} : k = 1, \ldots, n\}, \ n \in \mathbb{N}$ (cyclic)

(ii) $D_n = \{O_{k2\pi/n}, F_{k2\pi/n} : k = 1, \ldots, n\}, \ n \in \mathbb{N}$ (dihedral)

(iii) $D^*_1 = \{I, \text{diag}(-1, 1)\}$

(iv) $O(2)$

This is a full description of anisotropy...
Symmetry, exponents, anisotropy

**illustration:** $(2, n)$-OFBFs. All possible domain symmetry groups are

(i) $C_n = \{O_{k2\pi/n} : k = 1, \ldots, n\}, \ n \in \mathbb{N}$ (cyclic)

(ii) $D_n = \{O_{k2\pi/n}, F_{k2\pi/n} : k = 1, \ldots, n\}, \ n \in \mathbb{N}$ (dihedral)

(iii) $D_1^* = \{I, \text{diag}(-1, 1)\}$

(iv) $O(2)$

**open issue:** many OFBFs fall under one given symmetry group; do we need to refine our notion of anisotropy (e.g., within symmetry classes)?
Wavelet analysis
Wavelet analysis

How about multidimensional inference? Some references:

- scalar fields: Abry, Clausel, Jaffard, Roux and Vedel (2013); Lim, Meerschaert, and Scheffler (2014)

- vector processes: Becker-Kern and Pap (2008); Amblard and Coeurjolly (2011)

our focus: vector processes from a wavelet perspective
Wavelet analysis

wavelet coefficients/transform: $a \in \mathbb{N}, \ k \in \mathbb{Z}$

$$D_{a,k} = a^{-1} \int_{\mathbb{R}} \psi(a^{-1}t - k)X(t)dt = \left(d_p(a, k)\right)_{p=1,...,n}$$

where $\psi$ is a wavelet with compact support (Daubechies)
Wavelet analysis

• when $X(t) = \text{OFBM}$

$$ED_{a,k}D_{a,k}^* = a^H ED_{1,0}D_{1,0}^* a^{H*}$$

$\Rightarrow$ a wavelet regression method can be developed

• caveat: the Fourier and wavelet spectra are not equivalent. They do not blow up entrywise according to the same power law (up to a known constant). Equivalence holds under time reversibility
Wavelet analysis

assuming time reversibility...

• when $H$ is non-diagonal, as $a \to \infty$ there is a competition between power laws. A wavelet regression only captures the highest power law

• an extrapolation of the univariate framework yields that “scaling behavior is characterized by a power law at the origin of the spectrum”

However...
Wavelet analysis

why non-diagonal scaling? ($H$ is non-diagonal)

- blind source separation
- cointegration (Robinson (2008)): $Y(t) = PX(t)$, where $X$ is a long-memory vector
- how many others?...

under investigation: how to deal with non-diagonal scaling from a wavelet perspective
Summary

• multidimensionality can technically change the problem

• current goal: to develop the synthesis of domain and range multidimensionality

• integral representations: a powerful paradigm for the analysis of operator fractional fields

• interrelated ideas: symmetry groups, anisotropy, operator scaling, exponents
harmonizable representation:

\[ X_\psi(t) = \Re \left( \int_{\mathbb{R}^m} (e^{i\langle t,x \rangle} - 1)\psi(x)^{-q-\frac{H}{2}}W_2(dx) \right) \]

\[ \psi : [0, \infty)^m \to \mathbb{R} \text{ is a continuous, } E^*-\text{homogeneous function such that } \psi(x) \neq 0 \text{ when } x \neq 0 \]
Integral representations (Biermé et al. (2007))

moving average representation:

\[
X_\varphi(t) = \mathbb{R}\left( \int_{\mathbb{R}^m} (\varphi(t - u)^{H-q/2} - \varphi(-u)^{H-q/2}) W_2(dx) \right)
\]

\(\varphi : [0, \infty)^m \rightarrow \mathbb{R}\) is a \(E\)-homogeneous, \((\beta, E)\)-admissible function

- \((\beta, E)\)-admissibility: \(x \neq 0 \Rightarrow \varphi(x) > 0\), and there exists \(C > 0\) such that for any \(0 < A < B\)

\[
\tau(x) \leq 1 \Rightarrow |\varphi(x + y) - \varphi(y)| \leq C \tau(x)^\beta
\]
Wavelet analysis

\((D = H - (1/2)I)\)

**Fourier spectrum:**

\[
f_X(x) = \frac{1}{x^2}(x^D AA^* x^D - x^{-D} AA^* x^{-D})
\]

**Wavelet spectrum:**

\[
w(a) := ED_{a,k} D_{a,k}^* = a^H \left( \int_0^\infty \frac{\hat{\psi}(x)^2}{x^2} x^D \text{Re}(AA^*) x^{-D} \right) a^{H^*}
\]
Wavelet analysis

**problem:** the wavelet and Fourier spectra are not equivalent

**example:**

\[ H = \text{diag}(h_1, h_2), \quad AA^* = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \]

**Fourier spectrum:**

\[ x^D A A^* x^D^* = \begin{pmatrix} x^{-2d_1} & i x^{-(d_1+d_2)} \\ -i x^{-(d_1+d_2)} & x^{-2d_2} \end{pmatrix} \]

**wavelet spectrum (diagonal!):**

\[ w(a) = \text{diag}\left(a^{2h_1} \int_0^\infty \frac{\hat{\psi}(x)^2}{x^2} x^{-2d_1} dx, a^{2h_2} \int_0^\infty \frac{\hat{\psi}(x)^2}{x^2} x^{-2d_2} dx\right) \]
Wavelet analysis

**Theorem** (spectral equivalence): under time reversibility, diagonalizable $H$ with real roots, the spectra are equivalent. For a fixed entry $(m_1, m_2)$, either

- both $f_{m_1, m_2}(\cdot)$ and $w_{m_1, m_2}(\cdot)$ are identically zero; or

- for some $-1 < \delta < 1$,

  $f_{m_1, m_2}(x) \sim c_F |x|^{-\delta}, \quad x \to 0$

  $w_{m_1, m_2}(a) \sim c_W |a|^{|\delta+1}, \quad a \to \infty, \quad c_F, c_W \neq 0$
Wavelet analysis

**interest:** estimate the entry-wise powers “δ” based on the behavior of $w(a)$ as $a \to \infty$

simplifying assumption: entry-wise scaling, i.e.,

$(A)$ For $m_1, m_2 = 1, \ldots, n$, there exists $\eta_{m_1, m_2} \in (0, 2)$ such that

$$Ed_{m_1}(a, 0)d_{m_2}(a, 0) = a^{\eta_{m_1, m_2}}Ed_{m_1}(1, 0)d_{m_2}(1, 0),$$
Wavelet analysis

**Interest**: estimate the entry-wise powers \( \delta \) based on the behavior of \( w(a) \) as \( a \to \infty \)

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\((A)\) For \( m_1, m_2 = 1, \ldots, n \), there exists \( \eta_{m_1, m_2} \in (0, 2) \) such that

\[
Ed_{m_1}(a, 0) d_{m_2}(a, 0) = a^{\eta_{m_1, m_2}} Ed_{m_1}(1, 0) d_{m_2}(1, 0),
\]

Central idea behind the wavelet regression: take logs

\[
\log|Ed_{m_1}(a, 0) d_{m_2}(a, 0)| = \eta_{m_1, m_2} \log|a| + \log|Ed_{m_1}(1, 0) d_{m_2}(1, 0)|
\]
Wavelet analysis

regression equations \((n = 2)\):

\[ j = 1, \ldots, m \ (\# \text{ scales}), \ k = 1, \ldots, N_j \ (\# \text{ terms per scale}) \]

\[
\begin{pmatrix}
\log | Ed_1^2(a_j, k) | \\
\log | Ed_2^2(a_j, k) | \\
\log | Ed_1(a_j, k)d_2(a_j, k) |
\end{pmatrix}
= \begin{pmatrix}
\eta_1 \log a_j \\
\eta_2 \log a_j \\
\eta_{1,2} \log a_j
\end{pmatrix}
+ \begin{pmatrix}
\log | Ed_1^2(1, k) | \\
\log | Ed_2^2(1, k) | \\
\log | Ed_1(1, k)d_2(1, k) |
\end{pmatrix}
\]
Wavelet analysis

regression equations \((n = 2): (\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_{1,2})\) least squares

\(j = 1, \ldots, m\) (# scales), \(k = 1, \ldots, N_j\) (# terms per scale)

\[
\begin{pmatrix}
\log \left| \frac{1}{N_j} \sum_{k=0}^{N_j-1} d_1^2(a_j, k) \right| \\
\log \left| \frac{1}{N_j} \sum_{k=0}^{N_j-1} d_2^2(a_j, k) \right| \\
\log \left| \frac{1}{N_j} \sum_{k=0}^{N_j-1} d_1(a_j, k)d_2(a_j, k) \right|
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\eta_1 \log a_j \\
\eta_2 \log a_j \\
\eta_{1,2} \log a_j
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\log \left| E d_1^2(1, k) \right| \\
\log \left| E d_2^2(1, k) \right| \\
\log \left| E d_1(1, k)d_2(1, k) \right|
\end{pmatrix}
\]

+ stoch. error
Wavelet analysis

example: $H = \text{diag}(h_1, h_2)$

Then

$$ED_{a_j,0}D^*_{a_j,0} = \begin{pmatrix}
  a_j^{2h_1}Ed_1^2(1,0) & a_j^{h_1+h_2}Ed_1(1,0)d_2(1,0) \\
  a_j^{h_1+h_2}Ed_1(1,0)d_2(1,0) & a_j^{2h_2}Ed_2^2(1,0)
\end{pmatrix}$$

$\Rightarrow$ for entry $(1,1)$, $j = 1, \ldots, m$ (scales)

$$\log \left( \frac{1}{N_j} \sum_{k=0}^{N_j-1} d_1^2(a_j, k) \right) \overset{A}{=} 2h_1 \log a_j + \log(Ed_1^2(1,0)) + \sqrt{\frac{a(N)}{N}} N(0, \nu_j^1)$$

$\Rightarrow$ by least squares, $\hat{h}_1$
Figure 1: Bias vs sample size \((d_1 = -0.2, \ d_2 = 0.3)\)
Figure 2: Asymptotic normality: MC (solid black lines) vs best Gaussian fits (dashed red lines), $d_1 = -0.2$, $d_2 = 0.3$