Spectral properties of self-similar sets & measures

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1. Fuglede's spectral Problem on exponential ortho. basis

2. Spectrality of self-similar measures

3. Spectrality of self-similar tiles

Write $e(t) = e^{2\pi i t}$ for the complex exponential.

Basic fact: $L^2([0,1]^d)$ has an orthonormal basis $\{e(n \cdot x)\}_{n \in \mathbb{Z}^d}$

Question: What kind of (bounded) K or μ in $\mathbb{R}^d$ so that $L^2(K,\mu)$ will admit an exponential orthonormal basis $\{ce(\lambda \cdot x) : \lambda \in \Lambda\}$?

We call such μ spectral measure; Λ the spectrum.

K a spectral set if μ is the Lebesgue measure on K.

Background:

For a bounded connected domain $\Omega \subset \mathbb{R}^n$, The partial derivative operators $\{\partial_j\}_{j=1}^d$ is *commutative* on $C_c^\infty(\Omega)$ ($\subset L^2(\Omega)$).

Segal (58): Can these (unbounded) operators be extended to *commutative, self-adjoint* operators on $L^2(\Omega)$?

Fuglede (74): Segal's problem holds iff $L^2(\Omega)$ admits an exponential orthonormal basis $\{ce^{2\pi i\langle\lambda_k,\cdot\rangle} : \lambda_k \in \mathbb{R}^d\}$,
i.e, Ω is a spectral set.

- *lattice* (translational) tiles are spectral sets;
- $[0,1] \cup [2,3]$ is spectral set, and is a (non-lattice) tile;
- triangles and disks are not spectral set.

Fuglede Conjecture : K is a spectral set in $\mathbb{R}^d$ if and only if K is a (translational) tile.

Recall K is a (translational) tile if there exists a discrete set $\mathcal{T}$ such that

(i) $(K^\circ + t) \cap (K^\circ + s) = \emptyset$ *for* $t, s \in \mathcal{T},\ s \neq t$*;*

(ii) $\cup_{t\in\mathcal{T}}(K + t) = \mathbb{R}^n$.

There are positive results with additional assumptions

• Convexity tiles are lattice tiles (P. McMullen), hence they are spectral sets.

In $\mathbb{R}^2$, the converse is also true (Iosevich, Katz & Tao).

• Integer tiles: $A \subset \mathbb{Z} \ni A \bigoplus B \equiv \mathbb{Z}_n \ (\mathrm{mod} n)$, A tiles $\mathbb{Z}$ with $\mathcal{J} = B + n\mathbb{Z}$ as tiling set.

This tiling and spectral properties have been studied by Lagarias & Wang, Laba, Coven & Meyerowitz etc.

• Tao (03) gave a counter example of spectral set that is not a tile on $\mathbb{R}^d$, $d \geq 5$;

• Kolountzakis and Matolcsi (06) improved to $d \geq 3$, also disproved the other direction.

The Questions

- Fuglede conjecture still stands for dimension ≤ 2.

- Anything we can say for $L^2(K)$ where K is self-similar tiles?

- More generally, consider the spectral property for $L^2(K, \mu)$ where μ is self-similar measure.

Consider $(A, \mathcal{D})$: A is a $d \times d$ expanding integer matrix (all eigenvalues $|\lambda| > 1$), and $\mathcal{D} = \{0, d_1, \cdots, d_{N-1}\} \subset \mathbb{Z}^d$.

Self-affine set: $AK = K + \mathcal{D}$

Self-affine measure μ: For probability weight $\{w_i\}_{i=1}^N$,

$$\mu(E) = \sum_{i=1}^{N} w_i \mu(AE - d_i) \qquad \forall\ E \ \text{ Borel}$$

(Self-similar if A is a multiple of orthonormal matrix)

Basic example of self-similar measures

- Standard Cantor measure: $A = 3$, $\mathcal{D} = \{0, 2\}$, $w_1 = w_2 = 1/2$.

$$\mu(\cdot) = \frac{1}{2}\,\mu(3\,\cdot) + \frac{1}{2}\,\mu(3\cdot -2)$$

- ρ-Cantor measure μ_ρ: same as the above with $0 < \rho < 1/2$
- Bernoulli convolution μ_ρ: same as the above with $0 < \rho < 1$

Basic criterion (Jorgensen and Pedersen (98). μ p.m. on $\mathbb{R}^d$.
Then $\{e(\lambda \cdot x) : \ \lambda \in \Lambda\}$ is an *orthonormal* set of $L^2(\mu)$ iff

$$\widehat{\mu}(\lambda_1 - \lambda_2) = \int_K e^{2\pi i \langle \lambda_1 - \lambda_2, \ x \rangle} d\mu(x) = 0 \qquad \forall \quad \lambda_1 \neq \lambda_2.$$

i.e., $(\Lambda - \Lambda) \setminus \{0\} \subset \{\xi : \hat{\mu}(\xi) = 0\}$.

It is *complete* iff

$$\sum_{\lambda \in \Lambda} |\widehat{\mu}(t - \lambda)|^2 \equiv 1, \qquad t \in \mathbb{R}^n$$

Spectral property of self-affine measure μ

For μ with weight $1/N$, the Fourier transform is

$$\widehat{\mu}(\xi) = \frac{1}{N}P_{\mathcal{D}}((A^t)^{-1}\xi)\ \widehat{\mu}((A^t)^{-1}\xi) = \prod_{k=1}^{\infty}(\frac{1}{N}P_{\mathcal{D}}((A^t)^{-1}\xi))$$

where $P_{\mathcal{D}}(\xi) = \sum_{j=1}^{N} e^{2\pi i\langle d_j,\xi\rangle}$ is called the (mask polynomial).

- The zero set $\mathcal{Z}(\widehat{\mu}) = \{\xi : \widehat{\mu}(\xi) = 0\}$ is determined by $P_{\mathcal{D}}$.
- A spectrum Λ satisfies $0 \in \Lambda$, and

$$\Lambda - \Lambda \subset \mathcal{Z}(\widehat{\mu}) \cup \{0\}$$

and is maximal. But the converse is not true.

- We use this to construct the spectrum (might not exist)

The canonical spectrum Λ:

Let $\mathcal{B} = \{b_1, \cdots, b_N\} = A^{-1}\mathcal{D}$,

(i) choose $\Gamma \subset \mathbb{Z}^d$ such that $\#\Gamma = N$ and the $N \times N$ matrix

$$\frac{1}{N}[e(b \cdot \ell)]_{b \in \mathcal{B}, \ell \in \Gamma}$$

is an unitary matrix (not always exist).

(ii) Let

$$\Lambda = \{\ell_0 + A^t \ell_1 + \cdots + (A^t)^n \ell_n : \quad \ell_i \in \Gamma, \ n \in \mathbb{N}\}.$$

Then Λ is a natural candidate to be the spectrum of $L^2(\mu)$.

Jorgensen and Pedersen (98) : $\mu_{1/q}$ a Cantor measure with contraction ratio $\rho = 1/q$
Then μ is a spectral measure if q is an even integer;
but *not* a spectral measure if q is odd.

In particular $\mu_{1/4}$ is a spectral measure, but $\mu_{1/3}$ is *not* .

For $\mu_{1/4}$, $\mathcal{B} = \frac{1}{4}\{0, 2\}$, $L = \{0, 1\}$ and the Hadamard matrix is

$$\frac{1}{2}[e^{2\pi i \frac{jk}{2}}]_{j,k} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and $\Lambda = \{\ell_0 + 4\ell_1 + \cdots 4^n\ell_n : \quad \ell_i = 0, 1, \ n \in \mathbb{N}\}$.

• Strichartz (06) used this orthonormal basis to consider the Fourier series expansion.

• What about the Bernoulli convolution μ_ρ, $0 < \rho < 1$?

A complete solution to the Bernoulli convolution $\mu_\rho : \#\mathcal{D} = 2$, $0 < \rho < 1$

• (Hu-L, Adv.Math., 08) $L^2(\mu)$ contains an infinite exponential orthonormal set if and only if $\rho = \left(\frac{p}{q}\right)^{1/n}$ for some $n > 0$, where p is odd and q is even.

• (Dai, Adv.Math., 12) For the above μ_ρ to be a spectral measure if and only if $\rho = 1/q$ where q is even.

Theorem 1. (Dai-He-L) Let $\mathcal{D} = \{0, \cdots, N-1\}$, and $w_i = 1/N$. Then μ is a spectral measure iff

(i) $\rho = 1/q$ for some integer q,

(ii) N divides q.

Sufficient : Let $q = Nr$, $\mathcal{D} = \{0, \cdots, N-1\}$. Then

$$H = [e^{2\pi i \frac{jk}{N}}]_{0 \leq j,k \leq N}$$

is a Hardamard matrix, and $\frac{1}{q}\mathcal{D}$ and $\Gamma = r\{0, \cdots, N-1\}$ is a compatible pair. The spectrum is

$$\Lambda = \{\sum_{j=0}^{k} a_j q^j : \ a_j \in \Gamma, k \geq 0\}$$

Necessity: By exhaustion, i.e., excluding all the undesirable cases.

- The irrational ρ can be excluded by using some algebraic property from $\Lambda - \Lambda \subset \mathcal{Z}(\mu)$.

- For the rational ρ, the difficult case is $\rho = p/q$, $p \neq 1$ and $N|q$, we show that all the maximal orthogonal set cannot be complete, i.e.,

$$\sum_{\lambda \in \Lambda} |\widehat{\mu}(t-\lambda)|^2 \not\equiv 1, \qquad t \in \mathbb{R}^n$$

It depends on a characterization of the maximal orthogonal sets in (Dutkay-Han-Sun, Adv. Math., 11), (Dai-He-Lai, Adv. Math., 13)).

Q1. (Laba & Wang, 02): For $\mathcal{D} \subset \mathbb{N}$, $\gcd(\mathcal{D}) = 1$, and

$$\mu(\cdot) = \sum_{j=1}^{N} w_i \mu(\rho^{-1}(\cdot) - d_i),$$

to be a spectral measure, it is necessary and sufficient that

(i) $w_i = 1/N$ (ok, Dutkay & Lai);

(ii) $\rho = 1/q$ for some integer q;

(iii) $\mathcal{D} \oplus \mathcal{B} \equiv \{0, \cdots, q-1\} \pmod{q}$ for some $\mathcal{B}$.

Remark: Our previous theorem assumes that
$\mathcal{D} = \{0, 1, \cdots, N-1\}$ and $w_i = 1/N$

Q2. The standard Cantor measure $\mu_{1/3}$ is not a spectral measure. Does it has a Fourier frame/Riesz basis?

Fourier Frame of $L^2(\mu)$ (Duffin and Shaeffer): $\exists\ A, B > 0\ \ni$

$$A||f||^2 \leq \sum_{\lambda\in\Lambda} |\langle f, e^{2\pi i\lambda x}\rangle|^2 \leq B||f||^2 \quad \forall\ f \in L^2(\mu)$$

Self-affine tiles:

Theorem (Bandt) For $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\}$ and $|\det(A)| = b$. If $K^o \neq \emptyset$, then K is a self-affine tile.

- We call such $\mathcal{D}$ a tile digit set.

- It is known that (Bandt, Kenyon) on $\mathbb{R}$, if $\#\mathcal{D}$ is a prime, then $\mathcal{D}$ is a tile digit set iff $\mathcal{D}$ is a complete residue class (mod b). There are extensions to $\mathbb{R}^n$ under some additional conditions (Lagarias-Wang, He-L)

In such cases K is a $\mathbb{Z}^n$-tile, and hence a spectral set.

- Not much is know for $\#\mathcal{D} = |\det(A)| = b$ is non-prime

Problem: characterize (or find) tile digit sets for the non-prime cases, and consider the Fuglede problem.

- Product-Form : (*Odlyzko, Lagarias & Wang 95*):

Let

$$\mathcal{E} = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k,$$

and is a complete residue set w.r.t. A. Then

$$\mathcal{D} = \mathcal{E}_0 \oplus A^{\ell_1}\mathcal{E}_0 \cdots \oplus A^{\ell_k}\mathcal{E}_k$$

is a tile digit set.

Recall that for a self-similar measure on $\mathbb{R}$ with weight $1/b$, the Fourier transform is

$$\widehat{\mu}(\xi) = b^{-1} P_{\mathcal{D}}(x/b)\widehat{\mu}(\xi/b) = \prod_{k=1}^{\infty} (\frac{1}{b} P_{\mathcal{D}}(e^{2\pi i \xi / b^k}))$$

where $P_{\mathcal{D}}(x) = \sum_{d \in \mathcal{D}} x^d$ (mask polynomial).

- For the product form, $P_{\mathcal{D}}(x) = P_{\mathcal{E}_0}(x) P_{\mathcal{E}_1}(x^{b_1}) \cdots P_{\mathcal{E}_k}(x^{b_k})$

- Kenyon criterion : *$K(b, \mathcal{D})$ is a tile iff $\widehat{\mu}(m) = 0$ for all $m \neq 0$, i.e., for any m, there exists k such that*

$$P_{\mathcal{D}}(e^{2\pi i m / b^k}) = 0.$$

Modulo product-form $\mathcal{D}$ (Lai-Rao-L, Tran AMS 2013): Let

$$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \cdots \oplus \mathcal{E}_k \equiv \{0, \cdots, b-1\} (\mathrm{mod\ b}),$$

and $\ell_1 \leq \cdots \leq \ell_k$. $\mathcal{D} := \mathcal{D}^{(k)}$ is constructed by

$$\begin{cases} \mathcal{D}^{(0)} \equiv \mathcal{E}_0 \ (\mathrm{mod}\ n_0), \\ \vdots \\ \mathcal{D}^{(k)} \equiv \mathcal{D}^{(k-1)} \oplus b^{l_k} \mathcal{E}_k \ (\mathrm{mod}\ n_k), \end{cases}$$

where $n_i = l.c.m. \{s : \Phi_s(x) \mid \Theta_0(x) \cdots \Theta_i(x^{b^{\ell_i}})\}$ and $\Theta_i(x)$ is product of cyclotomic factors of $P_{\mathcal{E}_i}(x)$.

- Modulo product-forms are tile digit sets (use cyclotomic polynomials).

- We can further construct higher order modulo product-forms.

- This is used to characterize tile digit sets with $\mathcal{D} = p^\alpha, pq, p^\alpha q$.

Spectral property on self-similar tiles on $\mathbb{R}$ (Fu-He-L)

Theorem 2. Let $\mathcal{D}$ be a product-form associated with $\mathcal{E} = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b \ (mod\ b)$. Then

(i) $K(b, \mathcal{D}) = K(b, \mathcal{E}) \oplus \mathcal{A}$ and $\mathcal{A}$ is an integer tile.

(ii) $K(b, \mathcal{D})$ is a spectral set if and only $\mathcal{A}$ is a spectral measure.

- the sum in (i) is known in Lagarias and Wang (95)

- $K(b, \mathcal{E})$ is a spectral set as it is a lattice tile; the spectrum of $K(b, \mathcal{D})$ (if exists)is $\Gamma + \mathbb{Z}$, where Γ is the spectrum of $\mathcal{D}$.

- There are study of spectral property of $\mathcal{A}$ as integer tile (Laba (01)).

Theorem 3. Let $\mathcal{D}$ be a modulo product-form associated with $\mathcal{E} = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k = \mathbb{Z}_b$. Then the tile $K(b, \mathcal{D})$ is a spectral set.

- Idea of proof.
  (i) the corresponding product-form $K(b, \mathcal{D}')$ is a spectral set.
  (ii) $|K(b, \mathcal{D})| = |K(b, \mathcal{D}')|$, and $P_{\mathcal{D}'}(x)$ divides $P_{\mathcal{D}}(x)$.
  (iii) the spectrum of $K(b, \mathcal{D})$ is the same as $K(b, \mathcal{D}')$.

- If $\#\mathcal{D} = p^\alpha$ of pq, the tile digit sets are modulo product form of the above expression, hence $K(b, \mathcal{D})$ is a tile implies that it is a spectral set.

The converse, spectral set $\Rightarrow$ tile , seems to be a harder direction.

• A special case: $\mathcal{D} = 4$ and $\gcd(\mathcal{D}) = 1$, then $\mathcal{D}$ is a tile digit set if and only if

$$\mathcal{D} = (\{0,1\}(mod 2) \oplus 4^t\{0,2\}) \ (mod 4^{t+1}).$$

Simplify, $\mathcal{D} = \{0,\ \ell_1,\ 2^s\ell_2,\ \ell_1 + 2^s\ell_3\}$ where s and ℓ_i are odd integers.

Now if the self-similar set $K(b, \mathcal{D})$ is a spectral set, we can use Jorgensen-Pedersen's orthonormal criterion to show that $\mathcal{D}$ must be of the above form. Hence

Proposition 4. If $\#\mathcal{D} = 4$ and $\gcd(\mathcal{D}) = 1$, then $\mathcal{D}$ is a tile digit set iff it is a spectral set.

Some questions

Q1. Can tile digit set $\mathcal{D}$ be characterized by the higher order modulo product-forms?

Q2. Is modulo product-form tiles (associated with $\mathcal{E} = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b \ (mod\ \ b)$) spectral sets?

Q3. Higher dimension?

Thank You