Multifractality of Whole-Plane SLE

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Multifractal Analysis:

From Theory to Applications and Back

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Let $f(z) = \sum_{n\geqslant 0} a_n z^n$ be a holomorphic function in the unit disc $\mathbb{D}$. Further assume that f is injective. Then $a_1 \neq 0$ and *Bieberbach* proved in 1916 that $|a_2| \leqslant 2|a_1|$. In the same paper, he famously conjectured that $\forall n \geqslant 2,\ |a_n| \leqslant n|a_1|$, guided by the intuition that the *Koebe function*

$$\mathcal{K}(z) := -\sum_{n\geqslant 1} n(-z)^n = \frac{z}{(1+z)^2},$$

which is a holomorphic bijection between $\mathbb{D}$ and $\mathbb{C}\backslash[1/4, +\infty)$, should be *extremal*. This conjecture was finally proven in 1984 by *de Branges*. The earliest important contribution to the proof of Bieberbach's conjecture is that by *Loewner* in 1923 that $|a_3| \leqslant 3|a_1|$. *Oded Schramm* revived Loewner's method in 1999, introducing *randomness* into it, as driven by *standard Brownian motion.*

Whole-Plane SLE & LLE

$$\frac{\partial f_t}{\partial t} = z\frac{\partial f_t}{\partial z}\frac{\lambda(t)+z}{\lambda(t)-z},\ \ z \in \mathbb{D},$$

$$\lambda(t) = e^{i\sqrt{\kappa}B_t}\ \ [e^{i\xi L_t}].$$

The characteristic function of a Lévy process L_t has the form

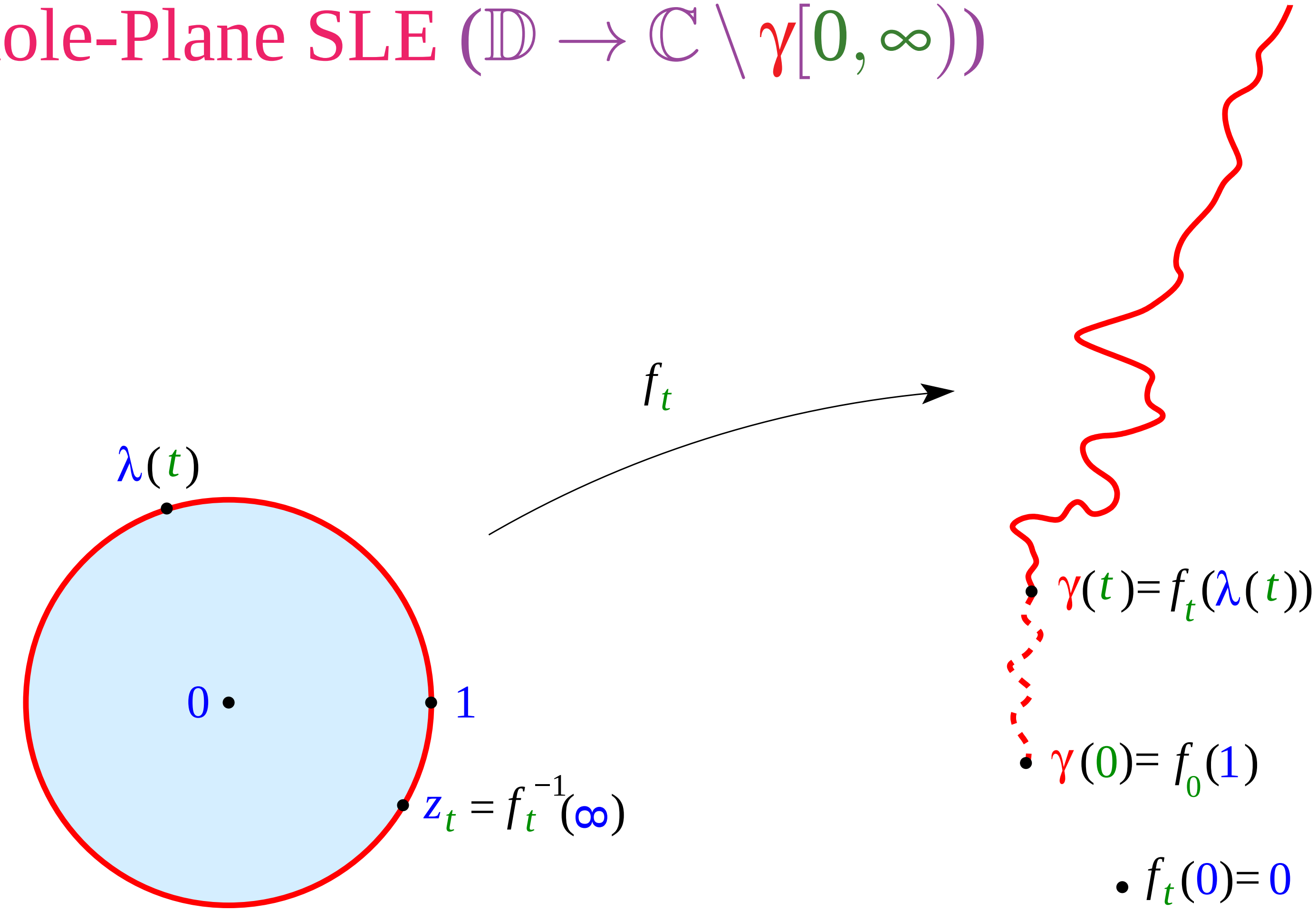
$$\mathbb{E}(e^{i\xi L_t}) = e^{-t\eta(\xi)},$$

where η the Lévy symbol. The function

$$\eta(\xi) = \kappa|\xi|^{\alpha}/2,\ \ \alpha \in (0,2]$$

is the Lévy symbol of the $\alpha-$*stable process*. The normalization here is chosen so that it is SLE_{κ} for $\alpha = 2$.

Whole-Plane SLE ($\mathbb{D} \to \mathbb{C} \setminus \gamma[0,\infty)$)



Loewner map $z \mapsto f_t(z)$ *from the unit disk* $\mathbb{D}$ *to the slit domain* $\Omega_t = C \setminus \gamma([t,\infty))$. *One has* $f_t(0) = 0, \forall t \geqslant 0$. *At* $t = 0$, $\lambda(0) = 1$, *so that the image of* $z = 1$ *is at the tip* $\gamma(0) = f_0(1)$ *of the curve.*

Series expansions

Let f_t *be the whole-plane evolution generated by the Lévy process* (L_t) *with Lévy symbol* η. *We write*

$$e^{-t} f_t(z) = z + \sum_{n=2}^{\infty} a_n(t) z^n; \; e^{-t/2} h_t(z) = z + \sum_{n \geqslant 1} b_{2n+1}(t) z^{2n+1}.$$

Then the conjugate *whole-plane LLE* $e^{-iL_t} f_t\left(e^{iL_t} z\right)$ *has the same law as* $f_0(z)$, *i.e.,* $e^{i(n-1)L_t} a_n(t) \overset{\text{(law)}}{=} a_n(0)$. *Similarly, the conjugate of the* oddified *whole-plane LLE* $h_t(z) := z\sqrt{f_t(z^2)/z^2}$, $e^{-(i/2)L_t} h_t\left(e^{(i/2)L_t} z\right)$, *has the same law as* $h_0(z)$, *i.e.,* $e^{inL_t} b_n(t) \overset{\text{(law)}}{=} b_n(0)$.

Loewner's method

Recall that

$$f_t(z) = e^t \Big(z + \sum_{n \geqslant 2} a_n(t) z^n \Big).$$

By expanding both sides of Loewner's equation as power series, and identifying coefficients, leads one to the set of *recursion equations for* $n \geqslant 2$

$$\dot{a}_n(t) - (n-1) a_n(t) = 2 \sum_{k=1}^{n-1} k a_k(t) \bar{\lambda}^{n-k}(t),$$

with $a_1 = 1$; *the dot means a t-derivative*, and $\bar{\lambda}(t) = 1/\lambda(t)$, with $\lambda(t) = e^{i\sqrt{\kappa} B_t}$ $[e^{i\xi L_t}]$.

Expected coefficients

Theorem 1. *Setting* $a_n := a_n(0)$ *and* $b_{2n+1} := b_{2n+1}(0)$*, we have*

$$\mathbb{E}(a_n) = \prod_{k=0}^{n-2} \frac{\eta_k - k - 2}{\eta_{k+1} + k + 1}, \quad n \geqslant 2,$$

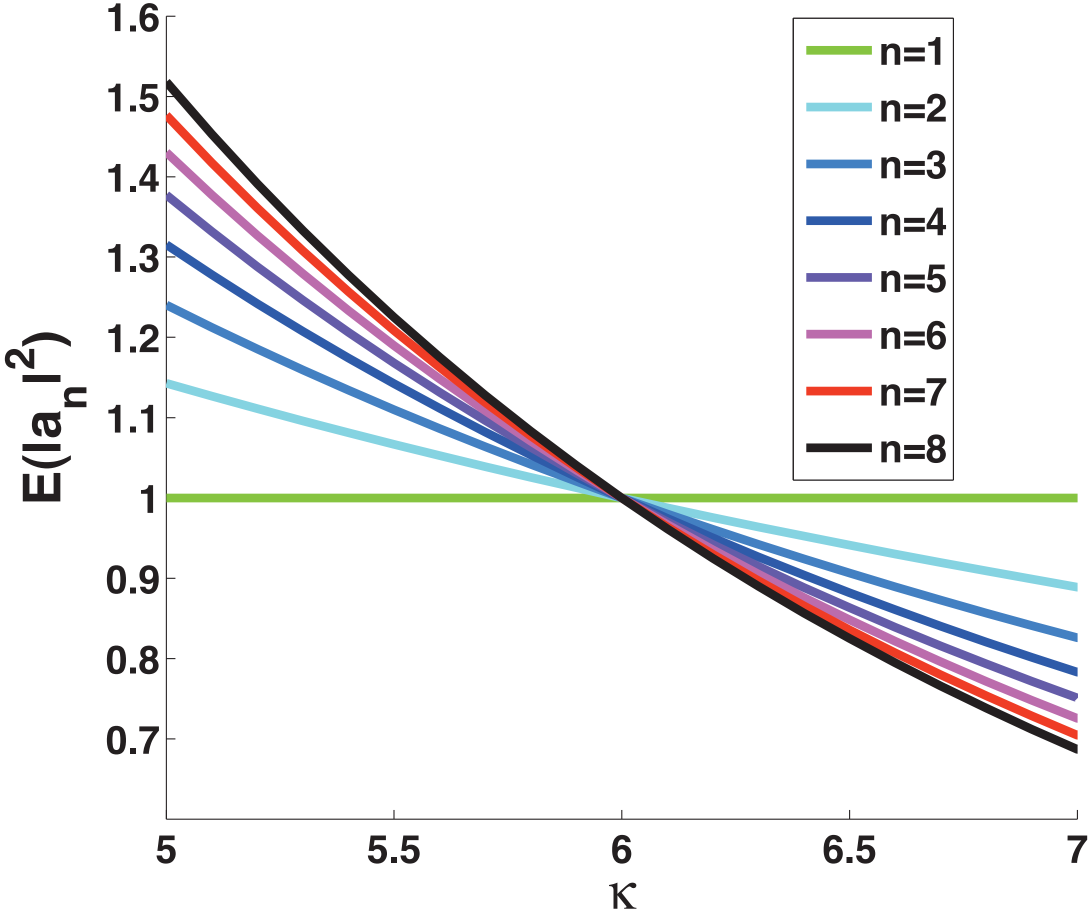
$$\mathbb{E}(b_{2n+1}) = \prod_{k=0}^{n-1} \frac{\eta_k - k - 1}{\eta_{k+1} + k + 1}, \quad n \geqslant 1.$$

Corollary 1. *If* $\eta_1 = 3$, $\mathbb{E}(f_0'(z)) = 1 - z$ *(SLE$_6$);*
if $\eta_1 = 1$ *and* $\eta_2 = 4$, $\mathbb{E}(f_0'(z)) = (1-z)^2$ *(SLE$_2$);*
if $\eta_1 = 2$, $\mathbb{E}(h_0'(z)) = 1 - z^2$ *(SLE$_4$).*

[See also Kemppainen '10 for expectations of SLE coefficient moments.]

The Surprise: Expected Square Coefficients $\mathbb{E}(|a_n|^2)$

Example: For SLE_6



Expected Square Coefficients

Example: For SLE_6

$$\mathbb{E}(|a_n|^2) = 1, \ \kappa = 6, \forall n$$

$$\mathbb{E}\left(|a_4|^2\right) = \frac{8}{9}\frac{\kappa^5 + 104\kappa^4 + 4576\kappa^3 + 18288\kappa^2 + 22896\kappa + 8640}{(\kappa+10)(3\kappa+2)(\kappa+6)(\kappa+1)(\kappa+2)^2}.$$

[Recursion: $n \leqslant 4$; Computer assisted: $n \leqslant 8$ (formal), $n \leqslant 19$ (num.)]

Theorem 2.

(i) *if* $\eta_1 = 3$, $\mathbb{E}(|a_n|^2) = 1, \forall n \geqslant 1$ *(SLE_6);*

(ii) *if* $\eta_1 = 1, \eta_2 = 4$, $\mathbb{E}(|a_n|^2) = n, n \geqslant 1$ *(SLE_2);*

(iii) *if* $\eta_1 = 2$, $\mathbb{E}(|b_{2n+1}|^2) = 1/(2n+1), n \geqslant 1$ *(SLE_4).*

Derivative Moments

Theorem 3. *The whole-plane* SLE_κ *map* $f_0(z)$ *has derivative moments*

$$\mathbb{E}\left[(f_0'(z))^{p/2}\right] = (1-z)^\alpha,$$
$$\mathbb{E}\left[|f_0'(z)|^p\right] = \frac{(1-z)^\alpha(1-\overline{z})^\alpha}{(1-z\overline{z})^\beta},$$

for the special set of exponents $p = p(\kappa) := (6+\kappa)(2+\kappa)/8\kappa$, *with* $\alpha = (6+\kappa)/2\kappa$ *and* $\beta = (6+\kappa)^2/8\kappa$. *[See also Loutsenko & Yermolayeva '12]*

Corollary 4. *$p = 2$ case: for* $\kappa = 6$:

$$\mathbb{E}(f_0'(z)) = 1-z, \ \mathbb{E}(|f_0'(z)|^2) = \frac{(1-z)(1-\overline{z})}{(1-z\overline{z})^3};$$

for $\kappa = 2$:

$$\mathbb{E}(f_0'(z)) = (1-z)^2, \ \mathbb{E}(|f_0'(z)|^2) = \frac{(1-z)^2(1-\overline{z})^2}{(1-z\overline{z})^4}.$$

The BS Equation

Beliaev and Smirnov (2005) obtained by martingale arguments the following equation for the *exterior whole-plane* case $(F(z) = F(re^{i\theta}), r \geqslant 1, \sigma = +1)$

$$p\left(\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}-\sigma\right)F+\frac{r(r^2-1)}{r^2-2r\cos\theta+1}F_r$$
$$-\quad\frac{2r\sin\theta}{r^2-2r\cos\theta+1}F_\theta+\Lambda F=0.$$

Proposition 1. *For the interior whole-plane Schramm (or Lévy)-Loewner evolution, the moments of the derivative modulus,* $F(z) := \mathbb{E}(|f_0'(z)|^p)$*, satisfy the same BS equation, but with* $\sigma = -1$*, and* $\Lambda = (\kappa/2)\partial^2/\partial\theta^2$ *the generator of the driving Brownian process (or of the Lévy process).*

Holomorphic Coordinates

Switch to $z, \bar{z}$ variables, instead of polar coordinates, and write $F(z)$ above as

$$F(z,\bar{z}) := \mathbb{E}(|f_0'(z)|^p) = \mathbb{E}[(f_0'(z))^{p/2}(\overline{f_0'}(\bar{z}))^{p/2}].$$

Using $\partial := \partial_z$, $\bar{\partial} := \partial_{\bar{z}}$, the equation then becomes

$$-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 F + \frac{z+1}{z-1} z\partial F + \frac{\bar{z}+1}{\bar{z}-1}\bar{z}\bar{\partial} F - p\left[\frac{1}{(z-1)^2} + \frac{1}{(\bar{z}-1)^2} + (\sigma - 1)\right] F = 0.$$

Exterior/Interior whole-plane: $\sigma = \pm 1$.

The action of the differential operator $\mathcal{P}(D)$ above on a function of the factorized form $F(z,\bar{z}) = \varphi(z)\bar{\varphi}(\bar{z})P(z,\bar{z})$ is, by Leibniz's rule, given by

$$\begin{aligned}
\mathcal{P}(D)[\varphi\bar{\varphi}P] = \quad & - \quad \frac{\kappa}{2}\varphi\bar{\varphi}(z\partial - \bar{z}\bar{\partial})^2 P - \kappa(z\partial - \bar{z}\bar{\partial})(\varphi\bar{\varphi})(z\partial - \bar{z}\bar{\partial})P \\
& + \quad \kappa(z\partial\varphi)(\bar{z}\bar{\partial}\bar{\varphi})P + \varphi\bar{\varphi}\frac{z+1}{z-1}z\partial P + \varphi\bar{\varphi}\frac{\bar{z}+1}{\bar{z}-1}\bar{z}\bar{\partial}P \\
& + \quad \left[-\frac{\kappa}{2}\bar{\varphi}(z\partial)^2\varphi - \frac{\kappa}{2}\varphi(\bar{z}\bar{\partial})^2\bar{\varphi} + \bar{\varphi}\frac{z+1}{z-1}z\partial\varphi + \varphi\frac{\bar{z}+1}{\bar{z}-1}\bar{z}\bar{\partial}\bar{\varphi}\right] P \\
& - \quad p\left[\frac{1}{(z-1)^2} + \frac{1}{(\bar{z}-1)^2} + \sigma - 1\right]\varphi\bar{\varphi}P.
\end{aligned}$$

- *For the the particular choice of a rotationally invariant $P(z,\bar{z}) := P(z\bar{z})$, the first line above vanishes.*
- *Study the algebra generated by the action of $\mathcal{P}(D)$ on $\varphi(z) = \varphi_\alpha(z) := (1-z)^\alpha$, and $P(z\bar{z}) := (1-z\bar{z})^{-\beta}$, $\forall \alpha, \beta$.*

Integral means spectrum

Definition 1. *The integral means spectrum of a conformal mapping f is the function defined on $\mathbb{R}$ by*

$$\beta(p) := \overline{\lim}_{r\to 1} \frac{\log\left(\int_{\partial D} |f'(rz)|^p |dz|\right)}{\log\left(\frac{1}{1-r}\right)}.$$

In the *stochastic* setting, one defines the *average* integral means spectrum

Definition 2.

$$\beta(p) := \overline{\lim}_{r\to 1} \frac{\log\left(\int_{\partial D} \mathbb{E}\, |f'(rz)|^p \,|dz|\right)}{\log\left(\frac{1}{1-r}\right)}.$$

Corollary 5. *For a Lévy-Loewner evolution with* $\eta_1 = 1, \eta_2 = 4$*, or* $\eta_1 = 3$ *(thus including SLE for* $\kappa = 2, 6$*), and for an oddified LLE with* $\eta_1 = 2$ *(thus including SLE for* $\kappa = 4$*), one has, respectively:*

$$\mathbb{E}\left(\frac{1}{2\pi}\int_0^{2\pi}|f'(re^{i\theta})|^2 d\theta\right) = \frac{1+4r^2+r^4}{(1-r^2)^4}; \frac{1+r^2}{(1-r^2)^3}; \frac{1+r^4}{(1-r^4)^2}.$$

This gives the values of the average integral means spectrum $\beta(2) = 4, 3$ for whole-plane LLE with $\eta_1 = 1, \eta_2 = 4$ or $\eta_1 = 3$ (thus whole-plane SLE with $\kappa = 2, 6$) respectively. For the oddified LLE with $\eta_1 = 2$ (thus the oddified whole-plane SLE_4), $\beta_2(2) = 2$.

• *They differ from the corresponding values at* $p = 2$ *of the SLE integral mean spectrum of Beliaev and Smirnov '05.*

Define

$$\beta_0(p,\kappa) := -p + \frac{4+\kappa}{4\kappa}\left(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\right),$$

$$\hat{\beta}_0(p,\kappa) := p - \frac{(4+\kappa)^2}{16\kappa}.$$

This is the average integral means spectrum $\bar{\beta}_0(p,\kappa)$ of the bulk of SLE$_\kappa$, as obtained in Beliaev & Smirnov '05:

$$\begin{aligned}\bar{\beta}_0(p,\kappa) &= \beta_0(p,\kappa),\ 0 \leqslant p \leqslant p_0^*(\kappa),\\ &= \hat{\beta}_0(p,\kappa),\ p \geqslant p_0^*(\kappa),\\ p_0^*(\kappa) &:= \frac{3(4+\kappa)^2}{32\kappa}.\end{aligned}$$

Integral means spectra

The whole-plane SLE$_\kappa$, $f_{t=0}(z), z \in \mathbb{D}$, and its m-fold transforms, $h_0^{(m)}(z) := z\left[f_0(z^m)/z^m\right]^{1/m}$, $m \geqslant 1$, have average integral means spectra $\beta_m(p,\kappa)$ that exhibit a *phase transition* and are given, for $p \geqslant 0$, by

$$\beta_1(p,\kappa) = \max\left\{\beta_0(p,\kappa), 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p}\right\},$$

$$\beta_2(p,\kappa) = \max\left\{\beta_0(p,\kappa), 2p - \frac{1}{2} - \frac{1}{2}\sqrt{1+\kappa p}\right\},$$

$$\beta_m(p,\kappa) = \max\left\{\overline{\beta}_0(p,\kappa), (1+2/m)\,p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p/m}\right\}.$$

The first spectrum β_1 has its transition point at

$$p^*(\kappa) := \frac{1}{16\kappa}\left((4+\kappa)^2 - 4 - 2\sqrt{4+2(4+\kappa)^2}\right) < p_0^*(\kappa).$$

Theorem 5. *The average integral means spectrum $\beta(p,\kappa)$ of the unbounded whole-plane SLE$_\kappa$ has a phase transition at $p^*(\kappa)$ and a special point at $p(\kappa) := (6+\kappa)(2+\kappa)/8\kappa$, such that*

$$\begin{aligned}
\beta(p,\kappa) &= \beta_0(p,\kappa),\ 0 \leqslant p \leqslant p^*(\kappa);\\
\beta(p,\kappa) &= 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p} > \beta_0(p,\kappa),\ p^*(\kappa) \leqslant p \leqslant \min\{1+\kappa/2, p(\kappa)\};\\
\beta(p,\kappa) &\geqslant 3p - 1/2 - (1/2)\sqrt{1+2\kappa p},\ \min\{1+\kappa/2, p(\kappa)\} \leqslant p \leqslant p(\kappa);\\
\beta(p(\kappa),\kappa) &= (6+\kappa)^2/8\kappa;\\
\beta(p,\kappa) &\leqslant 3p - 1/2 - (1/2)\sqrt{1+2\kappa p},\ p(\kappa) < p.
\end{aligned}$$

- *For $p > p^*(\kappa)$ the BS solution ceases to be uniformly positive.*
- *Existence of a subsolution/supersolution for the parabolic operator $\mathcal{P}(D)[\psi(z,\bar z)\ell_\delta(z\bar z)] \lesseqqgtr 0$ in some annulus of $\mathbb{D}$ whose boundary includes $\partial\mathbb{D}$, corresponding respectively to $p \lesseqqgtr p^*(\kappa)$. Trial functions: $\psi(z,\bar z) := (1-z\bar z)^{-\beta}|1-z|^{2\alpha}$, $\ell_\delta(z\bar z) := [-\log(1-z\bar z)]^\delta$.*

Integral means spectrum: *Inner whole-plane SLE*

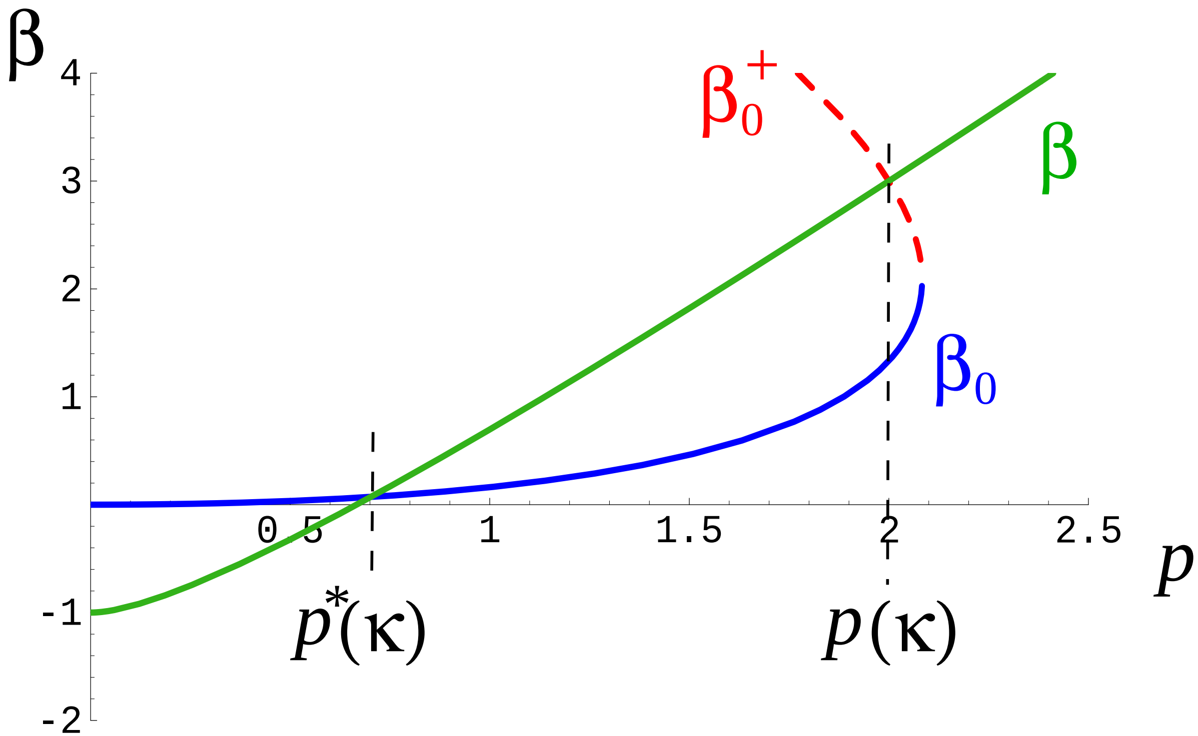


Figure 1: $\beta(p) = 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p}$

Integral means spectrum: *Outer whole-plane SLE*

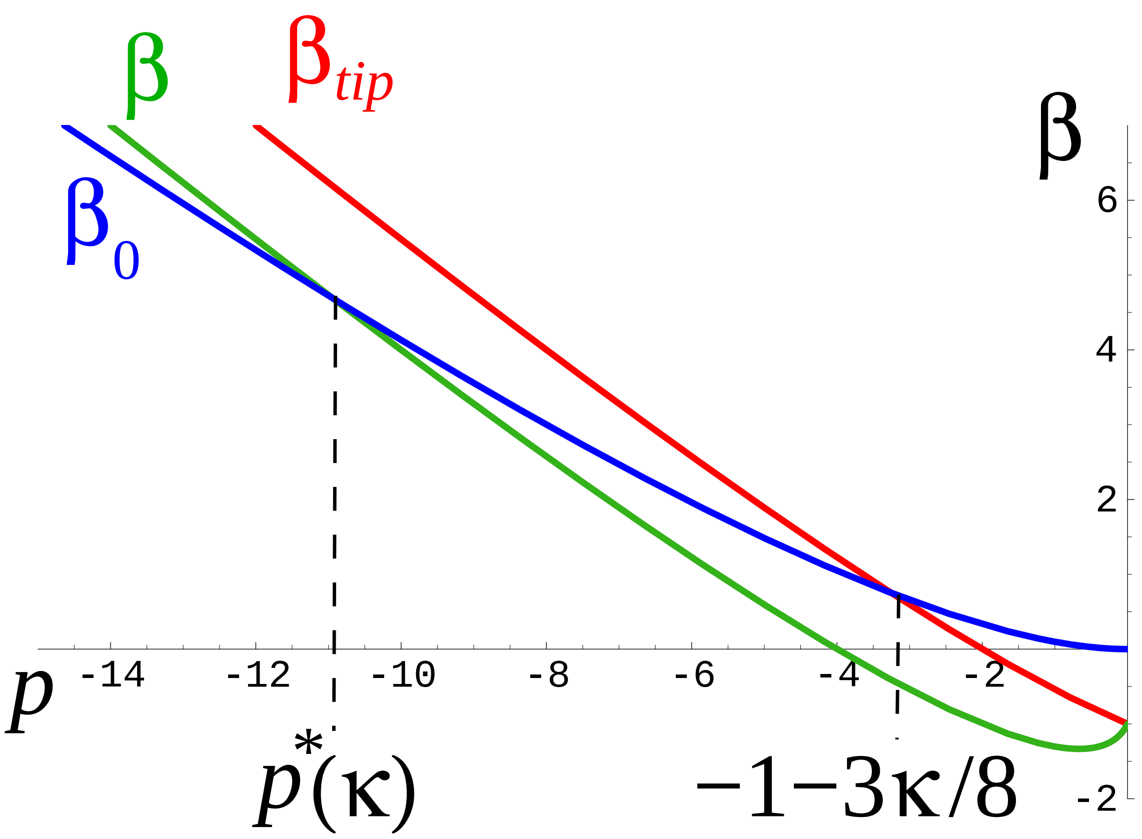


Figure 2: $\beta(p) = -p - \frac{1}{2} - \frac{1}{2}\sqrt{1-2\kappa p}$, $p^*(\kappa) = (4+\kappa)^2(8+\kappa)/128$ *(Beliaev, B.D., Zinsmeister '13)*

Packing Spectrum

The *packing spectrum [Makarov]* is defined as

$$s(p) := \beta(p) - p + 1.$$

For the unbounded whole-plane SLE$_\kappa$, we have for $p \geqslant p^*(\kappa)$

$$\begin{aligned} s(p,\kappa) &= \beta(p,\kappa) - p + 1 \\ &= 2p + \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p}. \end{aligned}$$

Consider its *inverse function*

$$\begin{aligned} p = p(s,\kappa) &:= \frac{s}{2} + \frac{\kappa}{8}\mathcal{U}_\kappa^{-1}(s), \\ \mathcal{U}_\kappa^{-1}(s) &:= \frac{1}{2\kappa}\left(\kappa - 4 + \sqrt{(4-\kappa)^2 + 16\kappa s}\right) \end{aligned}$$

(KPZ formula)

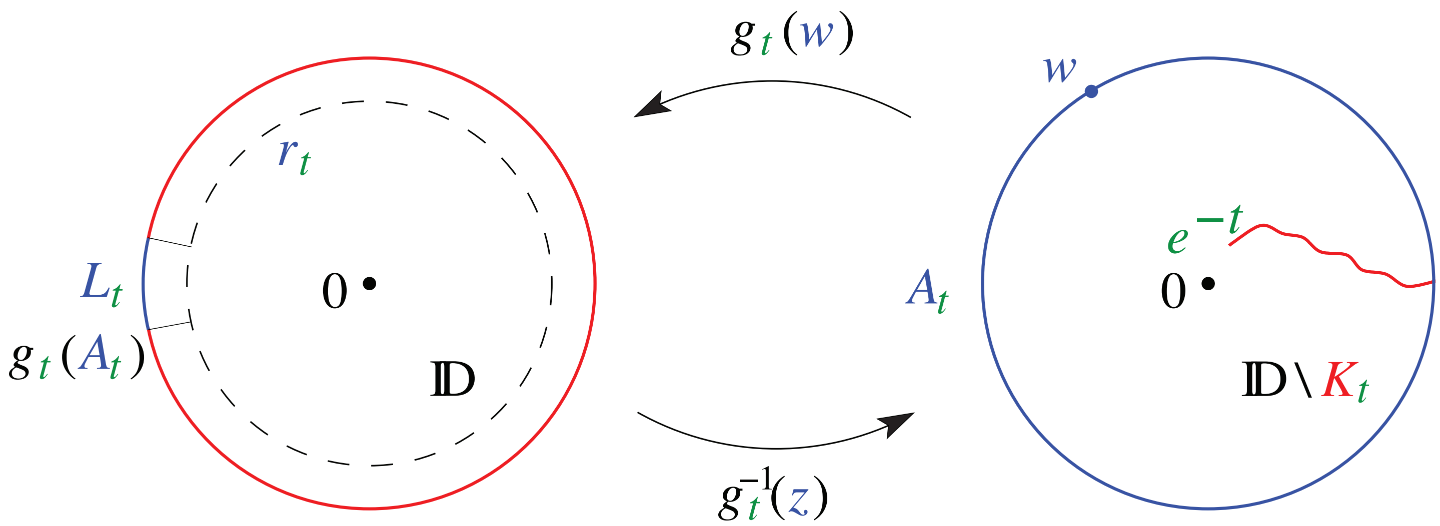
Relation to Tip & Derivative Exponents

(Non-standard) tip multifractal exponents obtained by *quantum gravity [D. '00]*, corresponding geometrically to the extremity of an SLE_κ path avoiding a packet of s independent Brownian motions.

Differ from the ones associated to the *standard SLE tip multifractal spectrum [Hastings '02, Beliaev & Smirnov '05, Johansson & Lawler '09]*.

Identical to the *derivative exponents* obtained for radial SLE_κ *[Lawler, Schramm & Werner '01]*.

(Inverse) Radial SLE Map



$$f_0(z) \overset{\text{(law)}}{=} \lim_{t\to+\infty}[e^t g_t^{-1}(z) =: \tilde{f}_t(z)].$$

Harmonic measure

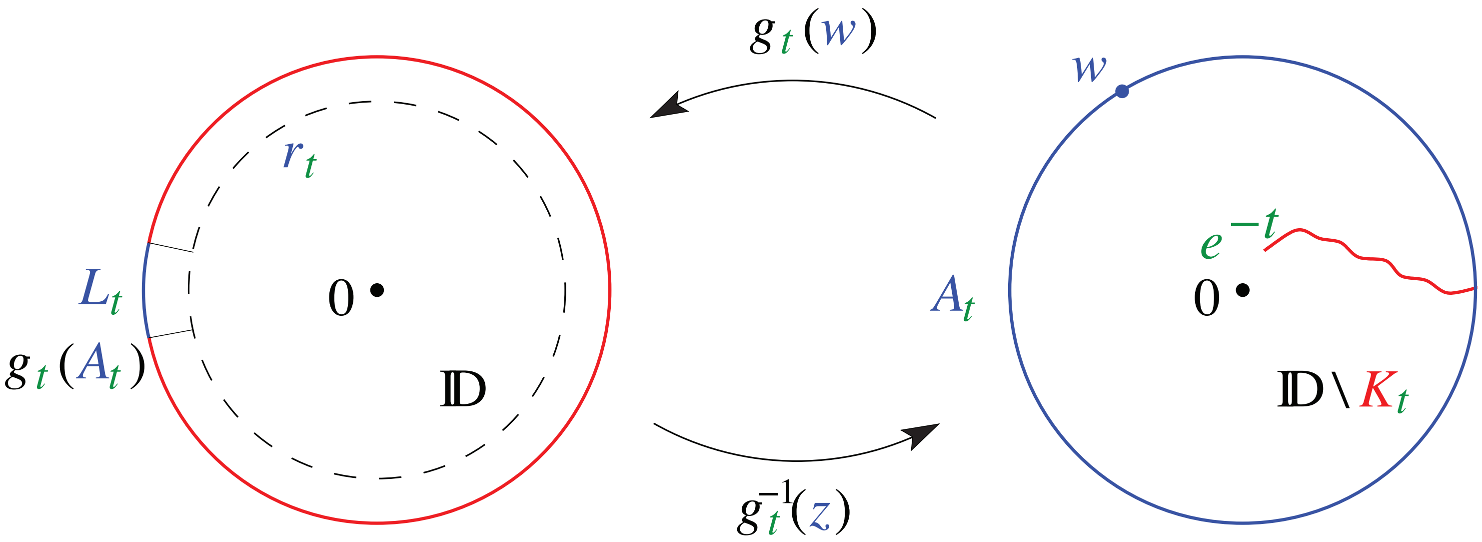


Figure 3: $f_0(z) \overset{(\text{law})}{=} \lim_{t \to +\infty} e^t g_t^{-1}(z)$, *where* $z \mapsto g_t^{-1}(z)$ *maps* $\mathbb{D}$ *to the slit domain* $\mathbb{D} \setminus K_t$ *(*K_t *SLE hull). The length* $L_t := |g_t(A_t)|$ *of the image of the boundary set* $A_t := \partial\mathbb{D} \setminus \overline{K_t}$ *is the* $(2\pi)\times$ *the harmonic measure of* A_t *as seen from* 0 *in* $\mathbb{D} \setminus K_t$*, with* $\mathbb{E}[L_t^s] \asymp e^{-p(s,\kappa)t}$ *for* $t \to +\infty$ *[LSW '01].*

Derivative exponents

Lemma 1. *(Lawler, Schramm, Werner '01) Let*

$$A_t := \partial\mathbb{D} \setminus \overline{K_t},$$

which is either an arc on $\partial\mathbb{D}$ or $A_t = \emptyset$. Let $s \geqslant 0$, and set

$$p = p(s,\kappa) := \frac{s}{2} + \frac{1}{16}\left(\kappa - 4 + \sqrt{(4-\kappa)^2 + 16\kappa s}\right).$$

Let $\mathcal{H}(\theta,t)$ denote the event $\{w = exp(i\theta) \in A_t\}$, and set

$$\begin{aligned} \mathcal{F}(\theta,t) &:= \mathbb{E}\left[\left|g_t'(\exp(i\theta))\right|^s 1_{\mathcal{H}(\theta,t)}\right], \\ q = q(s,\kappa) &:= \mathcal{U}_\kappa^{-1}(s) = \frac{\kappa - 4 + \sqrt{(4-\kappa)^2 + 16\kappa s}}{2\kappa}, \\ \mathcal{F}(\theta,t) &\asymp \exp(-pt)\big(\sin(\theta/2)\big)^q, \ \forall t \geqslant 1, \ \forall \theta \in (0,2\pi). \end{aligned}$$

Packing spectrum & derivative exponents

The average integral means spectrum involves evaluating, for the whole-plane SLE map $f_0(z)$, the integral

$$\mathbb{I}_p(r) := \int_{\partial D} \mathbb{E}\left[|f_0'(rz)|^p\right] |dz|,$$

on a circle of radius $r < 1$ concentric to $\partial\mathbb{D}$, and looking for the smallest $\beta(p)$ such that

$$(1-r)^{\beta(p)}\,\mathbb{I}_p(r) \overset{r\to 1}{<} +\infty.$$

For $p \geqslant p^*(\kappa)$, the integrand behaves like a distribution and the circle integral concentrates in the vicinity of the pre-image point of infinity by the whole-plane map, $z_0 := f_0^{-1}(\infty) \in \partial\mathbb{D}$. In the large-$t$ approximation to f_0, that is the neighborhood of $g_t(A_t)$.

Condensation

The circle integral there is the *restricted* integral in the image w-unit circle

$$I_p(t) \quad := \quad \int_{A_t} e^{pt} |g_t'(w)|^s \, |dw|; \; s = s(p) = \beta(p) + 1 - p,$$

From LSW's Lemma above

$$\mathbb{E}\left[I_p(t)\right] \asymp \int_0^{2\pi} \sin^q(\theta/2) d\theta < +\infty.$$

By defining the *stochastic radius* $r_t := 1 - L_t \to 0$, this can be recast as

$$\mathbb{E}\left[(1-r_t)^{\beta(p)} \int_{\partial\mathbb{D}} |\tilde{f}_t'(r_t z)|^p |dz|\right] \asymp 1, \; t \to +\infty,$$

where $f_0(z) \overset{\text{(law)}}{=} \lim_{t\to+\infty}[\tilde{f}_t(z) := e^t g_t^{-1}(z)]$. This is *(formally)* reminiscent of the definition of the average integral means spectrum, hinting at why *the derivative exponent* $p = p(s, \kappa)$ *is the inverse function of the unbounded whole-plane packing spectrum* $s(p, \kappa)$. □