Self-Similarity beyond Gaussian processes: Hermite processes and more

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Outline

- Review: SSSI processes
- Review: Fractional Brownian motion (FBM)
- Review: Hermite processes
- Evaluating the Rosenblatt distribution
- Generalized Hermite processes
- Multivariate limit theorems

Definition of self-similarity

Definition. A stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ is called *self-similar* (SS) or H-*self-similar* (H-SS) if there is $H > 0$ such that, for all $c > 0$,

$$\{X(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H X(t)\}_{t\in\mathbb{R}},$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. Thus, for any $c > 0$ and times $t_1, \ldots, t_m$ in $\mathbb{R}$ and integer $m \geq 1$,

$$(X(ct_1), \ldots, X(ct_m))$$

has the same finite-dimensional distributions as the vector

$$(c^H X(t_1), \ldots, c^H X(t_m)).$$

The parameter H is called the self-similarity parameter.

- Example: Brownian motion $B(t)$ is H-SS with $H = 1/2$.

Why self-similarity?

- Random fractals
- Appears naturally in limit theorems (Lamperti)
- Modelling

BUT: There are too many self-similar processes.

Theorem (Lamperti)

(i) If $\{X(t)\}_{t\geq 0}$ is H-SS, then $Y(t) = e^{-tH}X(e^t)$, $t \in \mathbb{R}$, is stationary.
(ii) Conversely, if $\{Y(t)\}_{t\in\mathbb{R}}$ is stationary, then $X(t) = t^H Y(\log t)$, $t > 0$, is H-SS.

• Example: $X(t)$ BM $\iff$ $Y(t)$ Ornstein-Uhlenbeck

• Hence there are as many SS processes as stationary processes.

• And this connection to stationary processes has not proved useful.

Let's focus on a convenient subclass

Which one? Stationary SS processes is natural but will not do.
Focus then on processes with *stationary increments (SI)*:
Definition. A process $\{X(t)\}_{t\in\mathbb{R}}$ has *stationary increments* if, for any $s \in \mathbb{R}$,

$$\{X(t) - X(s)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t-s) - X(0)\}_{t\in\mathbb{R}}.$$

• Convenient, because if $X(n)$ is a stationary sequence, then the limit of

$$\frac{1}{N^H}\sum_{n=1}^{[Nt]} X(n),$$

if it exists, will be an $H-SSSI$ process.
• Historically, this is how many $H-SSSI$ processes were obtained.
• The increments of a $SSSI$ processs are useful in modelling.

Simplest case: Gaussian $H - SSSI$ processes

Theorem

Suppose that $\{X(t)\}_{t\in\mathbb{R}}$ satisfies the following conditions:

- (i) *it is a Gaussian process with mean* 0, $X(0) = 0$,
- (ii) *it has stationary increments*
- (iii) $\mathbb{E}|X(t)|^2 = \sigma^2|t|^{2H}$ *for all* $t \in \mathbb{R}$ *and some* $\sigma > 0$ *and* $0 < H \leq 1$,

Then the covariance is determined:

$$\mathrm{Cov}(Z(s), Z(t)) = \Gamma_H(s,t) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

- *The process* $\{X(t)\}_{t\in\mathbb{R}}$ *is called* **fractional Brownian motion** *or FBM*
- FBM is denoted $B_H(t)$.

Perspectives

Two perspectives on characterizing processes:

- through their distribution;
- through their representation: an expression involving simpler processes.

Integral representation

$$\underbrace{X(t)}_{stoch.\ process} = (\text{or } \stackrel{d}{=}) \underbrace{\int_E}_{some\ space} \underbrace{h_t(u)}_{deterministic} \underbrace{M(du).}_{random\ measure}$$

We will consider first a simple integral with respect to the Wiener measure (or Brownian motion):

$$X(t) = (\text{or } \stackrel{d}{=}) \int_{\mathbb{R}} h_t(u) W(du).$$

where $W(du)$ is a Gaussian measure with Lebesgue control measure du, that is, heuristically, $\mathbb{E}((W(du)^2) = du$.

- It is well-defined if

$$\int_{\mathbb{R}} h_t(u)^2 du < \infty, \quad \text{and then} \quad \mathbb{E}((X(t)^2) = \int_{\mathbb{R}} h_t(u)^2 du.$$

Representation of FBM

Theorem

For $H \in (0,1)$, standard FBM admits the following time-domain integral representation:

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{c\int_{\mathbb{R}} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right)W(du)\right\}_{t\in\mathbb{R}},$$

where $x_+ = \max\{x,0\}$, c is a normalizing constant and $W(du)$ is a Gaussian measure with Lebesgue control measure du.

- "Time domain" because u is interpreted as time.
- We get back Brownian motion if $H = 1/2$.

Many representations are possible

The time domain representation is not unique. For example, one can have for any $a, b \in \mathbb{R}$,

$$\int_{\mathbb{R}} \Big(a\big((t-u)_+^{H-1/2}-(-u)_+^{H-1/2}\big)+b\big((t-u)_-^{H-1/2}-(-u)_-^{H-1/2}\big)\Big)W(du),$$

where $x_- = \max\{0, -x\}$.
• If $a \neq 0, b = 0$ as before, the representation is called *causal* (non-anticipative) because integration over $u \leq t$ is used for $t > 0$.
• If $a = b$, then it is called *well-balanced*. In this case, $x_+ + x_- = |x|$, and thus

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \Big\{a \int_R \big(|t-u|^{H-1/2} - |u|^{H-1/2}\big) W(du)\Big\}_{t\in\mathbb{R}}.$$

Multiple integrals with respect to Gaussian measures

If W is a *real-valued* Gaussian measure on $\mathbb{R}^k$ with Lebesgue control measure, these integrals are written as

$$\int_{\mathbb{R}^k}' h_t(u_1, \ldots, u_k) W(du_1) \ldots W(du_k) =: I_k(h_t),$$

where for each $t \in \mathbb{R}$, $h_t : \mathbb{R}^k \mapsto \mathbb{R}$ is a deterministic.

- The prime in $\int_{\mathbb{R}^k}'$ refers to the fact that integration excludes diagonals.
- It is well-defined if $h_t \in Ł^2(\mathbb{R}^k)$.
- One has $\mathbb{E} I_k(h_t) = 0$ and

$$\mathbb{E} I_k(h_t)^2 = \int_{\mathbb{R}^k} (h_t(u_1, \ldots, u_k))^2 du_1, \ldots, du_k < \infty.$$

Definition of the Hermite processes

Definition. Let $k \geq 1$ be an integer and

$$H \in \left(\frac{1}{2}, 1\right) \qquad \text{and} \qquad H_0 = 1 - \frac{1-H}{k} \in \left(1 - \frac{1}{2k}, 1\right),$$

so that $H = 1 - k(1 - H_0)$. The *Hermite process* $\{Z_H^{(k)}(t)\}_{t \in \mathbb{R}}$ *of order* k is defined as

$$\begin{aligned} Z_H^{(k)}(t) &= \int_{\mathbb{R}^k}' \left\{ \int_0^t \prod_{j=1}^k (s - u_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} ds \right\} W(du_1) \ldots W(du_k) \\ &= \int_{\mathbb{R}^k}' \left\{ \int_0^t \prod_{j=1}^k (s - u_j)_+^{H_0 - \frac{3}{2}} ds \right\} W(du_1) \ldots W(du_k), \end{aligned}$$

where $W(du)$ is a Gaussian measure with Lebesgue control measure du.

Remarks

- The Hermite process of order k is defined through the kernel function

$$h_t(u_1, \dots, u_k) = \int_0^t \prod_{j=1}^{k} (s - u_j)_+^{H_0 - 3/2} ds.$$

- The parameter H_0 is the one used in the fractional Brownian motion kernel.
- The parameter H is the self-similarity parameter of the Hermite process.
- The Hermite process of order $k = 1$ is fractional Brownian motion and in this case $H = H_0$.
- The Hermite process of order $k = 2$ is called the Rosenblatt process.

Heuristic structure of FBM and the Hermite processes

FBM:

$$
\begin{aligned}
B_H(t) &= \int_{\mathbb{R}^1} \left(\int_0^t (s-u)_+^{H_0 - \frac{3}{2}} ds \right) W(du) \\
&= \int_0^t \left(\int_{\mathbb{R}^1} (s-u)_+^{H_0 - \frac{3}{2}} W(du) \right) ds \\
&= \int_0^t \dot{B}_{H_0}(s) ds.
\end{aligned}
$$

Hermite process of order k:

$$
\begin{aligned}
Z_H^{(k)}(t) &= \int_0^t H_k \left(\int_{\mathbb{R}^1} (s-u)_+^{H_0 - \frac{3}{2}} W(du) \right) ds \\
&= \int_0^t H_k(\dot{B}_{H_0}(s)) ds.
\end{aligned}
$$

The Rosenblatt distribution

The Rosenblatt process $(k = 2)$

Three ways to view it:

$$
\begin{aligned}
Z_H^{(2)}(t) &= \int_{\mathbb{R}^2}' \left\{ \int_0^t \prod_{j=1}^{2} (s - u_j)_+^{-(1-H/2)} ds \right\} W(du_1) W(du_2) \\
&= \int_{\mathbb{R}^2}' \left\{ \int_0^t \prod_{j=1}^{2} (s - u_j)_+^{H_0 - \frac{3}{2}} ds \right\} W(du_1) W(du_2) \\
&= \int_0^t H_2(\dot{B}_{H_0}(s)) ds.
\end{aligned}
$$

Focus on the first representation, where $H \in (1/2, 1)$ is the self-similarity parameter of the process.

Why focus on the Rosenblatt process?

Because it is the most common limit after Brownian motion and fractional Brownian motion.

- One has its representation.
- One would like to have at least its marginal distribution.

The Rosenblatt distribution

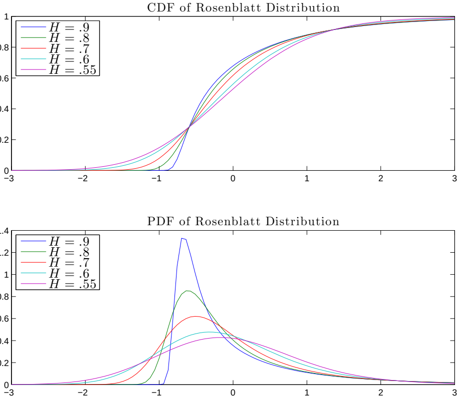
It is the marginal distribution of the Rosenblatt process $Z_H^{(2)}(t)$ at time $t = 1$, i.e. the distribution of the random variable

$$Z_H = Z_H^{(2)}(1),$$

standardized: $\mathbb{E}Z_H^{(2)}(1) = 0, \ \ \mathbb{E}(Z_H^{(2)}(1))^2 = 1.$

- Important because it appears often in limit theorems.
- Necessary for getting asymptotic confidence intervals.
- Unfortunately, the distribution is not known in closed form.
- In Veillette and Taqqu (2013), we developed a technique to evaluate it numerically.

Plots of the CDF and PDF of the Rosenblatt distribution



These are tabulated as well!

A mystery !!!

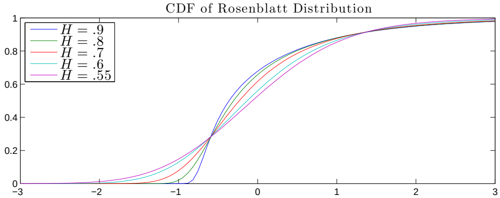
It seems that the CDFs of the Rosenblatt distributions intersect at the same point **for any** $H \in (1/2, 1)$. In fact, this is true even for extremes:
For $H = 1$: $Z_1 = \text{standardized chi squared}$
For $H = 1/2$: $Z_{1/2} = N(0,1)$:

$$P(Z_1 \le -0.6256) = P(Z_{1/2} \le -0.6256) = 0.2658.$$

Conjecture:

$$\forall \ \ 1/2 < H < 1 : \quad P(Z_H \le -0.6256) = 0.2658.$$

There seems to be a second common point: $P(Z_H \le 1.3552) = 0.9123$.



$$\begin{aligned} P(Z_1 \leq x) &= P(\chi^2 \leq \sqrt{2}\, x + 1) \\ P(Z_{1/2} \leq x) &= P(N(0,1) \leq x) \end{aligned}$$

$$\begin{aligned} P(Z_1 \leq -0.625589380332768) &= 0.265792153057490 \\ P(Z_{1/2} \leq -0.625589380332768) &= 0.265792153057490 \end{aligned}$$

$$\begin{aligned} P(Z_1 \leq 1.355216558592255) &= 0.912325714726415 \\ P(Z_{1/2} \leq 1.355216558592255) &= 0.912325714726415 \end{aligned}$$

Conjecture: This holds also for Z_H, $\forall \ 1/2 < H < 1$.

Beyond the Hermite processes:

♦

Generalized Hermite processes

Generalized Hermite kernel

In the definition of the Hermite process

$$Z_H^{(k)}(t) = a_{k,H_0} \int'_{\mathbb{R}^k} \left\{ \int_0^t \prod_{j=1}^{k} (s-u_j)_+^{H_0-\frac{3}{2}} ds \right\} W(du_1)\dots W(du_k),$$

replace the product of functions in the bracket, by

$$g(s-u_1,\cdots,s-u_k)1\{s\mathbf{1} > \mathbf{u}\}$$

where g is a multivariate homogeneous function.

• This idea goes back to Mori and Oodaira (1986) who use it to study the law of iterated logarithm.

Generalized Hermite kernels and processes

Definition. We say that a nonzero measurable function $g(\mathbf{x})$ defined on $\mathbb{R}_+^k$ is a *generalized Hermite kernel*, if it satisfies

A. $g(\lambda\mathbf{x}) = \lambda^\alpha g(\mathbf{x})$, $\forall\lambda > 0$, where $\alpha \in (-\frac{k+1}{2}, -\frac{k}{2})$;

B. $\int_{\mathbb{R}_+^k} |g(\mathbf{x})g(\mathbf{1}+\mathbf{x})|d\mathbf{x} < \infty$.

Definition. The process

$$Z(t) := \int_{\mathbb{R}^k}' \int_0^t g(s-x_1, \ldots, s-x_k) 1_{\{s>x_1,\ldots,s>x_k\}} ds\, W(dx_1)\ldots W(dx_k)$$

where g is a generalized Hermite kernel, is called a *generalized Hermite process*.

- Shorthand: $Z(t) = I_k(h_t)$ with $h_t(\mathbf{x}) = \int_0^t g(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1}>\mathbf{x}\}} ds$.

The generalized Hermite process is well defined

Let $g(\mathbf{x})$ be a generalized Hermite kernel. Then

$$h_t(\mathbf{x}) = \int_0^t g(s - x_1, \ldots, s - x_k) 1_{\{s>x_1,\ldots,s>x_k\}} ds$$

is well-defined in $L^2(\mathbb{R}^k)$, $\forall t > 0$, and the process

$$Z_t := I_k(h_t)$$

is an H-sssi process with

$$H = \alpha + k/2 + 1 \in (1/2, 1).$$

Example

Let

$$g(x_1,\ldots,x_k) = \max\left(\frac{x_1\ldots x_k}{x_1^{k-\alpha}+\ldots+x_k^{k-\alpha}},\ x_1^{\alpha/k}\ldots x_k^{\alpha/k}\right)$$

where $\mathbf{x} = (x_1,\ldots,x_k) \in \mathbb{R}_+^k$ and $\alpha \in (-k/2-1/2, -k/2)$.

- Note that g is homogeneous of degree α.

- The corresponding $H-SSSI$ generalized Hermite process with $H = \alpha + k/2 + 1$ is

$$\int_{\mathbb{R}^k}' \int_0^t \left(\frac{\prod_{j=1}^k (s-x_j)_+}{\sum_{j=1}^k (s-x_j)_+^{k-\alpha}}\right) \vee \left(\prod_{j=1}^k (s-x_j)_+^{\alpha/k}\right) ds \prod_{j=1}^k W(dx_j).$$

How to obtain an H-sssi process with $0 < H < 1/2$?

Theorem

The process $Z^\beta(t) = \int'_{\mathbb{R}^k} \int_{\mathbb{R}}$

$$\frac{1}{\beta}\left[(t-s)_+^\beta - (-s)_+^\beta\right] g(s-x_1,\dots,s-x_k) 1_{\{s>x_1,\dots,s>x_k\}} ds \prod_{j=1}^k W(dx_j),$$

is an H*-sssi process with* $H = \alpha + \beta + k/2 + 1 \in (0,1)$.

• g is homogeneous with exponent $\alpha \in (-k/2 - 1/2, -k/2)$ and

$$-1 < -\alpha - \frac{k}{2} - 1 < \beta < -\alpha - \frac{k}{2} < \frac{1}{2}.$$

These condition ensure that the integrand is in $L^2(\mathbb{R}^k)$.

• The case $\beta = 0$ leads back to the generalized Hermite process case.

Limit Theorems

Polynomial forms

Consider the polynomial form of order k:

$$X(n) = \sum_{0<i_1,\ldots,i_k<\infty}' a(i_1,\ldots,i_k)\epsilon_{n-i_1}\ldots\epsilon_{n-i_k},$$

where the prime $'$ indicates that one doesn't sum on the diagonals $i_p = i_q$ $p \neq q$ and where the $\sum'_{\mathbf{i}>\mathbf{0}} a(\mathbf{i})^2 < \infty$, so that $X(n)$ is well-defined in the $L^2(\Omega)$-sense. For simplicity, take

$$a(i_1,\ldots,i_k) = g(i_1,\ldots,i_k),$$

where g is a generalized Hermite kernel with some regularity conditions.

- Say that $X(n)$ is short-range dependent (SRD) if the sum of the covariances of $X(n)$ converges.
- Say that $X(n)$ is long-range dependent (LRD) if the sum of the covariances of $X(n)$ diverges.

Limit theorems

Consider then the polynomial form of order k:

$$X(n) = \sum_{0<i_1,\ldots,i_k<\infty}' g(i_1, \ldots, i_k)\epsilon_{n-i_1} \ldots \epsilon_{n-i_k}.$$

We are interested in the limit of

$$\frac{1}{N^H} \sum_{n=1}^{[Nt]} X(n)$$

for special choices of the kernel $a(i_1, \ldots, i_k)$. The limit can be:

- BM if $k = 1$ (linear process), SRD, $H = 1/2$
- BM if $k \geq 2$, (linear or non-linear process), SRD, $H = 1/2$
- $Z(t)$ if $k \geq 1$ (generalized Hermite), LRD, $H \in (1/2, 1)$
- $Z^\beta(t)$ if $k \geq 1$ (fractionally filtered generalized Hermite), SRD or LRD, $H \in (0, 1)$.

Consider a **vector** of such processes

The components are expressed as

$$\frac{1}{N^H}\sum_{n=1}^{[Nt]} X(n) = \frac{1}{N^H}\sum_{n=1}^{[Nt]} \sum_{0<i_1,\ldots,i_k<\infty}{}^{'} g(i_1,\ldots,i_k)\epsilon_{n-i_1}\ldots\epsilon_{n-i_k}.$$

The components have

- the same iid noise $\{\epsilon_k, k \in \mathbb{Z}\}$
- different kernels $g(i_1,\ldots,i_k)$
- different order k
- different exponents H
- SRD, LRD, fLRD

Question: Is there joint convergence? **Answer:** Yes

Multivariate limit theorem

Theorem

The joint convergence holds with limit vector

$$\Big(\mathbf{B}_1(t), \mathbf{B}_2(t), \mathbf{Z}(t), \mathbf{Z}^{\boldsymbol{\beta}}(t)\Big),$$

where

(i) $\mathbf{B}_1(t) = \mathbf{W}(t) := (\sigma_1 W(t), \dots, \sigma_{J_{S_1}} W(t))$ *defined by the same standard Brownian motion* $W(t)$. [$k = 1$, *SRD*]

(ii) $\mathbf{B}_2(t)$ *is a multivariate Brownian motion with joint covariance* 0 *if the orders* $k_p \neq k_q$. [$k \geq 2$, *SRD*]

(iii) $\mathbf{Z}(t)$ *is a multivariate generalized Hermite process and using the* $W(t)$ *in Point* (i) *as Brownian motion integrator.* [$k \geq 1$, *LRD*]

(iv) $\mathbf{Z}^{\beta}(t)$ *is a multivariate fractionally-filtered generalized Hermite process and using the* $W(t)$ *in Point* (i) *as Brownian motion integrator.* [$k \geq 1$, *SRD or LRD depending on* β]

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Thanks!

Idea of proof

For $\mathbf{B}_1(t), \mathbf{B}_2(t)$ use m-dependence and add

$$W_N(\Delta) := \frac{1}{\sqrt{N}} \sum_{n/N \in \Delta} \epsilon_n$$

as one of the components in Cramer-Wold.
To include the LRD components, view them as functionals of W_N with kernels converging in $Ł^2$ (Ref: Bai and Taqqu (2013b)).

Independence / dependence between the components

The limit vector is:

$$\Big(\mathbf{B}_1(t), \mathbf{B}_2(t), \mathbf{Z}(t), \mathbf{Z}^{\boldsymbol{\beta}}(t)\Big).$$

$\mathbf{B}_2(t)$ is always independent of $(\mathbf{B}_1(t), \mathbf{Z}(t), \mathbf{Z}^{\boldsymbol{\beta}}(t))$. Indeed:

- $\mathbf{B}_2 \perp\!\!\!\perp \mathbf{B}_1$ [$k \geq 2$ versus $k = 1$]
- The processes $\mathbf{B}_2(t), \mathbf{Z}(t)$ and $\mathbf{Z}^{\boldsymbol{\beta}}(t)$ involve the same integrator $W(\cdot)$ because they are defined in terms of the same ϵ_i's.
- $\mathbf{B}_2 \perp\!\!\!\perp W$ [$k \geq 2$ versus $k = 1$]
- $(\mathbf{Z}, \mathbf{Z}^{\boldsymbol{\beta}})$ involve only $W \implies \mathbf{B}_2 \perp\!\!\!\perp (\mathbf{Z}, \mathbf{Z}^{\boldsymbol{\beta}})$.

Spectral domain representation of the Hermite processes

$$Z_H^{(k)}(t) = b_{k,d} \int_{\mathbb{R}^k}'' \frac{e^{i(\nu_1+\ldots+\nu_k)t} - 1}{i(\nu_1 + \ldots + \nu_k)} |\nu_1|^{-d} \ldots |\nu_k|^{-d} \widehat{W}(d\nu_1) \ldots \widehat{W}(d\nu_k),$$

where

$$d = \frac{1}{2} - \frac{1-H}{k} \in \left(\frac{1}{2}(1 - \frac{1}{k}), \frac{1}{2} \right),$$

$\widehat{W}(\cdot)$ is a complex-valued Brownian motion (with real and imaginary parts being independent), and where the double prime on $\int_{\mathbb{R}^k}''$ indicates that the diagonals $\{x_i = \pm x_j, \ i \neq j\}$ are excluded in the integration.

- It is useful in Statistics
- It sheds light on the structure.

The spectral perspective

The L^2 Fourier transform of the integrand

$$h_t^\beta(\mathbf{x}) = \frac{1}{\beta}\left[(t-s)_+^\beta - (-s)_+^\beta\right] g(s-x_1,\ldots,s-x_k) 1_{\{s>x_1,\ldots,s>x_k\}} ds$$

is

$$\widehat{h}_t^\beta(\boldsymbol{\nu}) = \frac{(e^{it\langle\boldsymbol{\nu},\mathbf{1}\rangle}-1)}{i\langle\boldsymbol{\nu},\mathbf{1}\rangle)}(i\langle\boldsymbol{\nu},\mathbf{1}\rangle)^{-\beta}\widehat{g}(-\boldsymbol{\nu})\Gamma(\beta),\ a.e.,$$

where $\mathbf{x},\boldsymbol{\nu}\in\mathbb{R}^k$ and $\langle\boldsymbol{\nu},\mathbf{1}\rangle = \sum_{j=1}^k \nu_j$. Thus,

$$Z_H^{(k)}(t) = c\int_{\mathbb{R}^k}'' \frac{(e^{it\langle\boldsymbol{\nu},\mathbf{1}\rangle}-1)}{i\langle\boldsymbol{\nu},\mathbf{1}\rangle)}(i\langle\boldsymbol{\nu},\mathbf{1}\rangle)^{-\beta}\widehat{g}(-\boldsymbol{\nu})\widehat{W}(d\nu_1)\ldots\widehat{W}(d\nu_k),$$

One obtains back the Hermite process by setting $\beta = 0$ and

$$\widehat{g}(-\boldsymbol{\nu}) = c\prod_{j=1}^k |\nu_k|^{-d}.$$

Regularity conditions: Class (L)

We say that a generalized Hermite kernel g on $\mathbb{R}_+^k$ having homogeneity exponent α is of Class (L) (L stands for "limit" as in "limit theorems"), if

1. g is continuous a.e. on $\mathbb{R}_+^k$;
2. $|g(\mathbf{x})| \leq g^*(\mathbf{x})$ a.e. $\mathbf{x} \in \mathbb{R}_+^k$, where g^* is a finite linear combination of non-symmetric Hermite kernels: $\prod_{j=1}^k x_j^{\gamma_j}$, where $\gamma_j \in (-1, -1/2)$, $j = 1, \ldots, k$, and $\sum_{j=1}^k \gamma_j = \alpha \in (-k/2 - 1/2, -k/2)$.

- For example, $g^*(\mathbf{x})$ could be $x_1^{-3/4} x_2^{-5/8} + x_1^{-9/16} x_2^{-13/16}$ if $k = 2$. In this case, $\alpha = -11/8$.
- The limit theorems suppose that g is of Class (L).

The earlier example belongs to Class (L)

The following kernel g belongs to Class (L):

$$\begin{aligned} g(x_1, \dots, x_k) &= \max\left(\frac{x_1 \dots x_k}{x_1^{k-\alpha} + \dots + x_k^{k-\alpha}},\ x_1^{\alpha/k} \dots x_k^{\alpha/k} \right) \\ &= \max(g_1(\mathbf{x})/g_2(\mathbf{x}), g_3(\mathbf{x})). \end{aligned}$$

Max is continuous, g_1, g_3 are powers. For g_2 use the arithmetic-geometric mean inequality $k^{-1} \sum_{j=1}^{k} y_j \geq \left(\prod_{j=1}^{k} y_j \right)^{1/k}$ for $y_j > 0$.