# Local Limit Theorem in negative curvature

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$(M, g)$ is a closed Riemannian manifold with negative curvature,

$(\widetilde{M}, g)$ the universal cover

$\partial\widetilde{M}$ the geometric boundary at infinity of $\widetilde{M}$, sometimes identified with $S_x\widetilde{M}$ for any $x \in \widetilde{M}$.

$SM, S\widetilde{M}$ the unit tangent bundles, $\{g_t\}_{t\in\mathbb{R}}$ the geodesic flow.

$\Delta = \mathrm{Div}\nabla$ the Laplacian on $C^2$ functions,

$p(t,x,y)$ the heat kernel on $\widetilde{M}$.

In the weak sense, $p(t,x,y) = e^{t\Delta}(x,y)$.

In particular,

**Fact** *Let $\lambda_0$ be the bottom of the spectrum of $-\Delta$ in $L^2(\widetilde{M}, Vol)$. Then, for $x, y \in \widetilde{M}$:*

$$\lim_{t\to\infty} \frac{1}{t} \ln p(t,x,y) \;=\; -\lambda_0.$$

**Theorem** [L. - Lim]

$$\lim_{t \to \infty} t^{3/2} e^{\lambda_0 t} p(t,x,y) \;=\; C(x,y),$$

*where, for $x$ fixed, $C(x,y)$ is a harmonic function for of $\Delta + \lambda_0$.*

Remark: For all $x, y \in \widetilde{M}$, $C(x,y) = C(y,x)$.

**Example** The 3-dimensional hyperbolic space $\mathbb{H}^3$. Then, $\lambda_0 = 1$ and

$$p(t,x,y) = \left(\frac{1}{4\pi t}\right)^{3/2} \frac{d(x,y)}{\sinh d(x,y)}\, e^{-t} e^{-\frac{d(x,y)^2}{4t}}.$$

Therefore:

$$\lim_{t\to\infty} t^{3/2} e^t p(t,x,y) = \left(\frac{1}{4\pi}\right)^{3/2} \frac{d(x,y)}{\sinh d(x,y)}$$

and $y \mapsto \dfrac{d(x,y)}{\sinh d(x,y)}$ is the positive $\Delta + 1$ harmonic function that depends only on the distance to $x$, with maximum 1.

**Bougerol (1981)** If $\widetilde{M}$ is a symmetric space of non-compact type of rank $n$, then

$$\lim_{t\to\infty} t^{n+\frac{1}{2}} e^{\lambda_0 t} p(t,x,y) = C\Phi(x,y),$$

Where $C$ is some constant and $y \mapsto \Phi(x,y)$ is the positive $\Delta + \lambda_0$ harmonic function that depends only on the distance to $x$, with maximum 1.

This has been generalized to trees and buildings for isotropic random walks.

Non-isotropic random walks $(G, \mu)$ with $G$ a hyperbolic group. Expected result:

$$\lim_{n \to \infty} n^{3/2} R^n p^{(n)}(x, y) = c(x, y),$$

where $R^{-1}$ is the spectral radius of the random walk.

**Lalley (1993)** $G$ finitely generated free groups, $\mu$ finitely supported.
**Gouëzel - Lalley (2013)** $G$ surface group, $\mu$ finitely supported.
**Gouëzel (2014a) (2014b)** $G$ finitely gen. hyperbolic, $\mu$ has all exponential moments.

Our proof follows the Gouëzel - Lalley strategy (and arguments). Set, for $\lambda \leq \lambda_0, x \neq y$

$$G_\lambda(x,y) := (\Delta + \lambda)^{-1}(x,y) = \int_0^\infty e^{\lambda t} p(t,x,y)\,dt.$$

**Facts:** $G_\lambda(x,y) < \infty$ for $\lambda \leq \lambda_0, x \neq y$,
$\frac{\partial}{\partial \lambda} G_\lambda(x,y) < \infty$ for $\lambda < \lambda_0, x \neq y$,
$\frac{\partial}{\partial \lambda} G_\lambda(x,y) \nearrow \infty$ as $\lambda \nearrow \lambda_0, x \neq y$.

Enough to show, for $x \neq y$,

$$\lim_{\lambda \nearrow \lambda_0} \sqrt{\lambda_0 - \lambda}\, \frac{\partial}{\partial \lambda} G_\lambda(x,y) = C'(x,y) \qquad (1)$$

Then, by Tauberian Theorems, symmetry and positivity of $p(t,x,y)$, we get

$$\lim_{t \to \infty} t^{3/2} e^{\lambda_0 t} p(t,x,y) = \sqrt{\frac{2}{\pi}}\, C'(x,y).$$

**Goal** Find a function $P(\lambda), \lambda < \lambda_0$, such that:

$$P(\lambda) < 0, \;\; P(\lambda) \to 0 \text{ as } \lambda \to \lambda_0,$$
$$(-P(\lambda))\tfrac{\partial}{\partial\lambda}G_\lambda(x,y) \;\to\; C_1(x,y) \text{ as } \lambda \to \lambda_0,$$
$$(-P(\lambda))^3\tfrac{\partial^2}{\partial\lambda^2}G_\lambda(x,y) \;\to\; C_2(x,y) \text{ as } \lambda \to \lambda_0.$$

Then, if $F(\lambda) := \frac{\partial}{\partial\lambda}G_\lambda(x,y)$, we have $\dfrac{2F'(\lambda)}{F(\lambda)^3} \to \dfrac{2C_2}{C_1^3}$ and therefore (cf [GL2013])

$$\sqrt{\lambda_0 - \lambda}\,F(\lambda) \to \sqrt{\frac{C_1^3}{2C_2}}.$$

Lalley's renewal formula:

$$\begin{aligned}\frac{\partial}{\partial\lambda}G_\lambda(x,y) &= \int_{\widetilde{M}} G_\lambda(x,z)G_\lambda(z,y)d\mathsf{Vol}(z)\\ &= \int_0^\infty \left(\int_{S(x,R)} G_\lambda(x,z)G_\lambda(y,z)dz\right)dR\\ &= \int_0^\infty \left(\int_{S(x,R)} \frac{G_\lambda(y,z)}{G_\lambda(x,z)}(G_\lambda(x,z))^2 dz\right)dR,\end{aligned}$$

where $S(x,R), dz$ is the sphere of radius $R$ about $x$ and $dz$ the Lebesgue measure on it.

Let $\mu_{x,\lambda,R}$ be the measure on $S_xM$ such that if it is lifted to $S_x\widetilde{M}$ and then pushed forward to $S(x,R)$ by $v \mapsto \exp_x Rv$, one gets the measure $(G_\lambda(x,z))^2 dz$.

**Key fact:** *There exist a negative number $P(\lambda)$ and a finite measure $\mu_{x,\lambda}$ on $S_xM$ such that, as $R \to \infty$, the measures $e^{-P(\lambda)R}\mu_{x,\lambda,R}$ weak* converge towards $\mu_{x,\lambda}$.*

*Example:* $\widetilde{M}$ *is the hyperbolic space* $\mathbb{H}^n$*.* Then, if $d(x,y) = R$, $R$ large:

$$G_\lambda(x,y) \ \sim \ Ce^{-s(\lambda)R},$$

with $s(\lambda) = \frac{1}{2}\left(n - 1 + \sqrt{(n-1)^2 - 4\lambda}\right)$, whereas

$$dz \ \sim \ e^{(n-1)R} d\theta.$$

So with $P(\lambda) = -\sqrt{(n-1)^2 - 4\lambda} = -2\sqrt{\lambda_0 - \lambda}$, we do have $e^{-P(\lambda)R}\mu_{x,\lambda,R}$ converge towards C.Lebesgue on $S_x M$.

Let $v \in S_x\widetilde{M}$. As $z \to \infty$ in $\widetilde{M}$ along $\exp_x \mathbb{R}_+ v$, $\frac{G_\lambda(y,z)}{G_\lambda(x,z)}$ converges towards a continuous function $k_\lambda(x,y,v)$ (**Ancona 1985**).

The expression for $(-P(\lambda))\frac{\partial}{\partial\lambda}G_\lambda(x,y)$ becomes

$$-P(\lambda)\int_0^\infty e^{P(\lambda)R}\left(\int_{S_xM} k_\lambda(x,y,v)e^{-P(\lambda)R}d\mu_{x,\lambda,R}\right)dR.$$

**IF** we can exchange the limits in $R$ and $\lambda$, and **IF** $P(\lambda_0) = 0$, then $(-P(\lambda))\frac{\partial}{\partial\lambda}G_\lambda(x,y)$ converges, as $\lambda \nearrow \lambda_0$, towards

$$\int_{S_xM} k_{\lambda_0}(x,y,\xi)d\mu_{x,\lambda_0}(\xi) =: C_1(x,y)$$

if $k_{\lambda_0}(x,y)$ and $\mu_{x,\lambda_0}$ make sense.

Ingredients for the proof of the Key Fact, for a fixed $\lambda$:

1) Ancona's inequality and Martin boundary (**Ancona 85**)

*There is $C$ such that, if $x, y, z$ are, in that order, on the same geodesic, then:*

$$G_\lambda(x,z) \;\leq\; C\, G_\lambda(x,y) G_\lambda(y,z)$$

*There is $k_\lambda(x,y,\xi), x, y \in \widetilde{M},\ \xi \in \partial\widetilde{M}$, such that, $z \to \xi$ iff*

$$\frac{G_\lambda(y,z)}{G_\lambda(x,z)} \;\to\; k_\lambda(x,y,\xi).$$

2) Hölder regularity (**Hamenstädt, Kaimanovich, L.** )

For $v \in SM$, $\widetilde{v}$ a lift to $S\widetilde{M}$, set

$$\phi_\lambda(v) := \frac{d}{dt} \ln k_\lambda(\sigma_{\widetilde{v}}(0), \sigma_{\widetilde{v}}(t), \sigma_{\widetilde{v}}(+\infty)|_{t=0}$$

**Proposition** *There is* $\alpha = \alpha(C, \mathit{geometry})$ *positif such that the function* $\phi_\lambda$ *is* $\alpha$*-Hölder continuous on* $SM$*.*

3) Thermodyn. form. for $\varphi_\lambda := -2\phi_\lambda$.

- $P(\lambda) :=$ Pressure $(\varphi_\lambda)$,

- the Gibbs measure $m_\lambda$ is mixing for the geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ and

- there are *Patterson-Sullivan* measures $\mu_{x,\lambda}$ on $\partial\widetilde{M}$ such that:

$$\frac{d\mu_{y,\lambda}}{d\mu_{x,\lambda}}(\xi) = k_\lambda^2(x,y,\xi)e^{-P(\lambda)\beta(x,y,\xi)}.$$

4) **Margulis 69**'s argument for the volume of the spheres used only mixing of the geodesic flow for the measure of maximal entropy.

Its extension to (mixing) Gibbs measures yields that the measures $e^{-P(\lambda)R}\mu_{x,\lambda,R}$ weak* converge towards some measure $\mu_{x,\lambda}$ that is proportional to the Patterson-Sullivan measures of 3).

Proof of convergence of $(-P(\lambda))\frac{\partial}{\partial\lambda}G_\lambda(x,y)$ as $\lambda \nearrow \lambda_0$: all the previous steps can be made uniform in $\lambda$ up to $\lambda_0$.

Step 1: **Proposition** *The constant $C$ in Ancona inequality can be chosen independently of $\lambda \in [0, \lambda_0]$.*

The proof imitates the proof of the corresponding statement in [**Gouëzel 14a**]

Steps 2 and 3: *There is $\beta < \alpha$ such that the mapping $\lambda \mapsto \varphi_\lambda$ is continuous from $[0, \lambda_0]$ into the space of $\beta$-Hölder continuous functions.*

Step 4: Margulis's argument yields uniform convergence in $\lambda$ as soon as the rate of mixing is uniform in $\lambda$. We have:

**Proposition** *Let $(M, g)$ be a manifold with negative curvature, $\varphi_\lambda, \lambda \leq \lambda_0$ as above, $\beta > 0$ small enough. There exist $\varepsilon > 0, C, c$ such that if $f_1, f_2$ are $\beta$-Hölder continuous functions, then for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0], t > 0$,*

$$|\int f_1(v) f_2(g_t v) d\, m_\lambda(v) - \int f_1 d\, m_\lambda \int f_2 d\, m_\lambda| \leq$$
$$\leq \quad C \frac{\|f_1\|_\beta \|f_2\|_\beta}{1+t^c}.$$

*Proof* If we have topological rapid mixing, **Dolgopyat 98** proved the above for a fixed $\beta$-Hölder continuous potential $\varphi$. The constants $C$ and $c$ obtained in the proof are uniform in a small neighbourhood of $\varphi$.

A continuous flow $(X, d; \{g_t\}_{t\in\mathbb{R}})$ is topologically rapid mixing if there exist $t_0, \alpha_0$ such that for any two balls $B_1, B_2$ of radius $r > 0$, $g_t B_1 \cap B_2 \neq \emptyset$ for $t$ larger than $t_0$ and $r^{-\alpha_0}$.

By **Liverani 04**, a geodesic flow on a compact manifold of negative curvature is topologically rapid mixing.

This finishes the proof that

$$(-P(\lambda))\frac{\partial}{\partial\lambda}G_\lambda(x,y) \;\to\; C_1(x,y) \text{ as } \lambda \to \lambda_0,$$

with $C_1(x,y) = \int_{\partial\widetilde{M}} k_{\lambda_0}(x,y,\xi)d\mu_{x,\lambda_0}(\xi)$.

Remains to show that

$$(-P(\lambda))^3\frac{\partial^2}{\partial\lambda^2}G_\lambda(x,y) \;\to\; C_2(x,y) \text{ as } \lambda \to \lambda_0.$$

Further derivation yields Lalley's renewal formula for $\frac{\partial^2}{\partial\lambda^2} G_\lambda(x,y)$ :

$$2\int_{\widetilde{M}\times\widetilde{M}} G_\lambda(x,z)G_\lambda(z,w)G_\lambda(w,y)\,d\,\mathrm{Vol}(z)\,d\,\mathrm{Vol}(w).$$

Geometric reductions show that we have to compute the limit of the following expression as $R \to \infty$ and to verify that the limit is uniform in $\lambda$:

$$2\int_{S_xM} k_\lambda(x,y,v)\frac{1}{R}\left(\int_0^R u_\lambda(g_s v)\,ds\right) e^{-P(\lambda)R} d\mu_{x,\lambda,R},$$

where $u_\lambda$ is a positive $\beta$-Hölder continuous function with $\|u_\lambda - u_{\lambda_0}\|_\beta \to 0$ as $\lambda \nearrow \lambda_0$.

We have uniform 2-mixing by an extension of Dolgopyat's argument:

**Proposition** *Let* $(M, g)$ *be a manifold with negative curvature,* $\varphi_\lambda, \lambda \leq \lambda_0$ *as above,* $\beta > 0$ *small enough. There exist* $\varepsilon > 0, C, c$ *such that if* $f_1, f_2, f_3$ *are* $\beta$*-Hölder continuous functions, then for* $\lambda \in [\lambda_0 - \varepsilon, \lambda_0], t, s > 0$,

$$\Big| \int f_1 f_2(g_t \cdot) f_3(g_{t+s} \cdot) d\, m_\lambda - \int f_1 d\, m_\lambda \int f_2 d\, m_\lambda \int f_3 d\, m_\lambda \Big| \leq C \frac{\|f_1\|_\beta \|f_2\|_\beta \|f_3\|_\beta}{(1+t^c)(1+s^c)}.$$

By suitably extending the argument à la Margulis one gets that, as $\lambda \nearrow \lambda_0$,

$$(-P(\lambda))^3 \frac{\partial^2}{\partial \lambda^2} G_\lambda(x,y) \;\to\; 2 \int u_{\lambda_0} dm_{\lambda_0}\, C_1(x,y).$$

This Proves our Theorem with

$$C(x,y) = \frac{1}{\sqrt{2\pi \int u_{\lambda_0} dm_{\lambda_0}}} \int_{S_x M} k_{\lambda_0}(x,y,\xi) d\mu_{x,\lambda_0}(\xi).$$

**Remarks** Since $C(x,y)$ is an integral of $\Delta+\lambda_0$ harmonic functions,

$$\Delta_y C(x,y) = -\lambda_0 C(x,y)$$

(in particular, $C(x,y)$ is a smooth function).

Since $k_{\lambda_0}(x,y,\xi) = \sqrt{\frac{d\mu_{y,\lambda_0}}{d\mu_{x,\lambda_0}}(\xi)}$, we may use the halfdensity notation and write the function $\int_{\partial\widetilde{M}} k_{\lambda_0}(x,y,\xi)d\mu_{x,\lambda_0}(\xi)$ as

$$\int_{\partial\widetilde{M}} \sqrt{d\mu_{y,\lambda_0}}\sqrt{d\mu_{x,\lambda_0}},$$

in analogy with the Harish-Chandra function $\Phi$ of Bougerol's Local Limit Theorem.