# Invariant measures for $\mathscr{B}$-free systems and Möbius disjointness

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BIRS, 11-15.08.2014
on the occasion of 70th birthday of Anatole Katok

Joint work with Joanna Kułaga-Przymus and Benjamin Weiss

# Plan

# $\mathscr{B}$-free numbers (Erdös 1967)

$\mathscr{B} = \{b_k : \ k \geq 1\} \subset \mathbb{N} \setminus \{1\}$

- (b1) $\mathscr{B}$ is infinite,
- (b2) $(b_k, b_\ell) = 1$ whenever $k \neq \ell$,
- (b3) $\sum_{k \geq 1} \frac{1}{b_k} < +\infty$.

**$\mathscr{B}$-free numbers**

$\mathcal{F}_{\mathscr{B}} := \{n \in \mathbb{Z} : \ (\forall k \geq 1) \quad b_k \text{ does not divide } n\}$

**Examples**

- $\mathscr{B} = \{p_k^2 : \ k \geq 1\}$, $2 = p_1 < p_2 < \ldots$, consecutive prime numbers: $\mathcal{F}_{\mathscr{B}}$-square free numbers,
- $\mathscr{B} = \{p_k^r : \ k \geq 1\}$, $r \geq 2$,
- infinite $\mathscr{B}' \subset \mathscr{B}$, with $\mathscr{B}$ satisfying (b1)-(b3).

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1. $\mathscr{B}$-free numbers
2. $\mathscr{B}$-free systems
   - Canonical odometer
   - $\mathscr{B}$-free system
   - Admissible sequences
3. Entropy of $\mathscr{B}$-free systems
4. Intrinsic ergodicity of $\mathscr{B}$-free systems
5. Invariant (ergodic) measures for $\mathscr{B}$-free systems, $\nu \neq \delta_{(\dots,0,0,\dots)}$
6. Combinatorics
7. Sarnak's conjecture on Möbius disjointness

# Canonical odometer associated to $\mathscr{B}$

- $\Omega = \prod_{k\geq 1} \mathbb{Z}/b_k\mathbb{Z}$ - compact Abelian (monothetic) metric group (with coordinatewise addition),
- $\mathbb{P}$ - Haar measure on $\Omega$ (product of counting measures),
- $T\colon \Omega \to \Omega, \quad T(\omega_1, \omega_2, \ldots) = (\omega_1 + 1, \omega_2 + 1, \ldots)$.

**Ergodic properties of $(T, \Omega, \mathbb{P})$**

- $T$ is ergodic,
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- (by independence) $\mathbb{P}(C) = \mathbb{P}(\bigcap_{k\geq 1} C_k) = \prod_{k\geq 1}\left(1 - \frac{1}{b_k}\right) > 0$ (since $\sum_{k\geq 1}\frac{1}{b_k} < +\infty$),
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$$\varphi(\omega)(n)=\begin{cases} 0 & (\exists k\geq 1)\ \omega_k+n=0 \mod b_k \\ 1 & (\forall k\ \geq 1)\ \omega_k+n\neq 0 \mod b_k \end{cases}$$

- $S\circ\varphi=\varphi\circ T$,
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$X_\eta := \overline{\{S^i\eta : \ i \in \mathbb{Z}\}}$ - $\mathscr{B}$-free subshift

- square free system - Sarnak 2010,
- $x \in X_\eta$ if and only if every block occurring on $x$ occurs on $\eta$,
- $\varphi(\underline{0}) = \eta \in X_\eta$ but since $\varphi$ IS NOT continuous it is unclear whether $\varphi(\Omega) \subset X_\eta$ (if $2 \notin \mathscr{B}$ then $\varphi(1, 1, \ldots, 1, 1, \ldots)(1) = 1$ while $\varphi(1, 1, \ldots, 1, b_{k+1} - 1, b_{k+2} - 1, \ldots)(1) = 0$).

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$A \subset \mathbb{Z}$ is called $\mathscr{B}$-*admissible* if

$$|A \mod b_k| < b_k \text{ for each } k \geq 1.$$

- $x \in \{0,1\}^{\mathbb{Z}}$ is called *admissible* if $\operatorname{supp} x := \{n \in \mathbb{Z} : x(n) \neq 0\}$ is admissible,
- $D \in \{0,1\}^K$ (block) is called admissible if its support is admissible (to verify that a finite set is admissible, we check the condition only for "small" $b_k$),
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# The subshift of $\mathscr{B}$-admissible sequences

$$X_{\mathscr{B}} := \{x \in \{0,1\}^{\mathbb{Z}} : \ x \text{ is } \mathscr{B}\text{-admissible}\}$$

- $X_{\mathscr{B}}$ is closed and $S$-invariant (it is a subshift)

**Proposition**

$\varphi(\Omega) \subset X_{\mathscr{B}}$, in particular, $\eta = \varphi(\underline{0})$ is admissible, hence $X_\eta \subset X_{\mathscr{B}}$.

**Proof.** $\omega \in \Omega$; $n \in \operatorname{supp} \varphi(\omega) \Leftrightarrow \varphi(\omega)(n) = 1$

$$\Leftrightarrow (\forall k \geq 1)\ \omega_k + n \neq 0 \mod b_k$$

It follows, $-\omega_k \neq n$ mod $b_k$ for each $k \geq 1$, that is

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# Mirsky measure

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**Definition of the Mirsky measure**

$$\nu_{\mathscr{B}} := \varphi_*(\mathbb{P}).$$

**Proposition**

$(S, X_{\mathscr{B}}, \nu_{\mathscr{B}})$ is a factor of $(T, \Omega, \mathbb{P})$.

- $\varphi$ is not 1-1; indeed, e.g. $\varphi^{-1}(\dots,0,0,\dots)$ is UNCOUNTABLE(!), let $\mathbb{Z} \ni n \mapsto k_n \geq 1$ be 1-1, take $\omega \in \Omega$ so that $\omega_{k_n} + n = 0 \bmod b_{k_n}$ for each $n \in \mathbb{Z}$; then $\varphi(\omega) = (\dots,0,0,\dots)$.

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# Measure theory lemma

## Lemma

Suppose that $(\Omega, \mathbb{P})$ is a probability space; $(E_n)$ a sequence of events. Set $F_n = E_n^c$. Then for any finite disjoint sets $A, B \subset \mathbb{N}$, we have

$$\mathbb{P}\left(\bigcap_{n\in A} E_n \cap \bigcap_{m\in B} F_m\right) = \sum_{A\subset D\subset A\cup B} (-1)^{|D\setminus A|}\mathbb{P}(\bigcap_{d\in D} E_d).$$

# Mirsky measure - properties

$A, B \subset \mathbb{Z}$ - finite, $A \cap B = \emptyset$,

$$C_{A,B} := \{x \in X_{\mathscr{B}} :\ (\forall n \in A)\ x_n = 1,\ (\forall m \in B)\ x_m = 0\},$$

$C_{A,\emptyset} =: C^1_A,\ C_{\emptyset,B} =: C^0_B$.

**Lemma**

(a) $\nu_{\mathscr{B}}(C^1_A) = \prod_{k\geq 1}(1 - \frac{|A \bmod b_k|}{b_k})$,
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**Proof.** $\nu_{\mathscr{B}}(C^1_A) = \mathbb{P}(\varphi^{-1}(C^1_A)) = \mathbb{P}(\bigcap_{k\geq 1}\{\omega \in \Omega :\ (\forall n \in A)\ \omega_k + n \neq 0 \bmod b_k\})$. If $k \geq 1$ is FIXED,

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# Mirsky measure - properties

**Proposition**

$A, B \subset \mathbb{Z}$ finite, disjoint. The following are equivalent:
(i) $\nu_{\mathscr{B}}(C_{A,B}) > 0$,
(ii) $\nu_{\mathscr{B}}(C_A^1) > 0$,
(iii) $A$ is admissible.

# Hereditarity - first steps

Suppose that $A \subset \mathbb{Z}$ is admissible and $A' \subset A$. Then clearly $A'$ is also admissible. It follows that if

(*h*) $x \in X_{\mathscr{B}}$ and $\{0,1\}^{\mathbb{Z}} \ni x' \leq x$ (coordinatewise) then $x' \in X_{\mathscr{B}}$.

Subshifts satisfying this property are called *hereditary* (Kerr-Li, 2007, Kwietniak, 2013)

**Corollary**

The topological support of $\nu_{\mathscr{B}}$ is $X_{\mathscr{B}}$.

Indeed, $\nu_{\mathscr{B}}$ is concentrated on $X_{\mathscr{B}}$ (by definition) but not on a smaller (closed) set as EACH admissible block has positive measure.

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# Basic properties

**Theorem (Abdalauoi, L., de la Rue, 2013)**

(a) $(\exists\Omega_0 \subset \Omega, \mathbb{P}(\Omega_0) = 1)$ $\varphi|_{\Omega_0}$ is 1-1; in particular, $(T, \Omega, \mathbb{P})$ and $(S, X_{\mathscr{B}}, \nu_{\mathscr{B}})$ are isomorphic,
(b) $\eta$ is a generic point for the Mirsky measure $\nu_{\mathscr{B}}$,
(c) $X_\eta = X_{\mathscr{B}}$.

- For the square-free systems, the isomorphism in (a) has been proved by Cellarosi-Sinai 2012, Sarnak 2010 by different methods,
- (c) follows from (b).

# Basic properties

**Theorem (Abdalauoi, L., de la Rue, 2013)**

(a) $(\exists \Omega_0 \subset \Omega, \mathbb{P}(\Omega_0) = 1) \quad \varphi|_{\Omega_0}$ is 1-1; in particular, $(T, \Omega, \mathbb{P})$ and $(S, X_{\mathscr{B}}, \nu_{\mathscr{B}})$ are isomorphic,
(b) $\eta$ is a generic point for the Mirsky measure $\nu_{\mathscr{B}}$,
(c) $X_\eta = X_{\mathscr{B}}$.

- For the square-free systems, the isomorphism in (a) has been proved by Cellarosi-Sinai 2012, Sarnak 2010 by different methods,
- (c) follows from (b).

# Plan

# Entropy formula

**Proposition (Abdalaoui, L., de la Rue, 2013; Sarnak 2010, Peckner, 2012 - the square free system; Pleasants, Huck, 2013 - lattice systems)**

$h_{top}(S, X_\eta) = \log 2 \cdot \prod_{k=1}^{\infty}(1 - \frac{1}{b_k})(= \log 2 \cdot \mathbb{P}(C))$.

# Examples of invariant measure for $\mathscr{B}$-free systems

- $\delta_{(\ldots,0,0,\ldots)}$ (zero entropy),
- Mirsky measure $\nu_{\mathscr{B}}$ (zero entropy),
- $M : X_\eta \times \{0,1\}^{\mathbb{Z}} \to X_\eta$ coordinatewise multiplication:

$$\nu_{\mathscr{B}} \otimes \kappa \mapsto M_*(\nu_{\mathscr{B}} \otimes \kappa) =: \nu_{\mathscr{B}} * \kappa,$$

  where $\kappa$ is any (ergodic) full shift invariant measure. ($\nu_{\mathscr{B}} * \kappa$ need not be ergodic)
  We can compute that the entropy of $\nu_{\mathscr{B}} * B(1/2, 1/2)$ is equal to $\log 2 \cdot \prod_{k\geq 1}(1 - \frac{1}{b_k})$: hence, this convolution is a measure of maximal entropy.
- $\mathscr{B}' = \{b'_k : 1 < b'_k | b_k\}$, then $X_{\mathscr{B}'} \subset X_{\mathscr{B}}$ and we have a new Mirsky measure $\nu_{\mathscr{B}'}$, convolutions...
- $\mathscr{B} \subset \mathscr{B}''$, then also as above $X_{\mathscr{B}''} \subset X_{\mathscr{B}}$.

# Plan

# Main results

**Theorem (Kułaga-Przymus, L., Weiss, 2014)**

Each $\mathscr{B}$-free system is intrinsically ergodic.

- *Intrinsic ergodicity* - B. Weiss, 1970, Krieger (intrinsic Markov chains), 1964 - systems having exactly one measure with maximal entropy,
- Square free system is intrinsically ergodic - Peckner 2014 (different proof).

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There are hereditary systems which are not intrinsically ergodic.

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# Can we reverse $\varphi : \Omega \to X_\eta$?

$$Y := \{x \in X_\eta : (\forall i \geq 1)\ \ |\mathrm{supp}\, x \bmod b_i| = b_i - 1\}.$$

- $Y$ is a Borel set and $SY = Y$,
- $\eta \in Y$.

**Lemma (Peckner, 2012 for the square free system),**

Any measure $\nu$ with maximal entropy is concentrated on $Y$.

# Inverse of $\varphi$ on $Y$

**Recall:** The image of $\varphi : \Omega \to X_\eta$ is not "quite" included in $Y$ and the map itself is not 1-1.

$$\Omega_{k,z} := \{\omega \in \Omega : \ \omega_k = z\},$$
$$E_{k,z} := \{\omega \in \Omega : (\forall s \geq 1)\ \varphi(\omega)(-z + sb_k) = 0\}.$$

- $E_{k,z} \supset \Omega_{k,z}$ (i.e. points from $\Omega_{k,z}$ have some natural infinite arithmetic progressions of zeros).

$$\Omega_0 := \bigcap_{k\geq 1} \bigcap_{z\in\mathbb{Z}/b_k\mathbb{Z}} \left(E^c_{k,z} \cup \Omega_{k,z}\right).$$

**Lemma (Abdalauoi, L., de la Rue, 2013)**

We have $\mathbb{P}(\Omega_0) = 1$ and $\varphi|_{\Omega_0}$ is 1-1.

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We have $\mathbb{P}(\Omega_0) = 1$ and $\varphi|_{\Omega_0}$ is 1-1.

# Inverse of $\varphi$ on $Y$

Define a Borel map $\theta\colon Y \to \Omega$ by setting

$$\theta(y) = \omega \text{ if } -\omega_i \notin \operatorname{supp} y \bmod b_i \text{ for all } i \geq 1.$$

**Lemma**

- $\theta$ is equivariant, i.e. $T \circ \theta = \theta \circ S$.
- $\varphi(\Omega_0) \subset Y$ (in particular, $\theta \circ \varphi|_{\Omega_0} = id_{\Omega_0}$).
- For each $\omega \in \Omega$ and $y \in Y$ such that $\theta(y) = \omega$, we have $y \leq \varphi(\omega)$.
- For each $x \in X_\eta$ there exists $\omega \in \Omega$ such that $x \leq \varphi(\omega)$ ($\varphi(\Omega)$ is a symbolic model of the odometer and $X_\eta$ is a hereditary system "generated" by this model).
- If $u \leq \varphi(\omega)$ and $|\{r \in \mathbb{Z} : u(r) \neq \varphi(\omega)(r)\}| < +\infty$ then $u \in Y$.
- For each $z \in \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega_k\}$

$$\omega \in \Omega_0 \Rightarrow \left|\{s \geq 1 : -z + sb_k \in \operatorname{supp} \varphi(\omega)\}\right| = \infty.$$

- $[y \in Y,\ y \leq \varphi(\omega)] \Rightarrow \theta(y) = \omega$.
- whenever $\omega \neq \omega''$ are two points from $\Omega_0$ then $\varphi(\omega)$ and $\varphi(\omega'')$ are not $\leq$-comparable.

# Measure of maximal entropy projects on $\mathbb{P}$

Let $\nu$ be a measure of maximal entropy.

**Lemma**

$$\theta_*(\nu) = \mathbb{P}.$$

**Proof.** $\theta : Y \to \Omega$ is equivariant and $\nu$ is concentrated on $Y$. Since $(T, \Omega)$ is uniquely ergodic, the result follows.

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# Pull-back of $\mathcal{B}(\Omega)$ is contained in the Pinsker algebra of $(S, X_\eta, \nu)$

- $Q = (Q_0, Q_1)$, $Q_j = C_0^j \cap Y$, $j = 0, 1$ (this is a generating partition of $(S, Y, \nu)$),
- $Q^- := \bigvee_{-\infty}^{-1} S^j Q$ and $\mathcal{A} := \theta^{-1}(\mathcal{B}(\Omega))$. Since $Q$ is a generating partition, the $\sigma$-algebra $\bigcap_{k\geq 1}\left(\bigvee_{-\infty}^{-k} S^j Q\right)$ is the Pinsker $\sigma$-algebra of $(S, Y, \mathcal{B}(Y), \nu)$.

**Lemma**

$$\mathcal{A} \subset \bigcap_{m\geq 0} S^{-m} Q^- \text{ modulo } \nu.$$

It follows that a.e. atom of the partition corresponding to the Pinsker $\sigma$-algebra of $(S, Y, \mathcal{B}(Y), \nu)$ is contained in an atom of the partition of $Y$ corresponding to $\mathcal{A}$. We also have

$$\mathcal{A} \subset S^{-m} Q^- \text{ for } m \geq 0,$$

so, in other words, after removing a set of $\nu$-measure zero from $Y$, for the remaining points in $Y$ we have the following: for each $s \geq 1$

$$[y_1, y_2 \in Y, \ (\forall j \leq -s) \ y_1(j) = y_2(j)] \Rightarrow \theta(y_1) = \theta(y_2).$$

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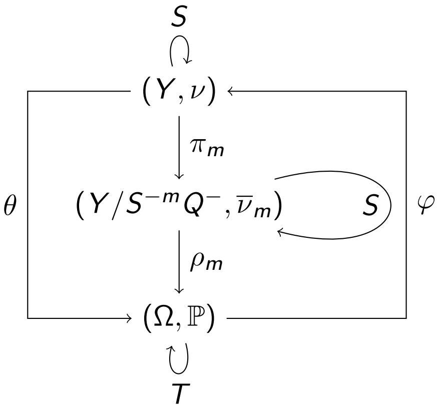



**Above:** $\theta, \pi_m$ and $\rho_m$ ($\rho_m\colon Y/S^{-m}Q^- \to \Omega$ is well defined since $\mathcal{A} \subset S^{-m}Q^-$) are equivariant and measure-preserving while $\varphi : \Omega \to Y$ is defined $\mathbb{P}$-a.e. and is not measure-preserving.

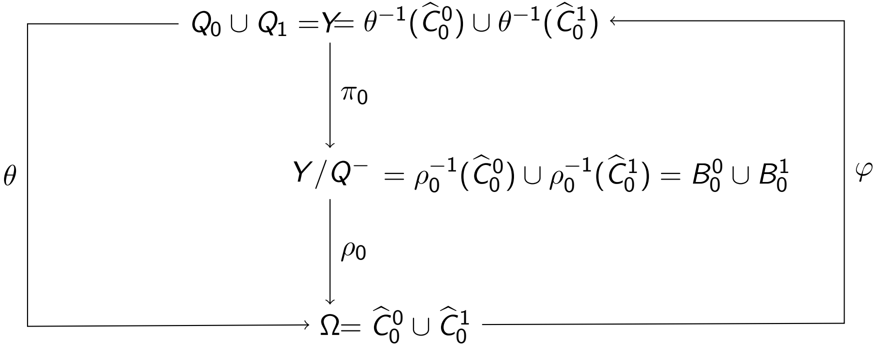
# Intrinsic ergodicity. Outline of the proof

- We will show that the conditional measures $\nu_\omega$ in the disintegration $\nu = \int_\Omega \nu_\omega \, d\mathbb{P}(\omega)$ of $\nu$ over $\mathbb{P}$ given by the mapping $\theta : Y \to \Omega$ are unique $\mathbb{P}$-a.e.
- **Definition of** $\mu$: Recall, for $\omega \in \Omega_0$, $y \in \theta^{-1}(\omega)$, we have $y[-k,k] \le \varphi(\omega)[-k,k]$ (as $\varphi(\omega)$ is the largest element in $\theta^{-1}(\omega)$). Therefore, there are at most $2^m$ blocks $u = (u_{-k}, \dots, u_k)$ on $\theta^{-1}(\omega)$, $m = \sum_{j=-k}^{k} \varphi(\omega)(j)$, obtained by replacing some of the 1s in $\varphi(\omega)[-k,k]$ by 0s. In fact, all such blocks do appear on $\theta^{-1}(\omega)$. For $u = (u_{-k}, \dots, u_k) \in \{0,1\}^{2k+1}$ denote by $[u]$ the corresponding cylinder set. If $u$ is such that $u \le \varphi(\omega)[-k,k]$, we set $\mu_\omega([u]) := 2^{-m}$, where $m$ has been defined above. Finally, we set $\mu = \int_\Omega \mu_\omega \, d\mathbb{P}(\omega)$.
- We will show that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mu_\omega = \nu_\omega$.
- To do it, we will show that for $A$ belonging to a countable dense family of subsets in $\mathcal{B}$, we have $\nu_\omega(A) = \mu_\omega(A)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.
- Recall that $\nu_\omega(A) = \mathbb{E}^\nu(A|\Omega)(\omega)$. To get the equality of conditional measures, we will step by step make use of the equality

$$\mathbb{E}^\nu(A|\Omega)(\omega) = \mathbb{E}^\nu\left(\mathbb{E}^\nu(A|Y/S^{-m}Q^-)(\overline{y}_m)\middle|\Omega\right)(\omega),$$

  where $A \in \bigvee_{t=-m}^{m} S^t Q$, $m \ge 0$, and show that $\mathbb{E}^\nu(A|Y/S^{-m}Q^-)(\overline{y}_m) = \mu_\omega(A)$ for all $\overline{y}_m$ having the same $\rho_m$-projection $\omega$.
- We first show that $\mu_\omega = \nu_\omega$ holds for $A \in Q$, that is, for $m = 0$.
- We show that this equality is satisfied for $A \in \bigvee_{t=-m}^{m} S^t Q$ for any $m \ge 0$.

# Partitions




Here: $\widehat{C}_0^j := \varphi^{-1}(C_0^j) = \{\omega \in \Omega : \varphi(\omega)(0) = j\}$ for $j = 0, 1$. Then $\Omega = \widehat{C}_0^0 \cup \widehat{C}_0^1$. Moreover, $Y = \theta^{-1}(\Omega) = \theta^{-1}(\widehat{C}_0^0) \cup \theta^{-1}(\widehat{C}_0^1)$. The partition of $Y$ is given by the fibers $\theta^{-1}(\omega)$ of $\theta$, according to the value at the zero coordinate of the biggest element $\varphi(\omega)$ in the fiber. Finally,

$$Y/Q^- = B_0^0 \cup B_0^1 \text{ with } B_0^j := \rho_0^{-1}(\widehat{C}_0^j) \text{ for } j = 0, 1$$

## $H_\nu(Q|Q^-)(\overline{y}) = 0$ whenever $\overline{y} \in B_0^0$

- If $y \in \theta^{-1}(\widehat{C}_0^0)$ then $y \in \theta^{-1}(\omega)$, where $\varphi(\omega)(0) = 0$, i.e. $\varphi(\omega) \in Q_0$.
- Since $y \leq \varphi(\omega)$, $y \in Q_0$.
- Hence, $\theta^{-1}(\widehat{C}_0^0) \subset Q_0$.
- For $\overline{y} \in B_0^0 = \rho_0^{-1}(\hat{C}_0^0)$, we have

$$\pi_0^{-1}(\overline{y}) \subset \pi_0^{-1}\rho_0^{-1}(\hat{C}_0^0) = \theta^{-1}(\hat{C}_0^0) \subset Q_0.$$

- $(Q_0, Q_1) \cap \pi_0^{-1}(\overline{y}) = (\pi_0^{-1}(\overline{y}), \emptyset)$.
- Since the measure $\overline{\nu}_0(\cdot|Q^-)(\overline{y})$ is concentrated on $\pi_0^{-1}(\overline{y})$, we obtain for each $\overline{y} \in B_0^0$

$$\big(\overline{\nu}_0(Q_0|Q^-)(\overline{y}), \overline{\nu}_0(Q_1|Q^-)(\overline{y})\big) = (1,0) =: (\lambda_0(Q_0), \lambda_0(Q_1)).$$

- $H_\nu(Q|Q^-)(\overline{y}) = 0$ whenever $\overline{y} \in B_0^0$.

## $H_\nu(Q|Q^-)(\overline{y}) = \log 2$ whenever $\overline{y} \in B_0^1$

- $\nu_{\mathscr{B}}(C_0^1) = \mathbb{P}(\widehat{C}_0^1) = \nu(\theta^{-1}(\widehat{C}_0^1))$,
-
$$\nu(\theta^{-1}(\widehat{C}_0^1)) \log 2 = h_{top}(S, X_\eta) = h_\nu(S, X_\eta)$$
$$= \int_{Y/Q^-} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y})$$
$$= \int_{B_0^0} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y}) + \int_{B_0^1} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y})$$
$$= \int_{B_0^1} H_\nu(Q|Q^-)(\overline{y})\, d\overline{\nu}_0(\overline{y}) \le \overline{\nu}_0(B_0^1) \log 2 = \nu(\theta^{-1}(\widehat{C}_0^1)) \log 2,$$
- It follows that for $\overline{\nu}_0$-a.e. $\overline{y} \in B_0^1$, we have $H_\nu(Q|Q^-)(\overline{y}) = \log 2$,
- Equivalently,
$$\left(\overline{\nu}_0(Q_0|Q^-)(\overline{y}), \overline{\nu}_0(Q_1|Q^-)(\overline{y})\right) = (1/2, 1/2) =: (\lambda_1(Q_0), \lambda_1(Q_1))\,.$$

# The end of the proof for toy model

Both

$$\left(\overline{\nu}_0(Q_0|Q^-)(\overline{y}), \overline{\nu}_0(Q_1|Q^-)(\overline{y})\right) = (1,0) =: (\lambda_0(Q_0), \lambda_0(Q_1))$$

and

$$\left(\overline{\nu}_0(Q_0|Q^-)(\overline{y}), \overline{\nu}_0(Q_1|Q^-)(\overline{y})\right) = (1/2,1/2) =: (\lambda_1(Q_0), \lambda_1(Q_1))\,.$$

do not depend on $\overline{y}$ itself but only on the value $\varphi(\rho_0(\overline{y}))(0)$. Since the $\sigma$-algebra $Q^-$ contains the $\sigma$-algebra $\theta^{-1}(\mathcal{B}(\Omega))$, this shows that if we denote by

$$\nu = \int_\Omega \nu_\omega \; d\mathbb{P}(\omega)$$

the disintegration of $\nu$ over $\overline{\nu}$ given by the mapping $\theta : Y \to \Omega$, then for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\nu_\omega(Q_i) = \mu_\omega(Q_i)$ for $i = 0, 1$.

# General case

Along the same lines + the chain rule for conditional probabilities

# Plan

# Borel partitions into Rokhlin towers

$$Y_{k,s_k} := \{x \in X_\eta : |\mathrm{supp}(x) \bmod b_k| = b_k - s_k\},$$

$k \geq 1$ and $1 \leq s_k \leq b_k - 1$.

$$Y_{k,s_k;a_1,\ldots,a_{s_k}} := \{x \in X_\eta : \mathrm{supp}(x) \bmod b_k = \mathbb{Z}/b_k\mathbb{Z} \setminus \{a_1, \ldots, a_{s_k}\}\}$$

$$\subset Y_{k,s_k}$$

$a_i \in \mathbb{Z}/b_k\mathbb{Z}$, $i = 1, \ldots, s_k$, with $a_i \neq a_j$ whenever $i \neq j$.

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- $Y_{k,s_k}$ is Borel and $SY_{k,s_k} = Y_{k,s_k}$,
- (by ergodicity) for each $k \geq 1$ there is exactly one $s_k$ such that $\nu(Y_{k,s_k}) = 1$,
- there exists $(a_1^k, \dots, a_{s_k}^k)$ such that $\nu(Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}) > 0$ (for each $k \geq 1$, any two sets of such a form are either disjoint or they coincide; moreover, their union gives $Y_{k,s_k}$);
- $SY_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k-1,\dots,a_{s_k}^k-1}$ (since $\mathrm{supp}(Sx) = \mathrm{supp}(x) - 1$),
- For $b_k' := \min\{j \geq 1 : \{a_1^k, \dots, a_{s_k}^k\} = \{a_1^k - j, \dots, a_{s_k}^k - j\}\}$ we have $b_k' \geq 2$,
- the sets $Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}, SY_{k,s_k;a_1^k,\dots,a_{s_k}^k}, \dots, S^{b_k'-1}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$ are pairwise disjoint and $S^{b_k'}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$,
- (by ergodicity) $\nu\left(\bigcup_{j=0}^{b_k'-1} S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}\right) = 1$.
- since $S^{b_k}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$, we have $b_k' | b_k$,

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- (by ergodicity) for each $k \geq 1$ there is exactly one $s_k$ such that $\nu(Y_{k,s_k}) = 1$,
- there exists $(a_1^k,\ldots,a_{s_k}^k)$ such that $\nu(Y_{k,s_k;a_1^k,\ldots,a_{s_k}^k}) > 0$ (for each $k \geq 1$, any two sets of such a form are either disjoint or they coincide; moreover, their union gives $Y_{k,s_k}$);
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- (by ergodicity) $\nu\left(\bigcup_{j=0}^{b_k'-1} S^j Y_{k,s_k;a_1^k,\ldots,a_{s_k}^k}\right) = 1.$
- since $S^{b_k}Y_{k,s_k;a_1^k,\ldots,a_{s_k}^k} = Y_{k,s_k;a_1^k,\ldots,a_{s_k}^k}$, we have $b_k' | b_k$,

# Invariant measures yield dynamical systems with infinite discrete part in the spectrum

**Proposition (Kułaga-Przymus, L., Weiss, 2014)**

Assume that $\mathcal{P}^e(S, X_\eta) \ni \nu \neq \delta_{(\dots,0,0,\dots)}$. The dynamical system $(S, X_\eta, \nu)$ has an infinite rational discrete spectrum. More precisely, the discrete spectrum part contains, for each $k \geq 1$, all $b'_k$-roots of unity for some $1 < b'_k | b_k$.

**Proof.** It suffices to notice that for $1 \leq s_k \leq b_k - 1$ and $\{a_1^k, \dots, a_{s_k}^k\}$ chosen so that $\nu(Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}) > 0$, and $b'_k$ given above, the partition of $Y_{k,s_k}$ into sets

$$S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k},\ 0 \leq j \leq b'_k - 1$$

is a Rokhlin tower fulfilling the whole space, whence the $b'_k$-root of unity is an eigenvalue of $(S, X_\eta, \nu)$.

# Invariant measures yield dynamical systems with infinite discrete part in the spectrum

**Proposition (Kułaga-Przymus, L., Weiss, 2014)**

Assume that $\mathcal{P}^e(S, X_\eta) \ni \nu \neq \delta_{(\dots,0,0,\dots)}$. The dynamical system $(S, X_\eta, \nu)$ has an infinite rational discrete spectrum. More precisely, the discrete spectrum part contains, for each $k \geq 1$, all $b'_k$-roots of unity for some $1 < b'_k | b_k$.

**Proof.** It suffices to notice that for $1 \leq s_k \leq b_k - 1$ and $\{a_1^k, \dots, a_{s_k}^k\}$ chosen so that $\nu(Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}) > 0$, and $b'_k$ given above, the partition of $Y_{k,s_k}$ into sets

$$S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k},\ 0 \leq j \leq b'_k - 1$$

is a Rokhlin tower fulfilling the whole space, whence the $b'_k$-root of unity is an eigenvalue of $(S, X_\eta, \nu)$.

# Subshifts of a $\mathscr{B}$-free dynamical system

**Theorem (Kułaga-Przymus, L., Weiss, 2014)**

- Each of the subshifts $\overline{Y}_{\underline{s},\underline{a}}$ is intrinsically ergodic.
- $h_{top}(S, \overline{Y}_{\underline{s},\underline{a}}) = \log 2 \cdot \prod_{k\geq 1} \left(1 - \frac{s_k}{b_k}\right)$.
- If $h_{top}(S, \overline{Y}_{\underline{s},\underline{a}}) = 0$ then $\mathcal{P}(S, \overline{Y}_{\underline{s},\underline{a}}) = \{\delta_{(\dots,0,0,0,\dots)}\}$. In particular, $\mathcal{P}(S, Y_{\underline{s},\underline{a}}) = \emptyset$.
- Let $1 < b'_k | b_k$ for $k \geq 1$. The following are equivalent:
  - (a) there exists a measure $\nu \in \mathcal{P}^e(S, X_\eta)$ such that the rational discrete spectrum of $(S, X_\eta, \nu)$ is equal to all $b'_1 \cdot \ldots \cdot b'_k$-roots of unity
  - (b) $\sum_{k\geq 1} 1/b'_k < +\infty$.
- In particular, no ergodic measure for the square-free subshift yields the dynamical system whose spectrum consists of all $p_1 \cdot \ldots \cdot p_k$-roots of unity, $k \geq 1$.

**Conjecture**

Each subshift of a $\mathscr{B}$-free dynamical system is of number theoretic origin.

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- Let $1 < b_k' | b_k$ for $k \geq 1$. The following are equivalent:
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# Invariant measures for $\mathscr{B}$-free dynamical systems and joinings

$M\colon X_\eta \times \{0,1\}^{\mathbb{Z}} \to X_\eta$, $M(x,u)(n) := x(n)\cdot u(n)$ for $n \in \mathbb{Z}$.

**Theorem (Kułaga-Przymus, L., Weiss, 2014)**

For any $\nu \in \mathcal{P}^e(S, X_\eta)$ there exist a $\mathscr{B}'$-free system and $\widetilde{\rho} \in \mathcal{P}^e(S \times S, X_\eta \times \{0,1\}^{\mathbb{Z}})$ such that $X_{\eta'} \subset X_\eta$, $\widetilde{\rho}|_{X_\eta} = \nu_{\mathscr{B}'}$ and $M_*(\widetilde{\rho}) = \nu$.

- Let $\lambda$ be an ergodic joining of the Mirsky measure $\nu_{\mathscr{B}'}$ of a $\mathscr{B}'$-free subshift contained in $(S, X_\eta)$ and an invariant measure $\kappa$ for the full shift $(S, \{0,1\}^{\mathbb{Z}})$ (this means that $\lambda$ is an $S \times S$-invariant ergodic measure on $X_\eta \times \{0,1\}^{\mathbb{Z}}$ such that $\lambda|_{X_\eta} = \nu_{\mathscr{B}'}$ and $\lambda|_{\{0,1\}^{\mathbb{Z}}} = \kappa$).
- Then the image $M_*(\lambda)$ of $\lambda$ via $M$ belongs to $\mathcal{P}^e(S, X_\eta)$.
- Such a measure is called a joining type measure. The theorem above says that each (ergodic) invariant measure is of joining type.
- When $\lambda = \nu_{\mathscr{B}'} \otimes \kappa$ then $M_*(\lambda)$ is called to be of product type; it is denoted $\nu_{\mathscr{B}'} * \kappa$.

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# Plan

**Proposition (Kułaga-Przymus, L., Weiss, 2014)**

Assume that $\mathscr{B} = \{b_k : k \geq 1\}$ and $\mathscr{B}' = \{b'_k : k \geq 1\}$. If $X_{\mathscr{B}} = X_{\mathscr{B}'}$ then $\mathscr{B} = \mathscr{B}'$.

We assume that $b_1 < b_2 < \ldots$ and $b'_1 < b'_2 < \ldots$.

# If $X_{\mathscr{B}} = X_{\mathscr{B}'}$ then $\mathscr{B} = \mathscr{B}'$

- $(T, \Omega, \mathbb{P})$ and $(T', \Omega', \mathbb{P}')$ the corresponding odometers and $\varphi : \Omega_0 \to X_{\mathscr{B}}$, $\varphi' : \Omega_0' \to X_{\mathscr{B}'}$ the relevant genuine embeddings.
- $\nu_{\mathscr{B}} * B(1/2, 1/2) = \nu_{\mathscr{B}'} * B(1/2, 1/2)$ since both measures are of maximal entropy on $X_\eta = X_{\eta'}$ and $(S, X_\eta)$ is intrinsically ergodic.
- $\varphi'(\Omega_0') \subset Y$ (if $\nu_{\mathscr{B}'}(Y) = 0$ then the hereditary system generated by a support of $\nu_{\mathscr{B}'}$ is disjoint of $Y$; but $M_*(\nu_{\mathscr{B}'} \otimes B(1/2, 1/2))$ is concentrated on it and the measure of maximal entropy is concentrated on $Y$, a contradiction).
- With no loss of generality, we can assume that $\varphi'(\Omega') \subset Y_0 = \varphi(\Omega_0)$ since $\theta_*(\nu_{\mathscr{B}'}) = \mathbb{P}$. Now, for $\omega' \in \Omega_0'$, there exists $\omega \in \Omega_0$ (in fact, $\omega = \theta(\varphi'(\omega'))$) such that $\varphi'(\omega') \le \varphi(\omega)$. Fixing now $\omega$ and reversing the roles, we find $\omega''' \in \Omega_0'$ such that $\varphi(\omega) \le \varphi'(\omega''')$. Thus, $\varphi'(\omega') \le \varphi'(\omega''')$ and then $\omega''' = \omega'$.
- It follows that $\varphi(\Omega_0) = \varphi'(\Omega_0')$.
- $\nu_{\mathscr{B}} = \nu_{\mathscr{B}'}$ (since $\theta_*(\nu_{\mathscr{B}}) = \mathbb{P} = \theta_*(\nu_{\mathscr{B}'})$ and $\theta|_{\varphi(\Omega_0)}$ is 1-1).
- $X_{\mathscr{B}} = X_{\mathscr{B}'} \ \Rightarrow \ b_1 = b_1'$ (indeed, suppose $b_1 < b_1'$; it is enough to notice that the block $C^1_{\{1,\dots,b_1'-1\}} \cap C^0_{\{b_k'\}}$ is $\mathscr{B}'$-admissible, while clearly it is not $\mathscr{B}$-admissible).
- Set $\widetilde{\mathscr{B}} = \mathscr{B} \setminus \{b_1\}$, $\widetilde{\mathscr{B}}' = \mathscr{B}' \setminus \{b_1'\}$.
- For any finite subset $A \subset \mathbb{N}$, $\nu_{\mathscr{B}}(C^1_A) = \left(1 - \frac{|A \bmod b_1|}{b_1}\right) \nu_{\widetilde{\mathscr{B}}}(C^1_A)$ with an analogous formula for $\nu_{\mathscr{B}'}$.
- $\nu_{\widetilde{\mathscr{B}}} = \nu_{\widetilde{\mathscr{B}'}}$.
- $X_{\widetilde{\mathscr{B}}} = X_{\widetilde{\mathscr{B}'}}$ (the Mirsky measure has full topological support).
- $b_2 = b_2'$, and by continuing, we conclude $\mathscr{B} = \mathscr{B}'$.

# Plan

# The Möbius function

Möbius function $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Prime number theorem (PNT) $\iff \frac{1}{N}\sum_{n\le N}\mu(n) \to 0$ when $N \to \infty$.
- $\mu(m \cdot n) = \mu(m) \cdot \mu(n)$ whenever $(m, n) = 1$ ($\mu$ is a multiplicative function).

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# Sarnak's conjecture

**Sarnak's conjecture (2010)**

$X$ - compact metric space, $T\colon X \to X$ homeomorphism of zero topological entropy, $x \in X$, $g \in C(X)$. Then

$$\sum_{n \le N} g(T^n x)\boldsymbol{\mu}(n) = \mathrm{o}(N).$$

**Orthogonality criterion: Katai 1986, Bourgain-Sarnak-Ziegler 2011**

$F\colon \mathbb{N} \to \mathbb{C}$ – a bounded sequence, $\sum_{n \le N} F(rn)\overline{F(sn)} = \mathrm{o}(N)$ for any sufficiently large primes $r \neq s$. Then

$$\sum_{n \le N} F(n)\nu(n) = \mathrm{o}(N)$$

for any multiplicative function $\nu$ with $|\nu| \le 1$.

Given a homeomorphism $T\colon X \to X$ we take

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$$F(n) = g(T^n x) \text{ for } n \in \mathbb{Z}, x \in X \text{ and } g \in C(X).$$

# Topological Möbius dynamical system (Sarnak, 2010)

**Definition of Möbius topological system**

$$\{-1,0,1\}^{\mathbb{Z}} \supset X_M :=$$
$$= \{x \in \{-1,0,1\}^{\mathbb{Z}}; (\forall 1 \leq k < \ell)(\exists r \geq 0)\ \ x(k,\ell) = \boldsymbol{\mu}(r+k, r+\ell)\}.$$

$X_M$ is invariant under $S$; $(S, X_M)$ is called the *topological Möbius system*.

# Square-free factor of the Möbius system

$$\Psi : \{-1,0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}},\ \psi((x(n))_{n\in\mathbb{Z}}) = (x^2(n))_{n\in\mathbb{Z}}.$$

- $\psi \circ S = S \circ \psi$,
- $\psi(X_M) = X_{\boldsymbol{\mu}^2} = X_{\mathscr{B}}$, $\mathscr{B} = \{p_i^2 :\ i \geq 1\}$ (the square-free system is a topological factor of $(S, X_M)$),
- $X_M \subset X_N := \psi^{-1}(X_{\boldsymbol{\mu}^2})$.

**Corollary**

$(S, X_N)$ is intrinsically ergodic.

Recall: $Y_{\underline{s}} = \bigcap_{k \geq 1} Y_{k,s_k}$

- $h_{top}(S, \overline{Y}_{\underline{s}}) = \log 2 \cdot \prod_{k \geq 1} \left(1 - \frac{s_k}{p_k^2}\right) < \log 2 \cdot \frac{6}{\pi^2} = h_{top}(S, Y)$ whenever $\underline{s} \neq (1, 1, \ldots)$,
- $h_{top}(S, \psi^{-1}(\overline{Y}_{\underline{s}})) = \log 3 \cdot \prod_{k \geq 1} \left(1 - \frac{s_k}{p_k^2}\right)$,
- $h_{top}(S, \psi^{-1}(\overline{Y}_{\underline{s}})) < h_{top}(S, \psi^{-1}(Y))$ whenever $\underline{s} \neq (1, 1, \ldots)$,
- If $\rho$ is a measure of maximal entropy for $(S, X_N)$ then $\rho(\psi^{-1}(Y)) = 1$, in particular, $(\theta \circ \psi)_*(\rho) = \mathbb{P}$,
- Repeat the proof of intrinsic ergodicity for $\mathscr{B}$-free systems.

# The Chowla and Sarnak conjectures

**The Chowla conjecture (1960)**

For $0 \leq a_1 < a_2 < \cdots < a_t$, $k_1, \ldots, k_t \in \{1, 2\}$ not all even, $\sum_{n \leq N} \boldsymbol{\mu}^{k_1}(n + a_1)\boldsymbol{\mu}^{k_2}(n + a_2) \ldots \boldsymbol{\mu}^{k_t}(n + a_t) = \mathrm{o}(N)$.

- The Chowla conjecture implies Sarnak's conjecture (Sarnak, 2010).
- The Chowla conjecture is equivalent to the fact that $\boldsymbol{\mu}$ is a generic point for the relatively independent extension of the Mirsky measure of the square-free system.
- The Chowla conjecture implies $X_M = X_N$ (Sarnak, 2010).

**Question**

Is $(S, X_M)$ intrinsically ergodic?

# Anzai skew products

$Tx = x + \alpha$ an irrational rotation on the (additive) circle $\mathbb{T}$,
$\eta : \mathbb{T} \to \mathbb{T}$ continuous

$$T_\eta : \mathbb{T}^2 \to \mathbb{T}^2, \quad T_\eta(x, y) = (x + \alpha, y + \eta(x)).$$

**Question**

Do homeomorphisms $T_\eta$ satisfy Sarnak's conjecture? (all irrational rotations do)

# Analytic cocycles

$f : \mathbb{R} \to \mathbb{R}$, 1-periodic,
$f \in L^2(\mathbb{T})$, $f(x) = \sum \widehat{f}(n)e^{2\pi inx}$
$f$ is not a trigonometric polynomial, $\widehat{f}(0) = 0$

**Theorem (Liu, Sarnak, 2013)**

If $f$ is analytic and such that $|\widehat{f}(n)| \gg e^{-\tau|n|}$ (for some $\tau > 0$) then for each $c \in \mathbb{Z} \setminus \{0\}$, $(x, y) \mapsto (x + \alpha, cx + y + f(x))$ satisfies Sarnak conjecture.

- $f$ very smooth + technical condition $|\widehat{f}(n)| \gg e^{-\tau|n|}$,
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# Smooth cocycles

**Theorem (Kułaga-Przymus, L. 2013)**

If $f \in C^{1+\delta}(\mathbb{T})$, $\delta > 0$ then for a generic $\alpha$ the automorphism $T_{c,f}$ of $\mathbb{T}^2$ given by $T_{c,f}(x,y) = (x+\alpha, y+cx+f(x))$ satisfies Sarnak's conjecture.

# Strategy of the proof

(a) Show that the ergodic components of $T_\eta^r \times T_\eta^s$ are pairwise disjoint closed sets filling the whole space.

(b) Show that the ergodic components are uniquely ergodic.

(c) Find $\mathcal{F} \subset C(\mathbb{T}^2)$: linearly dense, for each $g \in \mathcal{F}$, $g|_I \neq g|_I \circ T_\eta$, $\int g \otimes \overline{g}\, d\rho = 0$ for any $T_\eta^r \times T_\eta^s$-ergodic measure $\rho$, whenever $r \neq s$ are sufficiently large primes.

Leads to study $T_{\psi_{c_1}}$, where $\psi_{c_1}(x) = (\psi^{(r)}(rx), \psi^{(s)}(sx + c_1))$ and $\psi(x) = cx + f(x)$.

## Conditions (a) and (b)

- Plan: show that for a typical $\alpha$ each $T_{\psi_{c_1}}$ is (uniquely) ergodic
- $T_\eta$ is ergodic $\iff$ $\chi \circ \eta \neq \xi/\xi \circ T$ for $\widehat{T} \ni \chi \not\equiv 1$
- Need: $\chi \circ \psi_{c_1}(x) = e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+c_1))}$ is never a coboundary (for a typical $\alpha$), $r, s$ – relatively prime

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## Theorem (Katok, 1980th)

$f \in C^{1+\delta}(\mathbb{T})$, $Tx = x + \alpha$

- $\frac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|} \to 0$,
- $\frac{|\widehat{f}(q_n)|}{\sum_{k=1}^{\infty} |\widehat{f}(kq_n)|} > c > 0$,
- $\frac{|\widehat{f}(q_n)|}{|\widehat{f}(q_n)| + \sum_{k=1}^{\infty} |\widehat{f}(k\frac{r}{s}q_n)|} > c > 0$, whenever $s | q_n$.

Then for $\lambda \in \mathbb{S}^1$, $\lambda e^{2\pi i f(x)}$ is not a coboundary.

## Lemma

Given $f \in C^{1+\delta}(\mathbb{T})$, the above assumptions are true for a typical $\alpha$.

This ends the proof.

# Generic case – smooth Anzai skew products

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(b) Show that the ergodic components are uniquely ergodic.

Need: $\chi \circ \varphi_{c_1}(x) = e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+c_1))}$ is never a coboundary (for a typical $\alpha$), $r, s$ – relatively prime

## Theorem (Kułaga-Przymus, L., 2013)

$f \in C^{1+\delta}(\mathbb{T})$, $Tx = x + \alpha$

- $\dfrac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|} \to 0,$
- $\dfrac{|\widehat{f}(q_n)|}{\sum_{k=1}^{\infty} |\widehat{f}(kq_n)|} > c > 0,$
- $\dfrac{|\widehat{f}(q_n)|}{|\widehat{f}(q_n)| + \sum_{k=1}^{\infty} |\widehat{f}(k\frac{r}{s}q_n)|} > c > 0,$ whenever $s|q_n$.

Then for $\lambda \in \mathbb{S}^1$, $\lambda e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+c_1))}$ is not a coboundary.

## Lemma

Given $f \in C^{1+\delta}(\mathbb{T})$, the above assumptions are true for a typical $\alpha$.

This ends the proof.