# Stable transitivity of Heisenberg group extensions of hyperbolic systems

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# Summary


1. Motivation
2. Background
3. Group extensions
4. Results
5. Another transitivity criterion
6. The Semigroup Problem
7. Computations in nilpotent groups
8. Diophantine result

# One motivation: partial hyperbolicity

## Conjecture (Pugh-Shub, 1995)

*Among the $C^2$ volume-preserving partially hyperbolic diffeomorphisms of a compact manifold, the ergodic ones contain an open and dense set.*

- This has two parts
  - accessibility is open and dense
  - accessibility $\Longrightarrow$ ergodicity
- the second part (non-trivial extension of the Hopf argument) proved by Pugh & Shub (1995, with additional conditions), improved by Burns & Wilkinson (2010)
- the open dense accessibility proven in some cases
- skew extensions of hyperbolic systems are instances of partial hyperbolic systems

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- ergodicity (of an $f$-invariant probability measure $\mu$): the only invariant measurable sets are, up to $\mu$-measure zero, either $\emptyset$ or $X$
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# Hyperbolic basic set

- $f : M \to M$ smooth transformation
- $X \subset M$ a compact invariant set
- $f : X \to X$ is a hyperbolic basic set if
  - $X$ locally maximal
  - $f$ hyperbolic on $X$
  - $f$ transitive
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# Group extensions (a.k.a. skew extensions)

- $f : X \to X$
- $G$ group (G = connected Lie group in this talk)
- $\beta : X \to G$ (cocycle)
- extension of $f$ by $\beta$:
  $f_\beta : X \times G \to X \times G$ given by

$$(x, g) \mapsto (f(x), g\beta(x))$$

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# Q: How typical is transitivity?

- a homeomorphism is transitive if it has a dense orbit (one-sided/two-sided orbit gives same in our setting)

**Theorem**

*For locally compact perfect $X$: a homeomorphism $f : X \to X$ is transitive $\iff$ for any open $U, V \subset X$, there is $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$.*

Obstruction to transitivity of an extension:

**Example**

$\beta : X \to \mathbb{R}$, $\beta(x) \geq 0$ for all $x \in X \implies f_\beta$ not transitive on $X \times \mathbb{R}$

- $\beta$ takes values in a closed semigroup $S \subset G \implies f_\beta$ not transitive
- Same if $\beta$ cohomologous to such a cocycle:
  - $P : X \to G$ continuous
  - $\beta = (P \circ f)^{-1} \beta_0 P$
  - $\beta_0 : X \to S \subset G$, $S$ closed semigroup

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# Conjecture

- from now on all fibers are $G$ connected Lie group
- consider its proper closed semigroups
- $\mathcal{G}$ = the set of Hölder cocycles that are not cohomologous to a cocycle with values in such a semigroup

**Conjecture (Melbourne-N-Török = MNT)**

*Among the ($C^0$-small) cocycles in $\mathcal{G}$, there is a Hölder open and dense set of transitive cocycles.*

- $\beta$ is $C^r$-small $\iff$ $\beta$ is $C^r$-close to the identity of the group
- Conjecture true for compact extensions; no obstruction to avoid, no need for smallness

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# $G$ compact

Conjecture solved

- no obstructions to avoid
- Brin (1975): $f$ Anosov map $\implies C^1$-open, $C^r$-dense set of transitive cocycles
  (accessibility & recurrence $\implies$ transitivity)
- Adler, Kitchens & Shub (1996)
- Parry & Pollicott (1997)
- Field & Parry (1999)
- Burns & Wilkinson (1999)
- Field & N (2001)
- Field, Melbourne & Török (2005): over any hyperbolic basic set; $C^2$-open and $C^r$-dense within $C^r$ cocycles for $r \geq 2$

# Non-compact $G$ in general

MNT (2005): $r > 0$

- Existence: Any $G$ has a transitive $C^r$ cocycle ($C^r$-small)
- Open set:
  - $G$ perfect
  - $C := \{g \in G \mid g$ generates a compact subgroup in $G\}$ has non-empty interior

  $\Longrightarrow$ there is a Hölder-open set of transitive $C^r$ cocycles

  (e.g., $G = Sp(2n, \mathbb{R})$, $n \geq 1$)
- Open-dense set:
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# Non-compact $G$ in general

MNT (2005): $r > 0$

- Existence: Any $G$ has a transitive $C^r$ cocycle ($C^r$-small)
- Open set:
  - $G$ perfect
  - $\mathcal{C} := \{g \in G \mid g \text{ generates a compact subgroup in } G\}$ has non-empty interior

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- $\mathcal{H}_n \cong \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$ identified with matrices via

$$(a, b, c) := \begin{pmatrix} 1 & a^T & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{n+2}(\mathbb{R}).$$

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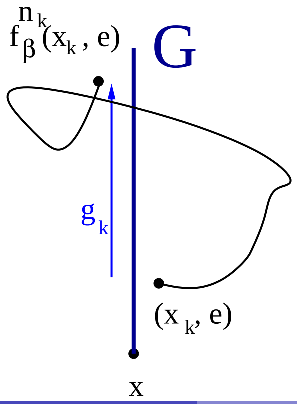
# Method: the semigroup $\mathcal{L}_\beta(x)$

- Brin (1975): accessibility & recurrence $\implies$ transitivity
- for non-compact fiber, recurrence is the main difficulty

**Definition** $\mathcal{L}_\beta(x) := \{g \in G \mid$ there exist $x_k \in X, n_k > 0$ such that $x_k \to x$ and $f_\beta^{n_k}(x_k, e) \to (x, g)\}$.



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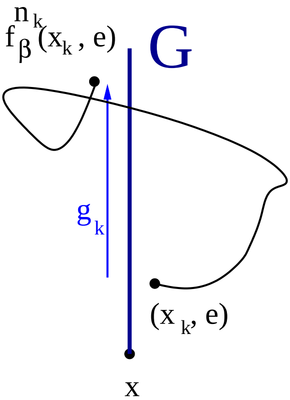

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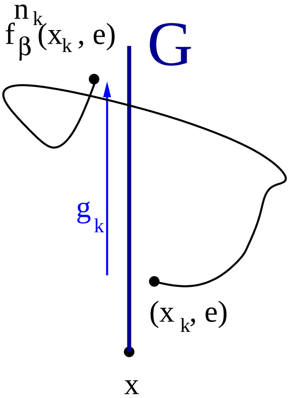


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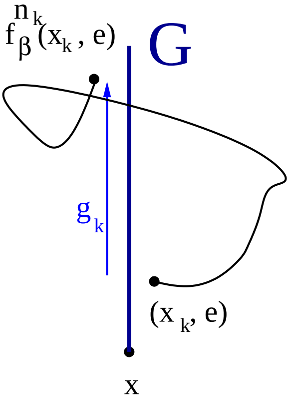


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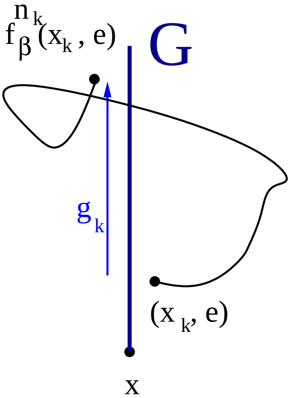


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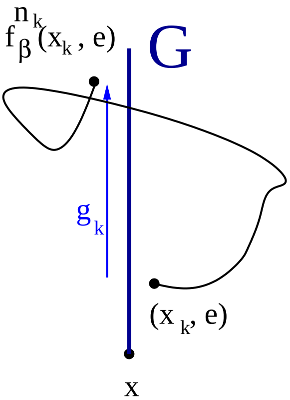


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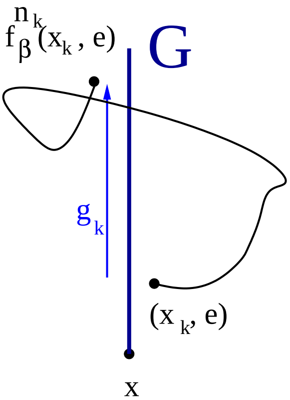


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# Transitivity criterion

$G$ = compact $\ltimes$ nilpotent, $\beta : X \to G$ Hölder

**Lemma**

*$\mathcal{L}_\beta(x) \subset G$ is a (closed) semigroup for each $x \in X$.*

**Theorem**

*there is $x_0$ such that $\mathcal{L}_\beta(x_0) = G \implies f_\beta$ is transitive.*
*[$\Longleftarrow$ obviously true]*

- Consequence: for some $x \in X$
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# The Semigroup Problem

- Question: given $F \subset G$, find conditions under which it (typically) generates $G$ as a semigroup
- Example: $F \subset \mathbb{R}^n$ not on one side of a hyperplane $\implies$

  closed **semigroup** generated by $F$ = closed **group** generated by $F$

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*Assume $S \subset \mathcal{H}_n$ a semigroup.*

- *projection of $S$ to $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ not on one side of a hyperplane $\implies$ closure of $S$ is a group*
- *In particular:*
  *projection of $S$ dense in $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ $\implies$ closure of $S$ is $\mathcal{H}_n$*

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$$\mathcal{L}_\beta(x) \text{ has dense image in } \mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$$

# The Semigroup Problem

- Question: given $F \subset G$, find conditions under which it (typically) generates $G$ as a semigroup
- Example: $F \subset \mathbb{R}^n$ not on one side of a hyperplane $\Longrightarrow$

  closed **semigroup** generated by $F$ = closed **group** generated by $F$

**Theorem (semigroup problem for $\mathcal{H}_n$)**

*Assume $S \subset \mathcal{H}_n$ a semigroup.*

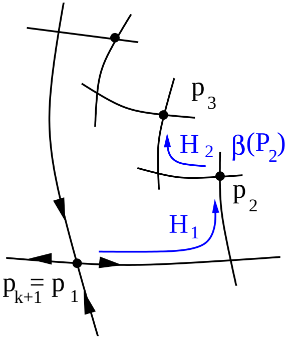
- *projection of $S$ to $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ not on one side of a hyperplane $\Longrightarrow$ closure of $S$ is a group*
- *In particular:*
  *projection of $S$ dense in $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ $\Longrightarrow$ closure of $S$ is $\mathcal{H}_n$*

- Consequence: enough to show that

$$\mathcal{L}_\beta(x) \text{ has dense image in } \mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$$

# Heteroclinic cycles

- Let $p_1 \in X$ be a periodic point. We will build elements in $\mathcal{L}_\beta(p_1)$.
- Pick periodic points $p_2, \ldots, p_k \in X$, take a heteroclinic cycle they determine




- $H_j$ = holonomy from $p_j$ to $p_{j+1}$
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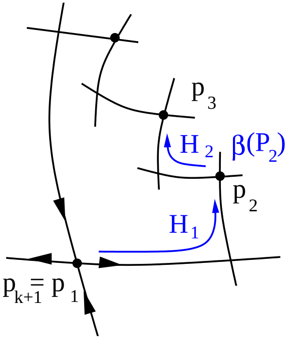


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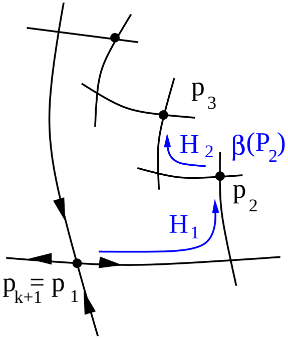


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# Element of $\mathcal{L}_\beta(p_1)$

**Theorem (holds for nilpotent groups, $K \ltimes \mathbb{R}^n$)**

*Let* $\boldsymbol{M} = (M_1, \ldots, M_{k+1}) \in \mathbb{N}^{k+1}$*. Define*

$$A(\boldsymbol{M}) = \beta(P_1)^{M_1} H_1\, \beta(P_2)^{2M_2} H_2 \cdots \beta(P_k)^{2M_k} H_k\, \beta(P_1)^{M_{k+1}}.$$

*If the limit* $A = \lim_{N\to\infty} A(\boldsymbol{M})$ *exists along an admissible sequence* $\boldsymbol{M}(1), \boldsymbol{M}(2), \ldots$*, then* $A \in \mathcal{L}_\beta(p_1)$*.*

(Sequence is admissible if all $M_i/M_j$ stay bounded.)

Idea of proof.

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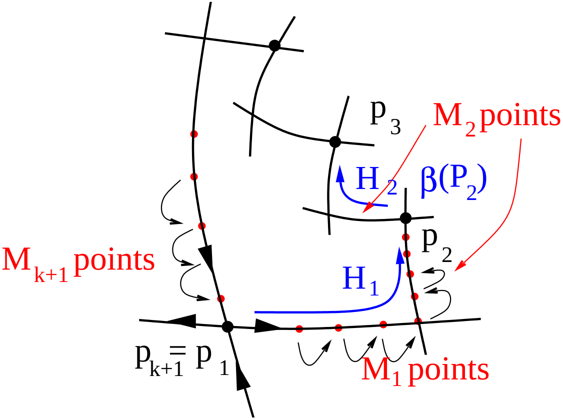
# Pseudo-orbit

For simplicity, assume $p_j$ are fixed points.



# Computations in $N$ nilpotent

- $Lie(N) \cong N$ via exp $\Longrightarrow$ product on $Lie(N)$
  (Baker-Campbell-Hausdorff formula)
- for $N$ step-2 nilpotent

$$X * Y = X + Y + \frac{1}{2}[X, Y]$$

- then

$$X^n = nX$$

$$X^{-1} = -X$$

$$X * Y * X^{-1} * Y^{-1} = [X, Y]$$

# The products $A(\boldsymbol{M})$

- for $X \in Lie(\mathcal{H}_n)$
  - $\pi(X)$ image in $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$
  - $\zeta(X)$ center component
- $A(\boldsymbol{M}) = (L(\boldsymbol{M}) + E, Q(\boldsymbol{M}) + L_Z(\boldsymbol{M}) + e) \in \mathbb{R}^{2n} \oplus \mathbb{R}$
  - $L(\boldsymbol{M}) = \sum_i M_i \pi(A_i) \in \mathbb{R}^{2n}$ linear
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  - the holonomies $H_i$ appear only in $L_Z$ and $E$, $e$
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- GOOD
  - the $\mathbb{R}^{2n}$-part mostly determined by the periodic data of $\hat{\beta} : X \to \mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$, which is transitive
  - in $L_Z$ have $\zeta(A_i)$ to combat the unknown holonomies
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# Diophantine result (to select $\boldsymbol{M}$)

**Theorem** *Assume, on $\mathbb{R}^d$:*

- *one (homogeneous) quadratic form $Q$*
- *$k$ (homogeneous) linear forms $L_1, L_2, \ldots, L_k$*
- *$Q|_{\cap \text{Ker } L_i}$ indefinite*
- rank $Q \geq 2k + 3$ *(number of squares in a diagonal form)*

*Then for a residual, full measure set of vectors*

$$\mathbf{v} \in \{Q = 0\} \cap \{L_i = 0, 1 \leq i \leq k\}$$

*for any $\varepsilon > 0$ there are $\mathbf{x}_n \in \mathbb{Z}^d$ such that:*

1. $\|\mathbf{x}_n\| \to \infty$,
2. $\text{dist}(\mathbf{x}_n, \mathbb{R}_+\mathbf{v}) \leq \varepsilon$ *(so $\mathbf{x}_n \to \infty$ along $\mathbb{R}_+\mathbf{v}$)*
3. $\sup |Q(\mathbf{x}_n)| < \infty$

*In particular, $|L_i(\mathbf{x}_n)| \leq C\varepsilon$, for all $1 \leq i \leq k$ and all $n$*

# How this helps

- $\hat{\beta} : X \to \mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ transitive
  $\implies$ its periodic data is dense in $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$, etc.
- can arrange for the periodic data of $\beta$ to be large in the center component (to overcome the holonomies)
- can pick heteroclinic cycle to satisfy hypotheses of Diophantine result
- pick periodic points to generate an $\varepsilon$-dense subgroup of $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$
- set up a heteroclinic cycle
- Diophantine result $\implies$ can get $A(\boldsymbol{M})$ to converge, to an $\varepsilon$-prescribed $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$-part
- repeat for smaller $\varepsilon$
- get $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ covered densely by $\mathcal{L}_\beta$
- $\implies \mathcal{L}_\beta = \mathcal{H}_n$ (by Semigroup Theorem)
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- set up a heteroclinic cycle
- Diophantine result $\implies$ can get $A(\boldsymbol{M})$ to converge, to an $\varepsilon$-prescribed $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$-part
- repeat for smaller $\varepsilon$
- get $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$ covered densely by $\mathcal{L}_\beta$
- $\implies \mathcal{L}_\beta = \mathcal{H}_n$ (by Semigroup Theorem)
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  $\implies$ its periodic data is dense in $\mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n]$, etc.
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