Free Energy and Complexity of Bipartite Spin Glasses

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Spin Glasses and Related Topics
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Sherrington-Kirkpatrick Model

For $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, +1\}^N$, Hamiltonian given by

$$H(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j} J_{ij} \sigma_i \sigma_j .$$

where $(J)_{ij}$ are i.i.d. standard Gaussians.

Mean field: all possible pair interactions between the N spins.

Mixed p-spin

H is a centered Gaussian process on $\{-1,+1\}^N$ with covariance given by

$$\mathbb{E}H(\sigma^a)H(\sigma^b) = N\xi\left(\frac{1}{N}\sum_{1\le i\le N}\sigma_i^a\sigma_i^b\right)$$

SK: $\xi(t) = \beta_2 t^2$
Pure p-spin: $\xi(t) = \beta_p t^p$.

These are actually "boundary" models of the mixed p-spin, when

$$\xi(t) = \sum_{p\ge 2}\beta_p t^p, \quad \beta_p \ge 0.$$

Temperature can be encoded in ξ through $\boldsymbol{\beta} = (\beta_2, \beta_3, \beta_4, \ldots)$.

The Gibbs measure

Here our goal is (1) to understand the sequence of Gibbs measures as a function of the temperature parameter when $N \to \infty$.

$$G_N(\sigma) = \frac{1}{Z_N} \exp\Big(- H_N(\sigma) \Big),$$

where the partition function

$$Z_N = \sum_{\sigma} \exp\Big(- H_N(\sigma) \Big).$$

And (2) determine limit theorems for the ground state energy

$$E_N := \min_{\sigma} H_N(\sigma).$$

The limit free energy - Parisi Formula

Theorem (M. Talagrand '06, D. Panchenko '10)

For all β the following limit exists a.s.:

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = \mathcal{P}(\beta).$$

- $\mathcal{P}(\beta)$ is given by a variational principle:

$$\mathcal{P}(\beta) = \min_{\mu\in\mathcal{M}} \left(\Phi_\mu(0,0) - \frac{1}{2}\int_0^1 s\xi''(s)\mu([0,s])ds \right).$$

  where Φ_μ solves in $(s,x) \in [0,1]\times\mathbb{R}$

$$\partial_s \Phi_\mu(s,x) = -\frac{\xi''(s)}{2}\left(\partial_x^2 \Phi_\mu(s,x) + \mu([0,s])\left(\partial_x \Phi_\mu(s,x)\right)^2\right),$$

  with terminal condition $\Phi_\mu(1,x) = \log\cosh x$.
- Understand $\mathcal{P}(\beta)$. What is the minimizer? What is the role of the minimizer?

The minimizer of the Parisi Functional

Parisi measure.

We call a measure that minimizes the previous variational problem a *Parisi measure*. We denote it by μ_P.

It is expected that:

- Uniqueness. For each ξ there exists a unique $\mu_P = \mu_P(\xi, \beta)$.
- Up to symmetry, μ_P is the limiting law of the overlap under $\mathbb{E}G_N^2$.
- The limit law of overlap fully describes the measure G_N as N goes to infinity.

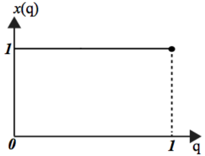
Important properties of μ_P:

- Decide if μ_P is atomic or not.
- If atomic, number of atoms.
- Minimum and maximum of its support.

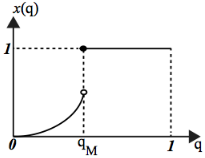
Predictions

$$x(q)=\mu[0,q]$$

SK model:

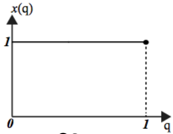


Replica Symmetric

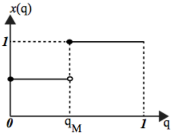


Full Replica Symmetry Breaking

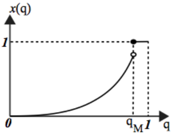
p-spin (p>2):



RS



1-RSB



FRSB

Ground State Energy

The ground state energy can be obtained through the bounds:

$$\frac{1}{N\beta}\log Z_N(\beta) - \frac{\log 2}{\beta} \le \min_{\sigma} H_N(\sigma) \le \frac{1}{\beta N}\log Z_N(\beta).$$

A second computation - Complexity

Fix $u \in \mathbb{R}$. Let

$$A(u) = \left\{ \sigma \in \{-1,+1\}^N : H(\sigma) \sim Nu \right\}.$$

We want to determine the size of of the set $A(u)$ and understand its relation to the Parisi Formula.

Waving hands big time:

$$Z_N(\beta) = \sum_{\sigma} \exp(-\beta H_N(\sigma) = \sum_{u} \exp(-\beta Nu)|A(u)|.$$

So if the size of $A(u) \sim \exp N\Theta(u)$ we get

$$\mathcal{P}(\beta) = \sup_{u}\{\Theta(u) - \beta u\}.$$

Difficult. But two ways to get there.
1) Spherical Model. 2) TAP Equations.

Complexity in the Spherical Model

Consider the same Hamiltonian but now on the configuration space

$$\sigma = (\sigma_1, \ldots, \sigma_N)$$

with

$$\sum_{i=1}^{N} \sigma_i^2 = N.$$

- Hypercube is a subset.
- Parisi formula still valid.
- H is now a random smooth function.

Complexity in the Spherical Model

Let

$$\mathrm{Crt}_0(u) = \#\Big\{\sigma : H(\sigma) \le u, \nabla H(\sigma) = 0, i(\nabla^2 H) = 0\Big\}.$$

Theorem (A., Ben Arous, Cerny '11, A., Ben Arous '12)

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\,\mathrm{Crt}_0(u) = \Theta(u).$$

The function $\Theta(u)$ is called the complexity (of local minima) of the model.

- Similar result for the number of critical points of index k.
- p-spin: predicted by Crisanti-Sommers.

First connection between Complextity and Free Energy

How to get Θ from $\mathcal{P}(\beta)$:

- Suppose the model is 1-RSB for $\beta > \beta_0$. (True for spherical p-spins, expected for some mixtures.) Let $\mu = m\delta_0 + (1-m)\delta_q$.
- For $m \in [0,1]$ set $\mathcal{P}(m) = \inf_q \mathcal{P}(\mu)$ and let $f_1(m) = \lim_{\beta\to\infty} \frac{1}{\beta}\mathcal{P}(\beta m)$.

Theorem (A. - Ben Arous '12)

$$\Theta(u) = \min_m \{um - mf_1(m)\}$$

- The duality between u and m was already predicted in the work of several physicists.

A finite temperature connection - TAP Solutions

Consider the TAP functional for the p-spin:

$$F_{\mathrm{TAP}}(\boldsymbol{m}) = \frac{1}{2^{1/2}N^{(p+1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} m_{i_1} \cdots m_{i_p} + B_{p,\beta}(q),$$

where, as usual, $J_{i_1,\ldots,i_p}$ are independent standard normal random variables and

$$B(q) = B_{p,\beta}(q) = -\frac{1}{2\beta}\log(1-q) - \frac{\beta}{4}\left(1 + (p-1)q^p - pq^{p-1}\right).$$

where $q = \frac{1}{N}\sum_i m_i^2$.

A finite temperature connection - TAP Solutions

The TAP equations are equations for critical points of the TAP functional,

$$\frac{\partial}{\partial m_i} F_{\text{TAP}}(\boldsymbol{m}) = 0.$$

- They are supposed to be satisfied if $m_i = \langle \sigma_i \rangle$ in the limit as N goes to infinity.
- Unique solution if β small.
- If β is sufficiently large the number of solutions should be exponentially large.
- True if one consider the average number of solutions $\mathcal{N}(u, \beta)$ at $F_{TAP} = Nu$.

Theorem (A. - Ben Arous)

$\exists \beta_c$ such that

$$\mathbb{E}\mathcal{N}(u, \beta) \sim \exp \Sigma(\beta, u) \quad \textit{with } \Sigma(\beta, u) > 0$$

if and only if $\beta > \beta_c$ and $u \in [a(\beta), b(\beta)]$.

- Now, for any temperature that satisfies $\beta > \beta_c$, the previous duality holds without taking $\beta \to \infty$:

$$\Sigma(\beta, u) = \min_m \{um - m\mathcal{P}_1(\beta^{-1}m)\}$$

- Duality is predicted even if FRSB and on the Hypercube.

- Bray-Moore '77, Crisanti-Parisi-Leuzzi-Rizzo '02.

Bipartite model

Divide the particles into 2 groups of size N_1 and N_2. Interactions only among pair of different groups.
Model is defined on $\{\pm 1\}^{N_1} \times \{\pm 1\}^{N_2}$ with

- $N_1 + N_2 = N$
- $N_1/N \to \gamma \in [0, 1]$.
- $$H(\sigma, \tau) = \frac{1}{\sqrt{N}} \sum_{1 \le i \le N_1} \sum_{1 \le j \le N_2} J_{ij} \sigma_i \tau_j$$
- $$EH(\sigma, \tau)H(\sigma', \tau') = N R_{N_1}(\sigma, \sigma') R_{N_2}(\tau, \tau')$$
- Guerra and coauthors (2010, 2012, 2013).

Mixed p,q-spin bipartite

As before, one considers a larger family of Gaussian processes:

$$H_{p,q}(\sigma,\tau) = \sum_{1\le i_1,\dots,i_p\le N_1}\ \sum_{1\le j_1,\dots,j_q\le N_2} J_{i_1,\dots i_p,j_1,\dots,j_q}\sigma_{i_1}\dots\sigma_{i_p}\tau_{j_1}\dots\tau_{j_q}$$

where all J variables are centered i.i.d. $\mathcal{N}(0, N/(N_1^p N_2^q))$.

Covariance:

$$\mathbb{E}H_{p,q}(\sigma^a,\tau^a)H_{p,q}(\sigma^b,\tau^b) = N(R_{N_1}(\sigma^a,\sigma^b))^p(R_{N_2}(\tau^a,\tau^b))^q.$$

The Hamiltonian of the mixed-spin bipartite spherical model is defined as

$$H_N(\sigma,\tau) = \sum_{p,q\ge 1}\beta_{p,q}H_{N,p,q}(\sigma,\tau).$$

$$\mathbb{E}H_N(\sigma^a,\tau^a)H_N(\sigma^b,\tau^b) = N\xi(R_{N_1}(\sigma^a,\sigma^b), R_{N_2}(\tau^a,\tau^b)).$$

Free Energy - Spherical Model

Define $P : [0,1)^2 \to \mathbb{R}$ by

$$P(a,b) = \frac{\gamma}{2}\left(h_1^2(1-a) + \frac{a}{1-a} + \log(1-a) + \xi(1,1) - \xi(a,b)\right)$$
$$+ \frac{1-\gamma}{2}\left(h_2^2(1-b) + \frac{b}{1-b} + \log(1-b) + \xi(1,1) - \xi(a,b)\right).$$

Theorem (A.-Chen, '14)

There exists $M_0 > 0$ such that whenever $\xi(1,1), |h_1|, |h_2| < M_0$.

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = \min_{a,b\in[0,1)} P(a,b).$$

- Minimum attained at the solution (a_0, b_0) of

$$h_1^2 + \frac{\partial_x \xi(a_0,b_0)}{\gamma} = \frac{a_0}{(1-a_0)^2}, \quad h_2^2 + \frac{\partial_y \xi(a_0,b_0)}{1-\gamma} = \frac{b_0}{(1-b_0)^2}.$$

Comments on the free energy.

- High temperature only.
- $P(a, b)$ is the interpolation of two Crisanti-Sommers solutions for the spherical mixed model.
- The matrix $\mathbb{E}J_{ij}^2$ is not positive definite and does not allow us to obtain a Guerra's bound, nor Ghirlanda-Guerra Identities for the free Energy. New ideas to try to analyze the low temperature region.
- On the hypercube, Barra-Genovese-Guerra produced a limiting free energy that has a min-max representation (still at high temperature).
- Phase diagram unknown. Full Replica Symmetry Breaking expected.
- The overlap of each parties is also concentrated at the minimum (a_0, b_0).

Complexity of the bipartite model

Recall the number of local minima:

$$\mathrm{Crt}_{N,0}(u) = \sum_{(\sigma,\tau):\nabla H_N(\sigma,\tau)=0} \mathbf{1}\{H_N(\sigma,\tau) \le Nu\}\mathbf{1}\{i(\nabla^2 H_N(\sigma,\tau)) = 0\},$$

Theorem (A.-Chen '14)

Assume that $\xi(x,y) \neq \beta xy$. *There exist continuous functions* $J, K : \mathbb{R} \to \mathbb{R}$ *such that*

$$J(t) \le \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\, \mathrm{Crt}_{N,0}(t) \le K(t).$$

Furthermore, $(\mathit{Im}\ J) \cap (0,\infty) \neq \emptyset$ *and* $\lim_{t\to-\infty} K(t) = -\infty$.

Idea of the Proofs for the bounds on the complexity

One starts with the Kac-Rice formula:

$$\mathbb{E}\,\mathrm{Crt}_{N,0}(u) = V_{N_1} V_{N_2} \int_{S^{N_1} \times S^{N_2}} \zeta(\sigma,\tau) d\lambda_{N_1} \times \lambda_{N_2}(\sigma,\tau).$$

where

$$\zeta(\sigma,\tau) =$$
$$c_\xi(N)\mathbb{E}\Big[|\det \nabla^2 H(\sigma,\tau)| \mathbf{1}\{H(\sigma,\tau) \le Nu, i(\nabla^2 H(\sigma,\tau)) = 0\} \,\Big|\, \nabla H(\sigma,\tau) = 0 \Big].$$

Upper bound on $\mathbb{E}\,\mathrm{Crt}$.

Upper bound turns out to be simple to obtain:

- Fischer's inequality says that if

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

  is positive definite, then

$$\det \mathcal{A} \leq \det A \det C.$$

- The upper bound $K(t)$ is just a linear combination of two complexity of mixed spherical models.

Lower bound on $\mathbb{E}\,\mathrm{Crt}$

Lower bound turns out to be more involved.

- For $a \in [0,1]$ let

$$H_1(a) = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} - a \begin{pmatrix} \frac{\partial_x \xi(1,1) u}{N_1} I_{N_1} & 0 \\ 0 & \frac{\partial_y \xi(1,1) u}{N_2} I_{N_2} \end{pmatrix},$$

$$H_2(a) = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix} - (1-a) \begin{pmatrix} \frac{\partial_x \xi(1,1) u}{N_1} I_{N_1} & 0 \\ 0 & \frac{\partial_y \xi(1,1) u}{N_2} I_{N_2} \end{pmatrix}.$$

- We use Minkowski determinant inequality:

$$|\det(A+B)| \mathbf{1}_{\{A \geq 0\}} \mathbf{1}_{\{B \geq 0\}} \geq (\det A + \det B) \mathbf{1}_{\{A \geq 0\}} \mathbf{1}_{\{B \geq 0\}}$$

Lower bound - ctd.

- The matrix $H_1(a)$ can be controlled again as in the mixed p-spin.
- To control the out diagonal matrix $H_2(a)$ one needs to insure it is positive definite. Let $\gamma_* = \max\{\gamma(1-\gamma)^{-1}, \gamma^{-1}(1-\gamma)\}$. If

$$(1-a)\min\left\{\frac{\xi_1' u}{\gamma}, \frac{\xi_2' u}{(1-\gamma)}\right\} < -(1+\sqrt{\gamma_*})$$

  then $\lim_{N\to\infty} \mathbb{P}(H_2(a) \geq 0) = 1$.
- And then optimize over the choice of u and a.

Thank You

Thank you.