High-dimensional landscapes

and random matrices[1]

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[1] Based on: **YVF** *arXiv:1307.2379* ; see also **YVF** & **P. Le Doussal** *J. Stat Phys.* **154** (2014), 466-490; **YVF** & **C. Nadal** *PRL* **109** (2012), 167203; **YVF** & **B.A. Khoruzhenko**, in progress.; **YVF**, **A. Lerario** & **E. Lundberg** *arXiv:1404.5349* ;

Counting Stationary points for Isotropic Gaussian Landscapes:

In the last decade there was a steady progress in counting $\&$ classifying the **mean number** of **stationary points** of smooth Gaussian random fields $V(\mathbf{x})$ on N-dimensional manifolds.

As is well-known the **total number** of stationary points in a domain D of $\mathbb{R}^N$ is given formally by the multidimensional **Kac-Rice** integral of the type:

$$\mathcal{N}_s(D) = \int_D \rho_s(\mathbf{x})\, d\mathbf{x}, \quad \rho_s(\mathbf{x}) = |\det\left(\partial^2_{k_1 k_2} V(\mathbf{x})\right)| \prod_{k=1}^N \delta\left(\partial_k V(\mathbf{x})\right)$$

and similarly for the **number of minima** $\mathcal{N}_m(D) = \int_D \rho_m(\mathbf{x})\, d\mathbf{x}$, with

$$\rho_m(\mathbf{x}) = |\det\left(\partial^2_{k_1 k_2} V(\mathbf{x})\right)| \theta\left(\partial^2_{k_1 k_2} V(\mathbf{x})\right) \prod_{k=1}^N \delta\left(\partial_k V(\mathbf{x})\right)$$

where $\theta(H) = 1$ for positive (semi)definite H, and zero otherwise.
Thus the problem amounts to studying the statistics of random **Hessian** matrix $H_{k_1 k_2} = \partial^2_{k_1 k_2} V(\mathbf{x})$ and random **gradient** vector $v_k = \partial_k V(\mathbf{x})$ with the discovered possibility of explicit evaluation of the expected values of those integrals:

YVF 2004; **Bray** $\&$ **Dean** 2007; **YVF** $\&$ **Williams** 2007; **Auffinger, Ben Arous** $\&$ **Cerny** 2011; **Auffinger, Ben Arous** 2012; **Nicolaescu** 2012-13 ; **YVF** $\&$ **Nadal** 2012; **YVF** $\&$ **Le Doussal** 2013

In particular, in this way explicit *finite-N* formulae for the **mean total number $\mathcal{N}_{tot}$ of stationary points** and the **mean number of minima $\mathcal{N}_{min}$** have been derived for arbitrary **isotropic** Gaussian field $V(\mathbf{x})$ on the sphere $|\mathbf{x}| = R$ such that

$$\mathbb{E}\left\{V(\mathbf{x})V(\mathbf{x}')\right\} = F(\mathbf{x}\cdot\mathbf{x}')$$

Those quantities turn out to be determined by the only control parameter

$$B = \frac{R^2 \cdot F''(R^2) - F'(R^2)}{R^2 \cdot F''(R^2) + F'(R^2)}$$

and evaluation of integrals can be directly related to the statistics of eigenvalues $\lambda_1, \ldots, \lambda_N$ of **random GOE** matrices H such that $\mathcal{P}(H) \propto \exp\left(-\frac{N}{4}\text{Tr}H^2\right)$. Namely:

$$\mathcal{N}_{tot} = 2N\left(\frac{1+B}{1-B}\right)^{N/2}\sqrt{1-B}\int_{-\infty}^{\infty} e^{-NB\frac{t^2}{2}}\rho_N(t)\,dt$$

$$\mathcal{N}_{min} = 2\left(\frac{1+B}{1-B}\right)^{N/2}\sqrt{1-B}\int_{-\infty}^{\infty} e^{-NB\frac{t^2}{2}}\frac{d\mathcal{F}_N}{dt}\,dt$$

Here $\rho_N(t) = \mathbb{E}\left\{\frac{1}{N}\sum_i \delta(t-\lambda_i)\right\}$ is the **mean eigenvalue density** and $\mathcal{F}_N(t) = \text{Prob}\left\{\lambda_{max} \le t\right\}$ is the distribution of the **maximal GOE eigenvalue**: $\lambda_{max} = \max(\lambda_1, \ldots, \lambda_N)$.

One can also give similar formulae for the mean number of stationary points of any **index**, and even specify their number to a given **height** of the landscape function (see **Auffinger** $\&$ **Ben Arous** 2011 $\&$ 2012)

High-dimensional landscapes: asymptotics for $N \gg 1$:

In the limit of large dimension $N \gg 1$ of the manifold we can employ the known asymptotics of the **mean eigenvalue density** (see e.g. **Forrester** '12):

$$\rho_{N\gg 1}(t) \approx \begin{cases} \frac{1}{\pi}\sqrt{2-t^2}, & |t| < \sqrt{2} \\ \frac{1}{2\sqrt{\pi N}} \frac{e^{-N\psi_+(t)}}{(t^2-2)^{1/4}(t+\sqrt{t^2-2})^{1/2}}, & t > \sqrt{2} \end{cases}$$

where $\psi_+ = \frac{t}{2}\sqrt{t^2-2} - \ln\frac{t+\sqrt{t^2-2}}{\sqrt{2}}$ and similar asymptotics of the density of the maximal GOE eigenvalue (**Borot et al.** '11). This yields leading $N \gg 1$ behaviour:

$$\mathcal{N}_{tot} \approx \begin{cases} 2, & B < 0 \\ 4N^{1/2}\sqrt{\frac{1-B}{\pi B}}\, e^{N\Sigma_{tot}(B)}, & B > 0 \end{cases}$$

where $\Sigma_{tot}(B) = \frac{1}{2}\ln\frac{1+B}{1-B} > 0$ for $B > 0$.
Simialrly, the mean number of **minima** is given asymptotically by $\mathcal{N}_{min}|_{N\gg 1} \approx 1$ for $B < 0$ and

$$\mathcal{N}_{min}|_{N\gg 1} = C_N(B)\, e^{N\Sigma_m(B)}, \quad \Sigma_m(B>0) = \frac{1}{2}\ln\frac{1+B}{1-B} - B > 0\,.$$

where the factor is given explicitly by

$$C_N(B) = const \cdot N^{-\frac{17}{36}}\, B^{\frac{23}{32}}\, \sqrt{1-B}\, e^{\frac{4\sqrt{2}}{3}N^{1/2}B^{3/2}}\,. \tag{1}$$

Abrupt topology detrivialization at a zero-temperature glass transition:

Consider the energy landscape given by the sum of random **linear form** and **polynomial** $p-$form

$$\mathcal{E}_h(\mathbf{x}) = \sum_{1\le i_1\le \ldots i_p \le N} J_{i_1\ldots i_p} x_{i_1}\ldots x_{i_p} + \sum_{i=1}^N h_i x_i,$$

over the $N-1$ dimensional **sphere** $x_1^2+\ldots x_N^2 = N$, assuming all J's and h_i are independent mean-zero real random Gaussian variables with the variances $\mathbb{E}\left\{h_i^2\right\} = \sigma^2$ and $\mathbb{E}\left\{J_{i_1\ldots i_p}^2\right\} = \frac{J^2}{pN^{p-1}}$. The problem is equivalent to studying the (zero-temperature) energy landscape of a **spherical** $p-$**spin glass** model.

It is easy to show that the main parameter B controlling the number of stationary points in this model is given by

$$B = \frac{J^2(p-2)-\sigma^2}{J^2 p+\sigma^2} \in (-1, \frac{p-2}{2})$$

This implies that as long as $p>2$ there exists a drastic change in the asymptotics of both $\mathcal{N}_{tot}$ and $\mathcal{N}_{min}$ at $\sigma=\sigma_c=J\sqrt{p-2}$, that is precisely at the point of (zero-temperature) **one-step RSB transition**. We conclude that such transition is associated with the **abrupt change** in the landscape topology: for $\sigma>\sigma_c$ the landscape is **topologically trivial**, with a single minimum and a single maximum, but for $\sigma<\sigma_c$ there abruptly emerges **exponentially many** minima.

Landscape topology detrivialization - critical crossover :

So far we have considered a fixed $B \neq 0$ and $N \to \infty$. In the properly defined **scaling vicinity** of the transition we may expect a non-trivial crossover behaviour which frequently turns out to be **universal**, i.e. insensitive to detail of the model.
The analysis shows that for small **negative** $B < 0$ the appropriate scaling is $B = -\frac{\kappa}{2N^{1/3}} \ll 1$ where $\kappa > 0$ is of order unity. The dominant contribution to the integral comes from the domain close to the spectral edge $|t - \sqrt{2}| \sim N^{-2/3}$. Accurate evaluation then gives:

$$\mathcal{N}_{tot}(\kappa) = 4e^{-\kappa^3/24} \int_{-\infty}^{\infty} e^{\frac{\kappa}{2}\zeta} \rho_{edge}(\zeta) d\zeta, \quad \kappa = -2B\, N^{1/3}$$

where

$$\rho_{edge}(\zeta) = \left[Ai'(\zeta)\right]^2 - \zeta \left[Ai(\zeta)\right]^2 + \frac{1}{2} Ai(\zeta) \left(1 - \int_{\zeta}^{\infty} Ai(\eta)\, d\eta\right)$$

with $Ai(\zeta) = \frac{1}{2\pi i} \int_{\Gamma} e^{\frac{v^3}{3} - v\zeta}$ being the **Airy function**. One easily shows that $\lim_{\kappa \to \infty} \mathcal{N}_{tot}(\kappa) = 2$ matching the $B < 0$, $N \to \infty$ result. On the other hand $\mathcal{N}_{tot}(\kappa \ll 1) \propto \kappa^{-3/2}$ indicating the **trend to essential growth** in the number of stationary points for $\kappa \to 0$. In the same **scaling limit** we also find

$$\mathcal{N}_{min}(\kappa) = 2e^{-\kappa^3/24} \int_{-\infty}^{\infty} e^{\frac{\kappa}{2}\zeta} \mathcal{F}'_{TW}(\zeta) d\zeta, \quad \kappa = -2B\, N^{1/3}$$

where $\mathcal{F}'_{TW}(\zeta)$ is the density of the **Tracy-Widom** distribution of the maximum GOE eigenvalue which can be written in terms of the **Painleve II** function: $q'' = xq + 2q^3$.

Landscape topology trivialization - universality :

- Consider a different model of the random landscape:
$$\mathcal{E}(\mathbf{x}) = \frac{\mu}{2}\sum_{i=1}^{N-1} x_i^2 + V(x_1,\ldots,x_{N-1}), \quad \mu>0,\ -\infty < x_i < \infty$$
where $V(\mathbf{x})$ is **stationary isotropic** random Gaussian potential with covariance
$$\mathbb{E}\{V(\mathbf{x})V(\mathbf{y})\} = Nf\left(\frac{1}{2N}(\mathbf{x}-\mathbf{y})^2\right)$$
The main control parameter turns out to be $B = 1 - \frac{\mu}{f''(0)}$. By the same methods we find e.g. for the mean number of minima:
$$\mathcal{N}_{min} \approx \begin{cases} 1, & B<0 \\ \sim\ e^{N\Sigma(B)}, & B>0 \end{cases}$$
where $\Sigma(B) = B - \frac{1}{2}B^2 - \ln(1+B) > 0$ for $B>0$ which is qualitatively similar to the spherical case. Moreover, after scaling $B = -\kappa/2N^{1/3}$ we find in the limit $N\to\infty$ **exactly** the same $\mathcal{N}_{tot}(\kappa), \mathcal{N}_{min}(\kappa)$ as in the sphere: **UNIVERSALITY**.
- Notice that for the spherical case with $p=2$ we have $B = -\frac{\sigma^2}{2J^2+\sigma^2} < 0$, so the system is essentially in a sort of **critical regime** for $\sigma \sim N^{-1/6}$. This special, yet non-trivial case of a random **quadratic form** can be studied in more detail. In particular, one can address the statistics of the **minimal energy** $\mathcal{E}_h(\mathbf{x})$ by a variant of the **replica trick**, see **YVF** & **P. Le Doussal** '14.

A Nonlinear Analogue of May-Wigner Instabilty Transition I:

"**Will a Large Complex System be Stable?**" - **Robert May** (1972) introduced a toy **linear** model for (in)stability of a large system of many interacting species:

$$\dot{\mathbf{x}} = -\mu\mathbf{x} + B\mathbf{x}, \quad \mu > 0,\ \mathbf{x} \in \mathbb{R}^N$$

where the real $N \times N$ random matrix B with mean zero and prescribed variance α^2 of entries mimics a complicated interaction between dynamics of different degrees of freedom individually relaxing to $\mathbf{x} = 0$. As typical eigenvalue of B with the largest real part grows as $\alpha\sqrt{N}$ the equilibrium $\mathbf{x} = 0$ becomes **unstable** when $\alpha\sqrt{N} > \mu$.

We suggest a natural **nonlinear extension** of the May's model:

$$\dot{x}_i = -\mu x_i + f_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N$$

where couplings $f_i(\mathbf{x})$ represent the components of $N-$dimensional vector field $\mathbf{f}$ and are chosen as a sum of a "**gradient**" and "**solenoidal**" contributions:

$$f_i(\mathbf{x}) = -\frac{\partial V(\mathbf{x})}{\partial x_i} + \sum_{j=1}^{N} \frac{\partial A_{ij}(\mathbf{x})}{\partial x_j}, \quad i = 1, \ldots, N$$

where we require the fields $A_{ij}(\mathbf{x})$ to be antisymmetric: $A_{ij} = -A_{ji}$. To make the model as simple as possible and amenable to a detailed analysis we choose the scalar potential $V(\mathbf{x})$ and the fields $A_{ij}(\mathbf{x})$ to be independent mean zero Gaussian random fields, with additional assumptions of **stationarity** and **isotropy**.

A Nonlinear Analogue of May-Wigner Instabilty Transition II:

Using Kac-Rice approach we are able to count the (mean) total number $\mathcal{N}_{tot}$ of all possible **equilibria** in the system of nonlinear ODEs , that is the number of simultaneous solutions of N equations $-\mu x_i + f_i(x_1, \ldots, x_N) = 0,\ i = 1, \ldots, N$. This turns out to be given by (**YVF** $\&$ **Khoruzhenko**, *in progress*):

$$\langle \mathcal{N}_{tot} \rangle = \frac{1}{m^N N^{N/2}} \int_{-\infty}^{\infty} \left\langle \left| \det \left((m + t\sqrt{\tau})\sqrt{N} - \mathbf{X} \right) \right| \right\rangle_X \frac{e^{-\frac{Nt^2}{2}} dt}{\sqrt{2\pi}}$$

where $m \propto \mu$ and the random **real asymmetric** matrix $\mathbf{X}$ is taken from the **Gaussian Elliptic Ensemble**:

$$\mathcal{P}(\mathbf{X}) = C_N(\tau) e^{-\frac{1}{2(1-\tau^2)} \left[\text{Tr}\mathbf{X}\mathbf{X}^T - \tau \text{Tr}\mathbf{X}^2 \right]}, \quad \tau \in [0, 1]$$

where parameter τ depends on the ratio of variances of **gradient** and **solenoidal** components of the field, so that **real Ginibre ensemble** with $\tau = 0$ corresponds to **purely solenoidal**, and GOE with $\tau = 1$ to **purely gradient** flow.

The above integral can be further related to the mean density of **real** eigenvalues of $\mathbf{X}$. Asymptotic analysis then reveals again the "**complexity explosion**" transition to **exponentially many equilibria** for $\mu < \mu_c$ with a (presumably) universal crossover function for $|\mu - \mu_c| \sim N^{-1/2}$. Similar transition was reported recently for dynamics of random neural networks (**G. Wainrib** and **J. Touboul** (2013)).

Topology of Random Algebraic Varieties :

Recently, the problem of computing the expectation of topological properties of random algebraic varieties has attracted a lot of interest (see e.g. the works by **Burgisser '07, Nazarov-Sodin '09, Gayet-Welshinger '11, Sarnak '11, Lerario-Lundberg '12, Sarnak-Wigman '13**) and others. An important class of problems addresses estimates for Betti numbers of "generic" (=random) real hypersurfaces given by **zero set** of real random homogenious polynomials of degree d in $n+1$ variables restricted to the unit sphere. E.g. for $d = 60$ and $n = 2$ a typical picture is:

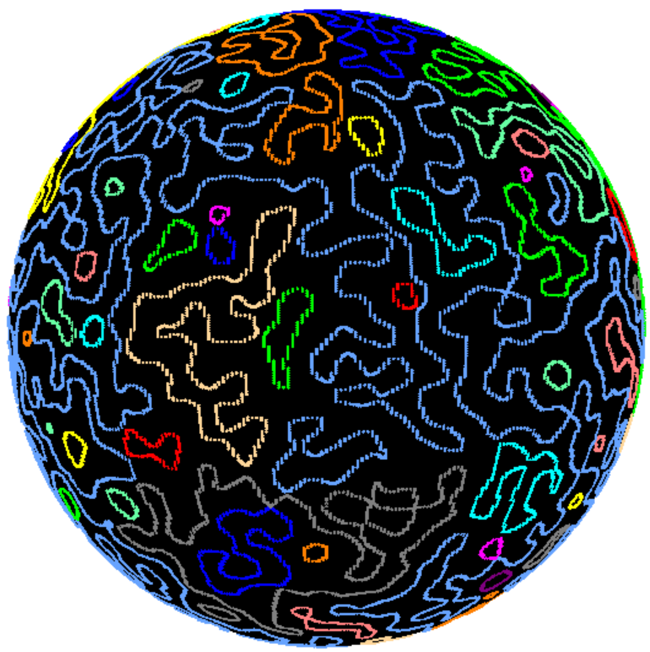

Figure 1: Zero locus of a random polynomial of degree $d = 60$ on the sphere (**M. Nastasescu**)

Upper bound on b_0 by Random Matrix Theory:

It turns out that the methods and results just exposed allow one to provide a useful **upper bound** to the **expected number of connected components** $b_0(f)$. Indeed, every component of the zero locus of the polynomials restricted to the sphere bounds a region where the function attains at least a maximum or a minimum, and consequently $\mathbb{E}\{b_0(f)\} \leq \mathbb{E}\{N_{min} + N_{max}\}$, where $N_{min/max}$ are numbers of minima/maxima on the sphere. Endowing polynomials with a rotationally-invariant Gaussian distribution we can find $N_{min/max}$ for any n and d from our formalism. We will mostly be interested in the limits $d \to \infty$ for a fixed n or $n \to \infty$ for a fixed d.

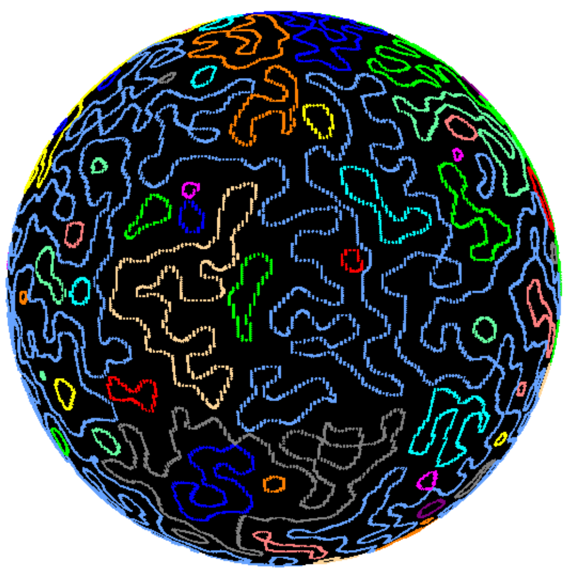

Figure 2: Zero locus of a random polynomial of degree $d = 60$ on the sphere (**M. Nastasescu**)

Upper bound on b_0 for Gaussian rotationally invariant polynomials:

Let $\{Y_l^j\}$ denote the standard basis of **spherical harmonics** of degree l on sphere S^n, then a random invariant Gaussian polynomial of degree d in $n+1$ variables can be constructed as (**Kostlan**):

$$f(\mathbf{x}) = \sum_{d-l\in 2\mathbb{N}} p_d(l) \sum_j \xi_l^j |\mathbf{x}|^{d-l} Y_l^j \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad p_d(l) \geq 0$$

where ξ_l^j are i.i.d. Gaussian coefficients, and nonnegative weights $p_d(d), p_d(d-2), \ldots,$ parametrize *all* invariant ensembles.

We assume that there exists such $0 < \lambda \leq 1$ that as $d \to \infty$ the polynomials assume the *scaling* form: $p_d(d^\lambda x) d^\lambda \to \psi(x) \neq 0$ pointwise. Further assuming that $\psi(x)$ is bounded by $c_1 e^{-c_2 x^2}$ we can prove the following **Theorem** (**YVF, Lerario, Lundberg**):

For any integrable $\psi(x) : (0,\infty) \to \mathbb{R}$ with subgaussian tails define the moments

$$\mu_k(\psi) = \int_0^\infty \psi^2(x)\, x^k\, dx, \quad k \in \mathbb{N}$$

Then there exists a constant $c > 0$ such that for any random hypersurface X

$$\lim_{d\to\infty} \textit{sup} \frac{\mathbb{E} b_0(X)}{d^{\lambda n}} \leq \lim_{d\to\infty} \frac{2\mathbb{E} N_{min}}{d^{\lambda n}} \sim c \left(\frac{\mu_3(\psi)}{\mu_2(\psi)}\right)^{n/2} n^{-\frac{n}{2}-\frac{17}{36}} e^{-n+\frac{4\sqrt{2}}{3}\sqrt{n}}$$