Central limit theorems for Ising model on random graphs

Cristian Giardinà

UNIMORE
UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA

Joint work (in progress) with

- Claudio Giberti (UniMoRe)
- Remco van der Hofstad (TU/e)
- Maria Luisa Prioriello (UniMoRe, TU/e)
- Sander Dommers (Bologna)

Plan of the talk

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1. Empirical networks and random graphs
2. Ising model on random graphs
3. Law of large numbers
4. Random-Quenched CLT
5. Average-Quenched and Annealed CLT's for special Configuration Models
6. Summary & Perspectives

Empirical networks and random graphs

Empirical networks

- Social networks (friendship, sexual, collaboration,..)
- Information networks (WWW, citation, ..)
- Technological networks (internet, airlines, roads, power grids,..)
- Biological networks (protein, neural, ...)

Empirical networks

Two emerging properties (among others)

- Scale free

- Small-world

Empirical networks

Two emerging properties (among others)

- Scale free

  Number of vertices with degree k is proportional to $k^{-\tau}$ for some $\tau > 1$

- Small-world

  Distance between most pairs of vertices are small

Empirical networks

	network	type	n	m	z	ℓ	α
social	film actors	undirected	449 913	25 516 482	113.43	3.48	**2.3**
	company directors	undirected	7 673	55 392	14.44	4.60	–
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–
	telephone call graph	undirected	47 000 000	80 000 000	3.16		**2.1**
	email messages	directed	59 912	86 300	1.44	4.95	**1.5/2.0**
	email address books	directed	16 881	57 029	3.38	5.22	–
	student relationships	undirected	573	477	1.66	16.01	–
	sexual contacts	undirected	2 810				**3.2**
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	**2.1/2.4**
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	**2.1/2.7**
	citation network	directed	783 339	6 716 198	8.57		**3.0/–**
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–
	word co-occurrence	undirected	460 902	17 000 000	70.13		**2.7**
technological	Internet	undirected	10 697	31 992	5.98	3.31	**2.5**
	power grid	undirected	4 941	6 594	2.67	18.99	–
	train routes	undirected	587	19 603	66.79	2.16	–
	software packages	directed	1 439	1 723	1.20	2.42	**1.6/1.4**
	software classes	directed	1 377	2 213	1.61	1.51	–
	electronic circuits	undirected	24 097	53 248	4.34	11.05	**3.0**
	peer-to-peer network	undirected	880	1 296	1.47	4.28	**2.1**
biological	metabolic network	undirected	765	3 686	9.64	2.56	**2.2**
	protein interactions	undirected	2 115	2 240	2.12	6.80	**2.4**
	marine food web	directed	135	598	4.43	2.05	–
	freshwater food web	directed	92	997	10.84	1.90	–
	neural network	directed	307	2 359	7.68	3.97	–

M.E.J. Newman, *The structure and function of complex networks* (2003)

Random graph $G_n = (V_n, E_n)$ models for empirical networks

- Configuration model
- Generalized random graph
- Preferential attachment model

Random graph $G_n = (V_n, E_n)$ models for empirical networks

- Configuration model
  Uniform graphs with a prescribed degree sequence

- Generalized random graph
  Independent edges with inhomogeneous probability

- Preferential attachment model
  Dynamical attachment proportional to degree

Random graph models

- Configuration model $CM_n(\mathbf{d})$:
  - Given a deterministic (or random) sequence $\mathbf{d} = (d_i)_{i \in [n]}$ with $\sum_i d_i$ even, assign d_i half-edges to each vertex $i \in V_n$.
  - Choose pairs of stubs at random and connect them together.

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  - Choose pairs of stubs at random and connect them together.

- Generalized random graph $GRG_n(\mathbf{w})$:
  - Given a deterministic (or random) sequence of weights $\mathbf{w} = (w_i)_{i \in [n]}$ with $w_i > 0$, define the edge-probability

$$p_{ij} = \frac{w_i w_j}{w_i w_j + \sum_i w_i} \qquad (i, j) \in E_n$$

  - Assign edges independently with occupation probabilities p_{ij}.

Local convergence to random trees

- A sequence of random graphs $\{G_n\}_{n\geq 1}$ is *locally tree-like* if

$$\lim_{n\to\infty} \mathbb{P}_n[B_U(t) \simeq \mathcal{T}] = \mathbb{P}[\mathcal{T}(P,\rho,t) \simeq \mathcal{T}].$$

  - $B_U(t)$ = ball of radius t in G_n centered at a uniformly chosen vertex $U \in [n]$
  - $\mathcal{T}(P,\rho,t)$ = rooted random tree with t generations
    - first generation: offspring distribution $P = (p_k)_{k\geq 1}$
    - further generations: size-biased law ρ

$$\rho_k = \frac{(k+1)p_{k+1}}{\sum_k kp_k}$$

Local Structure of $CM_n(\mathbf{d})$

- Let $D_n = d_U$ the degree of an uniformly chosen vertex $U \in [n]$. Assuming

  (a) $D_n \xrightarrow{\mathcal{D}} D$ with $\mathbb{P}(D = k) = p_k$,
  (b) $\mathbb{E}[D_n] \to \mathbb{E}[D] < \infty$,
  (c) $\mathbb{E}[D_n^2] \to \mathbb{E}[D^2] < \infty$,

- Then
  - The distribution of a random vertex is P

$$p_k^{(n)} = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{d_i = k\}} \longrightarrow p_k$$

  - The probability that the neighbor of a random vertex has degree $k+1$ equals the probability that a random stub is attached to a vertex with $k+1$ stubs:

$$\frac{(k+1) \sum_{i \in V_n} \mathbb{1}_{\{d_i = k+1\}}}{\sum_{i \in V_n} d_i} \longrightarrow \frac{(k+1) p_{k+1}}{\mathbb{E}[D]} = \rho_k$$

Local Structure of $GRG_n(\mathbf{w})$

- Let $W_n = w_U$ the weight of an uniformly chosen vertex $U \in [n]$. Assuming
  (a) $W_n \xrightarrow{\mathcal{D}} W$,
  (b) $\mathbb{E}[W_n] \to \mathbb{E}[W] < \infty$,
  (c) $\mathbb{E}[W_n^2] \to \mathbb{E}[W^2] < \infty$,
- Then
  - The degree distribution of a random vertex is *mixed Poisson*(W)

$$p_k^{(n)} = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{d_i = k\}} \longrightarrow p_k = \mathbb{E}\left[e^{-W} \frac{W^k}{k!}\right]$$

  - The probability that the neighbor of a random vertex has degree $k+1$ equals

$$\rho_k = \frac{1}{\mathbb{E}[W]} \mathbb{E}\left[e^{-W} \frac{W^{k+1}}{k!}\right]$$

Ising model on random graphs

Quenched and annealing

Denote by Q_n the average over all possible random graphs.
Let $\sigma = (\sigma_1, ..., \sigma_n) \in \Omega_n = \{-1, 1\}^n$.

- Random-quenched measure

$$\mu_{G_n}(\sigma) = \frac{\exp\left[\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B\sum_{i\in V_n} \sigma_i\right]}{Z_{G_n}(\beta, B)}$$

- Averaged-quenched measure

$$\overline{P}_n(\sigma) = Q_n\left(\frac{\exp\left[\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B\sum_{i\in V_n} \sigma_i\right]}{Z_{G_n}(\beta, B)}\right)$$

- Annealed measure

$$\widetilde{P}_n(\sigma) = \frac{Q_n\left(\exp\left[\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B\sum_{i\in V_n} \sigma_i\right]\right)}{Q_n(Z_{G_n}(\beta, B))}$$

Quenched and annealing

Correspondingly

- Random-quenched pressure

$$\psi_n(\beta, B) := \frac{1}{n} \log Z_n(\beta, B)$$

- Averaged-quenched pressure

$$\overline{\psi}_n(\beta, B) := \frac{1}{n} Q_n[\log Z_n(\beta, B)]$$

- Annealed pressure

$$\widetilde{\psi}_n(\beta, B) := \frac{1}{n} \log Q_n[Z_n(\beta, B)]$$

Remark:

$$\overline{\psi}_n \leq \widetilde{\psi}_n \qquad \lim_{n\to\infty} \psi_n = \lim_{n\to\infty} \overline{\psi}_n$$

Theorem (quenched pressure)

Dembo, Montanari [AAP'10], Dommers, G., van der Hofstad [JSP'10]

Assume $\{G_n\}_{n \geq 1}$ is uniformly sparse and *locally tree-like* with degree distribution P, where P has finite mean. Let $D \sim P$ and $K \sim \rho$. Then:

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$$\lim_{n\to\infty} \psi_n = \psi \qquad Q_n - a.s.$$

$$\begin{aligned}\psi(\beta, B) &= \frac{\mathbb{E}(D)}{2}\log\cosh(\beta) - \frac{\mathbb{E}(D)}{2}\mathbb{E}[\log(1+\tanh(\beta)\tanh(h_1)\tanh(h_2))]\\ &+ \mathbb{E}\left[\log\left(e^B\prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B}\prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right]\end{aligned}$$

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$$h_1 \overset{\mathcal{D}}{=} B + \sum_{i=1}^{K} \operatorname{arctanh}(\tanh(\beta)\tanh(h_i))$$

Remarks

- *key idea*: compare to the Ising model on the random Bethe tree and show pressures are the same
- *key tools*: stochastic recursion, ferromagnetic ineq. (GKS, GHS)
- Consequences: TD-limit of magnetization, susceptibility,...

$$M(\beta, B) := \lim_{n\to\infty} \frac{1}{n} \sum_{i\in V_n} \mu_{G_n}[\sigma_i] = \lim_{n\to\infty} \frac{\partial \psi_n}{\partial B} = \frac{\partial \psi}{\partial B}$$

$$\chi(\beta, B) := \lim_{n\to\infty} \frac{1}{n} \sum_{(i,j)\in V_n\times V_n} \left(\mu_{G_n}[\sigma_i\sigma_j] - \mu_{G_n}[\sigma_i]\mu_{G_n}[\sigma_j]\right) = \frac{\partial^2 \psi}{\partial B^2}$$

- Phase transition: $\beta_c^{\mathrm{qu}} = \operatorname{atanh}(1/\nu)$ where $\nu = \frac{\mathbb{E}(D(D-1))}{\mathbb{E}(D)}$

Theorem (Critical exponents)

Dommers, G., van der Hofstad [CMP '14]

The critical exponents $\boldsymbol{\beta}, \boldsymbol{\delta}, \gamma$ defined by:

$$M(\beta, 0^+) \asymp (\beta - \beta_c)^{\boldsymbol{\beta}}, \qquad \text{for } \beta \searrow \beta_c;$$
$$M(\beta_c, B) \asymp B^{1/\boldsymbol{\delta}}, \qquad \text{for } B \searrow 0;$$
$$\chi(\beta, 0^+) \asymp (\beta_c - \beta)^{-\gamma}, \qquad \text{for } \beta \nearrow \beta_c;$$

satisfy

	$\tau \in (3,5)$	$\mathbb{E}[D^4] < \infty$
$\boldsymbol{\beta}$	$1/(\tau-3)$	$1/2$
$\boldsymbol{\delta}$	$\tau - 2$	3
γ	1	1

For the boundary case $\tau = 5$ there are logarithmic corrections

$$M(\beta, 0^+) \asymp \left(\frac{\beta - \beta_c}{\log 1/(\beta - \beta_c)}\right)^{1/2} \quad \text{for } \beta \searrow \beta_c, \qquad M(\beta_c, B) \asymp \left(\frac{B}{\log(1/B)}\right)^{1/3} \quad \text{for } B \searrow 0.$$

Law large numbers

Theorem (Quenched LLN)

G., Giberti, van der Hofstad, Prioriello [in progress]

Assume that the random graph sequence $(G_n)_{n\geq 1}$ is uniformly sparse and locally tree-like with degree distribution with finite mean.

Let $S_n = \sum_{i\in[n]} \sigma_i$. For all $(\beta, B) \in \mathcal{U}^{qu}$ and for almost all graph sequences

$$\frac{S_n}{n} \xrightarrow{\text{exp}} M \qquad \text{w.r.t. } \mu_{G_n}, \qquad \text{as } n \to \infty,$$

where

$$M = M(\beta, B) = \frac{\partial \psi}{\partial B}(\beta, B)$$

and

$$\mathcal{U}^{qu} = \{(\beta, B) \; : \; B \neq 0, \beta \geq 0 \quad \text{or} \quad B = 0, 0 < \beta < \beta_c^{\mathrm{qu}}\}$$

Proof

- Study the random-quenched cumulant generating functions of S_n and prove the differentiability of the limiting function in 0.

$$
\begin{aligned}
c_n(t) &= \frac{1}{n}\log(\mu_{G_n}[\exp(tS_n)]) \\
&= \frac{1}{n}\log\sum_{\sigma\in\Omega_n}\frac{\exp\left(\beta\sum_{(i,j)\in E_n}\sigma_i\sigma_j+(B+t)\sum_{i\in V_n}\sigma_i\right)}{Z_n(\beta,B)} \\
&= \frac{1}{n}\log\frac{Z_n(\beta,B+t)}{Z_n(\beta,B)}
\end{aligned}
$$

Then

$$
c(t) := \lim_{n\to\infty} c_n(t) = \psi(\beta,B+t)-\psi(\beta,B) \quad a.s.
$$

- Moreover

$$c'(t) = \frac{\partial}{\partial t}\left[\psi(\beta, B+t) - \psi(\beta, B)\right] = \frac{\partial}{\partial B}\left[\psi(\beta, B+t)\right]$$

and hence

$$c'(0) = \frac{\partial}{\partial B}\left[\psi(\beta, B)\right] = M(\beta, B)$$

By large deviations argument for any $\varepsilon > 0$ there exists a number $L = L(\varepsilon) > 0$ such that

$$\mu_{G_n}\left(\left|\frac{S_n}{n} - M(\beta, B)\right| > \varepsilon\right) \leq \epsilon^{-nL} \qquad \text{for all sufficiently large n}$$

- Corollary

$$\frac{S_n}{n} \xrightarrow{\mathbb{P}} M(\beta, B) \qquad \text{w.r.t. } \overline{P}_n, \quad \text{as } n \to \infty$$

since

$$\overline{P}_n\left(\left|\frac{S_n}{n} - M(\beta, B)\right| > \varepsilon\right) = Q_n\left[\mu_{G_n}\left(\left|\frac{S_n}{n} - M(\beta, B)\right| > \varepsilon\right)\right]$$

Central limit theorems

Theorem (Random-quenched CLT)
G., Giberti, van der Hofstad, Prioriello [in progress]

- Assume that the random graph sequence $(G_n)_{n\geq 1}$ is uniformly sparse and locally tree-like with degree distribution with finite mean.
- For almost all graph sequences and for all $(\beta, B) \in \mathcal{U}^{qu}$

$$\frac{S_n - \mu_{G_n}[S_n]}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \chi) \qquad \text{w.r.t. } \mu_{G_n} \qquad \text{as } n \to \infty,$$

where

$$\chi = \chi(\beta, B) = \frac{\partial^2 \psi}{\partial B^2}(\beta, B)$$

Proof

- Define

$$V_n = \frac{S_n - \mu_{G_n}[S_n]}{\sqrt{n}}$$

and prove

$$\lim_{n\to\infty} \mu_{G_n}(\exp(tV_n)) = \exp\left(\frac{1}{2}\chi t^2\right)$$

for all $t \in [0, \alpha)$.

- Cumulant generating function

$$c_n(t) = \frac{1}{n}\log(\mu_{G_n}[\exp(tS_n)])$$

gives

$$c_n'(t) = \frac{1}{n}\frac{\mu_{G_n}[S_n \exp(tS_n)]}{\mu_{G_n}[\exp(tS_n)]}$$

and

$$c_n''(t) = \frac{1}{n}\frac{\mu_{G_n}[S_n^2 \exp(tS_n)]}{\mu_{G_n}[\exp(tS_n)]} - \left(\frac{\mu_{G_n}[S_n \exp(tS_n)]}{\mu_{G_n}[\exp(tS_n)]}\right)^2$$

Therefore

$$c_n'(0) = \mu_{G_n}\left[\frac{S_n}{n}\right] \qquad c_n''(0) = \frac{1}{n}\left(\mu_{G_n}[S_n^2] - \mu_{G_n}^2[S_n]\right)$$

- For $t > 0$

$$\log \mu_{G_n}(\exp(tV_n)) = \log\left[\mu_{G_n}\left(\exp\left[\frac{t}{\sqrt{n}}S_n\right]\right)\right] - \frac{t}{\sqrt{n}}\mu_{G_n}[S_n]$$
$$= n\left[c_n\left(\frac{t}{\sqrt{n}}\right) - \frac{t}{\sqrt{n}}c_n'(0)\right]$$

- By using $c_n(0) = 0$ and applying Taylor's theorem with Lagrange remainder

$$\log \mu_{G_n}(\exp(tV_n)) = \frac{t^2}{2}c_n''(t_n^*)$$

for some $t_n^* \in \left[0, \frac{t}{\sqrt{n}}\right]$.

- Therefore we need to prove

$$\lim_{n\to\infty} c_n''(t_n^*) = c''(0) = \chi < \infty$$

- The proof is completed by convexity and ferromagnetic inequalities. Assume $B > 0$ then:
  - $t \mapsto c_n(t)$ is convex by GKS and has a differentiable limit. Hence there exists $\alpha > 0$ s.t.

$$c_n'(t) \longrightarrow c'(t) \qquad \text{for all } -\alpha < t < \alpha\,, \qquad \text{as } n \to \infty$$

  - $t \mapsto c_n'(t)$ is concave on $[-B, \infty)$ by GHS inequality since

$$c_n'(t) = \mu_{G_n}\left[\frac{S_n}{n}\right](\beta, B+t)$$

  - As a consequence for any $0 \le t_n < \alpha$ with $t_n \to 0$

$$c_n''(t_n) \longrightarrow c''(0) = \chi \qquad \text{as } n \to \infty$$

Remarks

- Average-quenched measure cumulant generating fct.

$$\overline{c}_n(t) \quad = \quad \frac{1}{n} \log \overline{P}_n \left[\exp\left(tS_n\right)\right] = \frac{1}{n} \log Q_n \left[\frac{Z_n(\beta, B+t)}{Z_n(\beta, B} \right]$$

  Annealed cumulant generating fct.

$$\tilde{c}_n(t) \quad = \quad \frac{1}{n} \log \widetilde{P}_n \left[\exp\left(tS_n\right)\right] = \frac{1}{n} \log \frac{Q_n[Z_n(\beta, B+t)]}{Q_n[Z_n(\beta, B)]}$$

  However the strategy does not work since in general GHS is not known for average-quenched measure $\overline{P}_n$ or annealing $\widetilde{P}_n$

- By the law of total variance:

$$\mathrm{Var}_{\overline{P}_n}\left(\frac{S_n}{\sqrt{n}}\right) = Q_n\left(\mathrm{Var}_{\mu_{G_n}}\left(\frac{S_n}{\sqrt{n}}\right)\right) + \mathrm{Var}_{Q_n}\left(\mu_{G_n}\left(\frac{S_n}{\sqrt{n}}\right)\right)$$

  Therefore, if an average-quenched CLT holds, then

$$\overline{\chi}(\beta, B) \geq \chi(\beta, B)$$

Special Configuration Models

- $CM_n(\mathbf{2})$: the Configuration Model with $d_i = 2$ for all $i \in [n]$, i.e. the *2-regular random graph*.
- $CM_n(\mathbf{1}, \mathbf{2})$: the Configuration Model with $d_i \in \{1, 2\}$ for all $i \in [n]$ in which, for a given $p \in [0, 1]$, we have $n_1 = n - \lfloor pn \rfloor$ vertices with degree 1 and $n_2 = \lfloor pn \rfloor$ vertices with degree 2.
- The Ising model on such special random graphs is always in the uniqueness regime since $\nu \leq 1$.
- The 1-d Ising model naturally appears.

$CM_n(\mathbf{2})$ Configuration Model

- the random graph splits into disjoint cycles (tori)

$$\#\,\text{tori} = K_n^t = \sum_{j=1}^{N} I_j \qquad I_j = \text{Bern}\left(\frac{1}{2N-2j+1}\right)$$

$$L_n(i) \quad := \text{length of the } i\text{-th cycle}$$

$$Z_n(\beta, B) = \prod_{i=1}^{K_n^t} Z^{(t)}_{L_n(i)}(\beta, B)$$

where

$$Z^{(t)}_m(\beta, B) = \lambda_+^m(\beta, B) + \lambda_-^m(\beta, B)$$

$$\lambda_\pm = \lambda_\pm(\beta, B) = \epsilon^\beta \left[\cosh(B) \pm \sqrt{\sinh^2(B) + \epsilon^{-4\beta}}\right]$$

$CM_n(\mathbf{1},\mathbf{2})$ Configuration Model

- the random graph splits into tori + lines

$$\#\,\text{lines} = K_n^{(l)} = \frac{n_1}{2} = \frac{1-p}{2}n \qquad \#\,\text{tori} = K_n^{(t)} = K_{\bar{n}}^{(t)}$$

$L_n^{(l)}(i)$:= length of the i-th line $\quad$ $L_n^{(t)}(i)$:= length of the i-th torus

$$Z_n(\beta,B) = \prod_{i=1}^{K_n(l)} Z^{(l)}_{L_n^{(l)}(i)}(\beta,B) \cdot \prod_{i=1}^{K_n(t)} Z^{(t)}_{L_n^{(t)}(i)}(\beta,B),$$

where

$$Z_m^{(l)}(\beta,B) = A_+(\beta,B)\lambda_+^m(\beta,B) + A_-(\beta,B)\lambda_-^m(\beta,B)$$

$$A_\pm = A_\pm(\beta,B) = \frac{\epsilon^{-2\beta}\epsilon^{\pm B} + (\lambda_+ - \epsilon^{\beta+B})^2\epsilon^{\mp B} \pm 2\epsilon^{-\beta}(\lambda_+ - \epsilon^{\beta+B})}{[\epsilon^{-2\beta} + (\lambda_+ - \epsilon^{\beta+B})^2]\lambda_\pm}$$

Theorem [Averaged-quenched CLT for $CM_n(\mathbf{2})$]

G., Giberti, van der Hofstad, Prioriello [in progress]

- Consider a sequence of $CM_n(\mathbf{2})$ graphs. Then, for all $\beta > 0$, $B \in \mathbb{R}$,

$$\frac{S_n - P_n(S_n)}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \overline{\chi}_{\mathbf{2}}), \qquad \text{w.r.t. } \overline{P}_n, \qquad \text{as } n \to \infty,$$

where $\overline{\chi}_{\mathbf{2}}(\beta, B) = \chi_{\mathbf{2}}(\beta, B)$. Moreover

$$\overline{\chi}_{\mathbf{2}}(\beta, B) = \chi^{ISING-1d}(\beta, B) = \frac{\partial^2}{\partial B^2} ln\lambda_+(\beta, B) = \frac{\cosh(B) e^{-4\beta}}{(\sinh(B) + e^{-4\beta})^{3/2}}$$

Theorem [Averaged-quenched CLT for $CM_n(\mathbf{1},\mathbf{2})$]

G., Giberti, van der Hofstad, Prioriello [in progress]

- Consider a sequence of $CM_n(\mathbf{1},\mathbf{2})$ graphs. Then, for all $\beta > 0$, $B \in \mathbb{R}$,

$$\frac{S_n - P_n(S_n)}{\sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \overline{\chi}_{\mathbf{1},\mathbf{2}}\right), \qquad \text{w.r.t. } \overline{P}_n, \qquad \text{as } n \to \infty,$$

where $\overline{\chi}_{\mathbf{1},\mathbf{2}}(\beta, B) = \chi_{\mathbf{1},\mathbf{2}}(\beta, B) + \sigma_G^2(\beta, B)$

$$\chi_{\mathbf{1},\mathbf{2}}(\beta, B) = \chi^{ISING-1d}(\beta, B) + \lim_{n\to\infty} \sum_s Q_n(p_s^{(n)}) \frac{\partial^2 f_s}{\partial B^2}(\beta, B)$$

$$\sigma_G^2(\beta, B) = \lim_{n\to\infty} n \sum_{s,t} Cov_{Q_n}(p_s^{(n)}, p_t^{(n)}) \frac{\partial f_s}{\partial B}(\beta, B) \frac{\partial f_t}{\partial B}(\beta, B)$$

with

$$p_s^{(n)} := \frac{1}{n} \sum_{i=1}^{K_n(l)} \mathbb{1}_{\{L_n^{(l)}(i)=s\}} \qquad f_s(\beta, B) := \log\left(A_+(\beta, B) + A_-(\beta, B)\left(\frac{\lambda_-(\beta, B)}{\lambda_+(\beta, B)}\right)^s\right)$$

Theorem [Annealed CLT for $CM_n(\mathbf{2})$]
G., Giberti, van der Hofstad, Prioriello [in progress]

- Consider a sequence of $CM_n(\mathbf{2})$ graphs. Then, for all $\beta > 0$, $B \in \mathbb{R}$,

$$\frac{S_n - \widetilde{P}_n(S_n)}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \widetilde{\chi}_{\mathbf{2}}), \qquad \text{w.r.t. } \widetilde{P}_n, \qquad \text{as } n \to \infty,$$

where $\widetilde{\chi}_{\mathbf{2}}(\beta, B) = \chi_{\mathbf{2}}(\beta, B)$. Moreover

$$\widetilde{\chi}_{\mathbf{2}}(\beta, B) = \chi^{ISING-1d}(\beta, B)$$

Theorem [Annealed CLT for $CM_n(\mathbf{1},\mathbf{2})$]

G., Giberti, van der Hofstad, Prioriello [in progress]

- Consider a sequence of $CM_n(\mathbf{1},\mathbf{2})$ graphs. Then, for all $\beta > 0$, $B \in \mathbb{R}$,

$$\frac{S_n - \widetilde{P}_n(S_n)}{\sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \widetilde{\chi}_{\mathbf{1},\mathbf{2}}\right), \qquad \text{w.r.t. } \widetilde{P}_n \quad \text{as } n \to \infty,$$

where $\widetilde{\chi}_{\mathbf{1},\mathbf{2}}(\beta, B) = \chi_{\mathbf{1},\mathbf{2}}(\beta, B) + [\sigma^*_G(\beta, B)]^2$

$$\chi_{\mathbf{1},\mathbf{2}}(\beta, B) = \chi^{ISING-1d}(\beta, B) \; + \lim_{n\to\infty} \sum_s Q^*_n(p_s^{(n)}) \frac{\partial^2 f_s}{\partial B^2}(\beta, B)$$

$$[\sigma^*_G(\beta, B)]^2 = \lim_{n\to\infty} n \sum_{s,t} Cov_{Q^*_n}(p_s^{(n)}, p_t^{(n)}) \frac{\partial f_s}{\partial B}(\beta, B) \frac{\partial f_t}{\partial B}(\beta, B)$$

with

$$Q^*_n(\cdot) := \frac{Q_n(\cdot \, e^{n\psi_n})}{Q_n(e^{n\psi_n})}$$

Summary & Perspectives

Summary

- Locally-tree like random graphs: random-quenched CLT with variance $\chi(\beta, B)$

- $CM_n(\mathbf{2})$: CLT's with a unique variance

$$\chi_{\mathbf{2}}(\beta, B) = \overline{\chi}_{\mathbf{2}}(\beta, B) = \widetilde{\chi}_{\mathbf{2}}(\beta, B)$$

- $CM_n(\mathbf{1}, \mathbf{2})$: CLT's with different variances

$$\chi_{\mathbf{1},\mathbf{2}}(\beta, B) < \overline{\chi}_{\mathbf{1},\mathbf{2}}(\beta, B) \neq \widetilde{\chi}_{\mathbf{1},\mathbf{2}}(\beta, B)$$

Perspective: Annealing on $GRG(w)$
Dommers, G., Giberti, van der Hofstad, Prioriello [in progress]

- The annealed pressure can be computed thanks to edges independence (nice variational formula).
- Annealed phase transition at $\beta_c^{an} < \beta_c^{qu}$.
- Annealed CLT in the region

$$\mathcal{U}^{an} := \{(\beta, B) : \beta \geq 0, B \neq 0 \text{ or } 0 < \beta < \beta_c^{\text{an}}, B = 0\}.$$

- Non-classical limit theorems at criticality: $B = 0, \beta = \beta_c^{an}$.

THANK YOU!

Perspective: Annealing on $GRG(w)$

Dommers, G., Giberti, van der Hofstad, Prioriello [in progress]

- Non-classical limit theorems at criticality: $B = 0, \beta = \beta_c^{an}$

$$\mathbb{E}[D^4] < \infty \quad \Longrightarrow \quad \frac{S_n}{n^{3/4}} \xrightarrow{\mathcal{D}} X \quad \text{with density} \quad \frac{e^{-c_1 x^4}}{\int e^{-c_1 x^4} dx}$$

$$\tau \in (3,5) < \infty \quad \Longrightarrow \quad \frac{S_n}{n^{\frac{\tau-2}{\tau-1}}} \xrightarrow{\mathcal{D}} X \quad \text{with density} \quad \frac{e^{-c_2 x^{\tau-1}}}{\int e^{-c_2 x^{\tau-1}} dx}$$