Asymptotic Spectral Distribution of Random Matrices

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joint work with:

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BIRS Conference

"Spin Glasses and Related Topics"

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Topics

- Spectral Universality for Products of Non Hermitian Matrices
- Optimal Error Bounds for Convergence in Mean to the Semicircular Law

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Elliptical Random Matrix Ensembles

- let $\mathbf{X}_n(\omega) = \{X_{ij}(\omega)\}_{i,j=1}^n$ with condition (**C0**):

  a) (X_{jk}, X_{kj}) mutually independent for $1 \leq j < k \leq n$;

  b) for any $j, k = 1, ..., n$

$$\mathrm{E}\, X_{jk} = 0 \text{ and } \mathrm{E}\, X_{jk}^2 = 1;$$

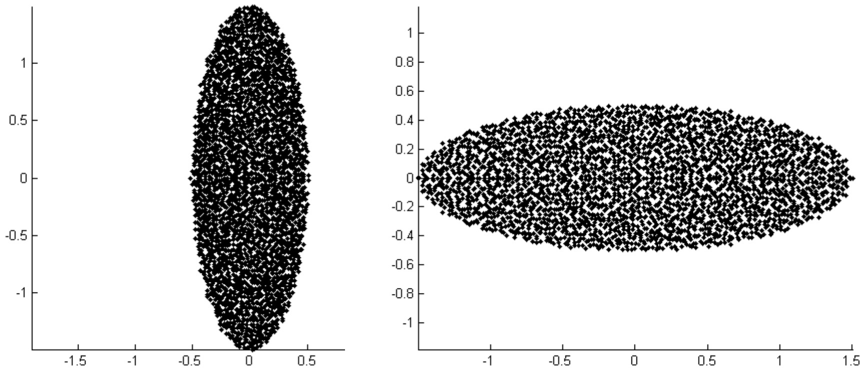
  c) for any $1 \leq j < k \leq n$

$$\mathrm{E}(X_{jk} X_{kj}) = \rho, \quad |\rho| \leq 1;$$

- $\rho = 1$: symmetric matrix: Wigner ensemble

  $\rho = 0$ and $\mathbf{X}_{jk}$ Gaussian: Ginibre ensemble.

Examples Elliptical Law



$n = 3000, \qquad \rho = -0.5 \qquad\qquad \rho = 0.5$

Elliptical Law

- $\lambda_1, ..., \lambda_n$ eigenvalues of $n^{-1/2}\mathbf{X}_n$
- Empirical spectral measure of $B \in \mathcal{B}(\mathbb{T})$, $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{C}$

$$\mu_n(B) = \frac{1}{n}\#\{i : \lambda_i \in B\},$$

Theorem (Naumov (2012))

$\mathbf{X}_n$ *with* (**C0**), $|\rho| < 1$: $X_{jk}, j < k$ *i.i.d.* $\mathbf{E}X_{jk}^4 < \infty$.

Then $\mu_n \xrightarrow{weak} \mu$ *in prob. with density:*

$$g(x,y) = \begin{cases} \frac{1}{\pi(1-\rho^2)}, & x, y \in \left\{u, v \in \mathbb{R} : \frac{u^2}{(1+\rho)^2} + \frac{v^2}{(1-\rho)^2} \le 1\right\}, \\ 0, & \textit{elsewhere.} \end{cases}$$

- Girko: 1985;
- If (X_{ij}, X_{ji}) - **i.i.d.** + assumption (**C0**), i.e. $\mathbf{E}X_{jk}^2 = 1$:

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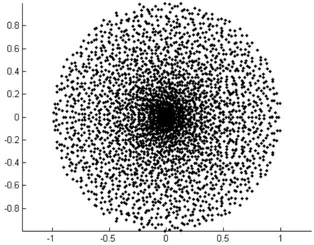
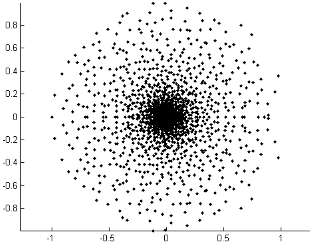
Examples: Product Laws for Elliptical Matrices



i.i.d. $m = 2$ $n = 3000$, $\rho = 0$



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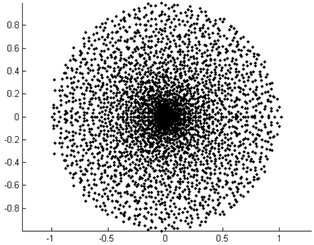

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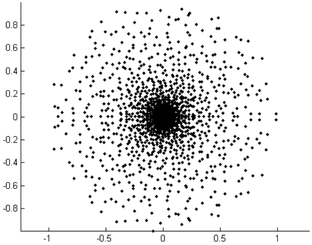

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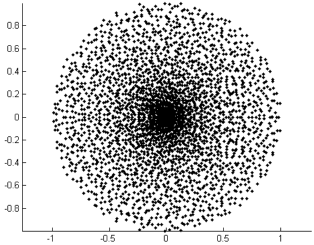

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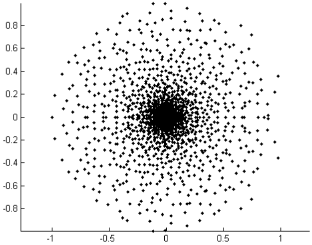

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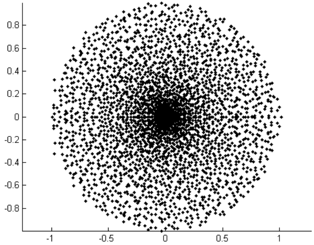

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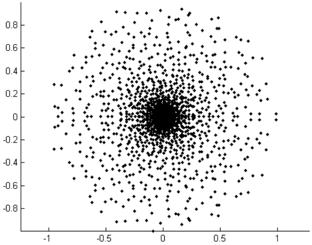

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Product Laws for Elliptical Wigner Matrices

- **Th.** **(non i.i.d. product case)** (G.-Naumov-Tikhomirov, (2013)).
  Let $\mathbf{X}_n^{(q)}, q \geq 2$ be independent $n \times n$ random matrices,
  Assume (**C0**) and $|\rho| \leq 1$ and condition:

$$\textbf{(L*)} \qquad \sup_{q,j,k} \mathrm{E}\,|X_{jk}^{(q)}|^2 I(|X_{jk}^{(q)}| \geq M) \to 0 \text{ as } n \to \infty.$$

Let $\mathbf{V} = n^{-m/2} \prod_{q=1}^{m} \mathbf{X}_n^{(q)}$, $m \geq 2$ and

μ_n - empirical spectral measure of the eigenvalues of $\mathbf{V}$.

Then $\mathrm{E}\,\mu_n \to \mu$, with density:

$$g(x,y) = \begin{cases} \frac{1}{\pi m (x^2+y^2)^{\frac{m-1}{m}}}, & x, y \in \{u, v \in \mathbb{R} : u^2 + v^2 \leq 1\}, \\ 0, & \text{elsewhere.} \end{cases}$$

which is independent of $|\rho| \leq 1$! Extension of (G.-Tikhomirov (2010))

Gaussian case: Akemann, Burda (2010,2012).

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Girko's Hermitization

Singular values versus spectral values:

$$|\det(\mathbf{A})| = \prod_{i=1}^{n} |\lambda_i(\mathbf{A})| = \prod_{i=1}^{n} s_i(A).$$

$\nu_n(\cdot, z)$: empirical measure of singular values of $(n^{-1/2}\mathbf{X}_n - z\mathbf{I})$.

$$U_{\mu_n}(z) = -\int_{\mathbb{C}} \log|z-w|\,\mu_n(dw) = -\frac{1}{n}\log\left|\det\left(\frac{1}{\sqrt{n}}\mathbf{X}_n - z\mathbf{I}\right)\right|$$

$$= -\frac{1}{2n}\log\det\left(\frac{1}{\sqrt{n}}\mathbf{X}_n - z\mathbf{I}\right)^*\left(\frac{1}{\sqrt{n}}\mathbf{X}_n - z\mathbf{I}\right) = -\int_0^\infty \log x\,\nu_n(dx)$$

Complex spectra of :

$\mathbb{F}(\mathbf{X}_n)$: $\nu_{\mathbb{F}(\mathbf{X})}$

$\mathbb{F}(\mathbf{Y}_n)$: $\nu_{\mathbb{F}(\mathbf{Y})}$ for Gaussian matrices $\mathbf{Y}_n = (\mathbf{Y}_n^{(1)}, \ldots, \mathbf{Y}_n^{(m)})$

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Weak convergence of $\nu_{\mathbb{F}(\mathbf{X})}$ and $\nu_{\mathbb{F}(\mathbf{Y})}$ to some limit $\nu_{\mathbb{F}}$:

condition (C1): *Log-potential of **shifted** singular distr. $\nu_{\mathbb{F}(\mathbf{X})}$ and $\nu_{\mathbb{F}(\mathbf{Y})}$*

$$U_{\mathbf{X}}(z) = -\int_{\mathbb{C}} \log|z-\zeta|\,\nu_{\mathbb{F}(\mathbf{X})}(d\zeta) = -\log\det|\mathbb{F}(\mathbf{X}) - z\mathbf{I}|.$$

converge to the log. potential of $\nu_{\mathbb{F}} - z\mathbf{I}$ such that:

- log *is uniformly integrable, i.e.*
  $\lim_{t\to\infty}\limsup_{n\to\infty} \Pr\left\{\left|\int_0^\infty \log(x)\,\nu_{\nu_{\mathbb{F}}(\mathbf{X})-z\mathbf{I}}(dx)\right| > t\right\} = 0.$
- *for **all** $z \in \mathbb{C}$.*
  $\lim_{n\to\infty} U_{\mathbf{X}}(z) = \lim_{n\to\infty} U_{\mathbf{Y}}(z) = U_{\mathbb{F}}(z)$ *in probability*
  with log potential determining distribution

G.-Tikhomirov (2007), Tao and Vu (2010), methods: Rudelson (2006):

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Universality of Complex Spectra

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  with log potential determining distribution

G.-Tikhomirov (2007), Tao and Vu (2010), methods: Rudelson (2006):

(C_0) yields unif. log-integr. for $m = 1$ and $\mathbb{F}(\mathbf{X}) = \mathbf{X}^{(1)}$ (Circular law)

Lemma (Bordenave-Chafai (2009))

Weak convergence of $\nu_{\mathbb{F}(\mathbf{X})}$ *and* $\nu_{\mathbb{F}(\mathbf{Y})}$ *to some limit* $\nu_{\mathbb{F}}$*:*

condition (C1): *Log-potential of **shifted** singular distr.* $\nu_{\mathbb{F}(\mathbf{X})}$ *and* $\nu_{\mathbb{F}(\mathbf{Y})}$

$$U_{\mathbf{X}}(z) = -\int_{\mathbb{C}} \log|z-\zeta|\,\nu_{\mathbb{F}(\mathbf{X})}(d\zeta) = -\log\det|\mathbb{F}(\mathbf{X}) - z\mathbf{I}|.$$

converge to the log. potential of $\nu_{\mathbb{F}} - z\mathbf{I}$ *such that:*

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Universality of Complex Spectra

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Smallest Singular Values

- Let $s_k := s_k(\mathbf{W})$ singular values of $\mathbf{W} = \mathbf{X} + \mathbf{M}_n$, $s_n(\mathbf{W}) \leq ... \leq s_1(\mathbf{W})$, i.e.

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Theorem ((G.-Naumov-Tikhomirov (2013)))

$\mathbf{X}$ $n \times n$ with condition **(C0)***. $\mathbf{M}_n$ $n \times n$ non-random. Assume*

$$\textbf{(L*)} : \quad \max_{j,k} \mathrm{E}\,|X_{jk}|^2 \mathbb{I}\{|X_{jk}| > M\} \to 0 \quad as \quad M \to \infty.$$

and $||\mathbf{M}_n|| \leq Kn^Q$, $K > 0$, $Q \geq 0$.

Exist $C, A, B > 0$ depending on K, Q s. th.

$$\mathbb{P}(s_n \leq n^{-B}) \leq Cn^{-A}.$$

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Uniform Integrability for Products

Condition (C1) for classes of $\mathbb{F}$:

$$\mathbb{F}(\mathbf{X}_l, \dots \mathbf{X}_m, X_l^*, \dots, \mathbf{X}_m^*) = \prod_{l=1}^{m} F_l(\mathbf{X}_l, \mathbf{X}_l^*)$$

Examples: $F_l(\mathbf{X}_l) = \mathbf{X}_l$ or $F_l(\mathbf{X}_l, \mathbf{X}_l^*) = X_l(\mathbf{X}_l \mathbf{X}_l^*)^{-\frac{1}{2}}$.

for the latter: minimal singular value $s_n(F_l) \geq s_1(\mathbf{X}_l)/s_n(\mathbf{X}_l)$.

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the uniform integrability of $\log |s_{n-k}(\mathbb{F}(\mathbf{X}) - z\mathbf{I})|$ is reduced to that of

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Smallest Singular Value Bounds

- Rudelson-Vershynin type decompositions (compressible/incompressible) of the *n*-sphere
- Distance bounds of first row to the span of others (with iterations for 2nd moment)
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Optimal Rates of Convergence to the Semicircular Law

Assume condition $\mathbf{C}_0$ with $\rho = 0$

i.e. (real/complex) hermitian Wigner matrices $\mathbf{X}_n = (X_{jk}, 1 \le j, k \le n)$.

Theorem (G.-Tikhomirov (2014))

Let $\mathrm{E}\, X_{jk} = 0$, $\mathrm{E}\, |X_{jk}|^2 = 1$ *and* $\lambda_j, 1 \le j \le n$ *eigenvalues of* $n^{-1/2}\mathbf{X}_n$ *with*

$\mu_n(x)$: *empirical spectral d.f.* $W(x)$: *Wigner half-circle d.f.*

$$\beta_k := \sup_{n \ge 1} \sup_{1 \le j,k \le n} \mathrm{E}\, |X_{jk}|^k.$$

If $\beta_8 < \infty$, *there is a constant* $C(\beta_8)$ *such that for all* $n \ge n_0$,

$$\Delta_n := \sup_x |\mathrm{E}\, \mu_n(x) - W(x)| \le C(\beta_8) n^{-1}.$$

sufficient: $\beta_4 < \infty$ *and* $\sup_{1 \le j,k \le n} |X_{jk}| \le D_0\, n^{\frac{1}{4}}$ *for all* n *and some* $D_0 < \infty$

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Previous Results

Let $\Delta_n^* := \sup_x |\mu_n(x) - W(x)|$. Assume $\beta_4 < \infty$.

$\Delta_n^* = O_P(n^{-\frac{1}{2}})$, G.-Tikhomirov (GT) (2003),

$\Delta_n = O(n^{-1/2})$, Bai (2002), GT (2002). $\Delta_n = O(n^{-1/4})$, Bai (1993).

$\Delta_n^* = O_P(n^{-2/3})$, Poincaré inequality for X_{jk}, Bobkov-GT (2010),

Gaussian case: $\Delta_n = O(n^{-1})$, GT (2005).

lower bound: $\Delta_n^* = \Omega(n^{-1}\sqrt{\log n})$, Gustavsson (2005).

Assume $\Pr\{|X_{jk}| > t\} \le A\exp\{-t^{\varkappa}\}$, for $t \ge 1$, $\varkappa > 0, A > 0$.

Erdös, Yau and Yin (2010) showed (rigidity of eigenvalue locations):

$$\mathrm{E}\,\Delta_n^* \le C\,n^{-1}(\log n)^{C \log\log n}.$$

Similar results: Tao (2010), provided $\mathrm{E}\,X_{jk}^3 = 0$.

Sharper bounds: $\Delta_n^{**} = O_P(n^{-1}\log n)$, Schlein-Maltseva (2013)

Used in our proof: ”vertical” recursion bound and methods from GT (03,09).

Ideas of Proof

Stieltjes transforms of μ_n and W: $m_n(z)$ and $s(z)$,

$z = u + iv, \quad u, v \in \mathbb{R}, \qquad m_n(z) = \frac{1}{n} \sum_j R_{jj}(z) := \frac{1}{n} \operatorname{Tr} \mathbf{R}.$

For z in region

$$\mathbb{G}_n = \{z = u + iv \in \mathbb{C}_+ : -2 + cn^{-2/3} \le u \le 2 - cn^{-2/3},\ v \ge cn^{-1}/\sqrt{2 - |u|}\}$$

show: $$|\operatorname{E} m_n(z) - s(z)| \le \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} \left|z^2 - 4\right|^{\frac{1}{4}}}.$$

Main recursion:

$$R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j R_{jj},$$

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Main recursion

Use scale factor $s_0 > 2$ for "decent" from $v = 4$ to $v = cn^{-1}$:

$$|R_{jj}(u + iv/s_0)| \leq s_0 \, |R_{jj}(u + iv)|.$$

Assuming for $z = u + iv$, $p = s_0^q$

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ok for e.g. $v = 4$.

Burkholder's martingale bound for $2p$th moment results in a pth moment of another quadratic form in the independent variables of a new column/row.

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Similarly: $\mathrm{E}\,|\frac{1}{m_n(z) + z}|^{8} \leq C_0^{8}$, for all $Im(z) = v \geq cn^{-1}$.

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Let

$$\Lambda_n := m_n(z) - s(z).$$

Using conditional independence by "shelling" the matrix **R** and expectations of martingale decompositions:

$$\mathrm{E}\,|\Lambda_n|^2 \le \frac{C}{nv}\,\mathrm{E}^{\frac{1}{2}}\,|\Lambda_n|^2 + \frac{C}{n^2 v^2}.$$

This recursion results in:

$$\mathrm{E}\,|\Lambda_n|^2 \le C(nv)^{-2}.$$

Thank you for your attention!

Bounds For Singular values

Compressible and incompressible vectors

- As in Rudelson-Vershynin partition $\mathcal{S}^{(n-1)} := \{\mathbf{x} : ||\mathbf{x}||_2 = 1\}$ in compressible and incompressible vectors.
- For $\delta, r \in (0,1)$, $x \in \mathbb{R}^n$ is called δ-**sparse** if $|\operatorname{supp}(\mathbf{x})| \le \delta n$.
- $\mathbf{x} \in \mathcal{S}^{(n-1)}$ is called (δ, r)-**compressible**
  if $\mathbf{x}$ is in r-distance from δ-sparse vectors: $(\mathbf{x} \in \mathcal{C}(\delta, r))$
- $\mathbf{x} \in \mathcal{S}^{(n-1)}$ is (δ, r)-**incompressible** if it's **not** (δ, r)-compressible: $(\mathbf{x} \in \mathcal{IC}(\delta, r))$

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- Exist set $\sigma(\mathbf{x}) \in [n]$ of cardinality $|\sigma(x)| \geq \frac{1}{2}n\delta$ such that

$$\frac{r}{\sqrt{2n}} \leq |x_k| \leq \frac{1}{\sqrt{\delta n}} \quad \text{for any } k \in \sigma(x),$$

Subgaussian Distribution

$$\text{Recall} \qquad \mathbf{W} = \mathbf{X} + \mathbf{M}_n$$

- for compressible vectors

$$\inf_{\mathbf{x}\in\mathcal{C}(\delta,r)} ||\mathbf{W}\mathbf{x}||_2 \geq n^{1/2} \quad \text{with high probability}$$

- For incompressible vectors

$$\mathbb{P}\left(\inf_{\mathbf{x}\in\mathcal{IC}(\delta,r)} ||\mathbf{W}\mathbf{x}||_2 < \varepsilon n^{-1/2}\right) \leq \frac{1}{\delta n}\sum_{k=1}^{n} \mathbb{P}(\mathrm{dist}(\mathbf{W}_k, H_k) < r^{-1}\varepsilon),$$

where $\mathbf{W}_1, \ldots, \mathbf{W}_n$ are columns of $\mathbf{W}$ and

$\mathcal{H}_k =$ span of column vectors except the kth.

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Bound Via Distances

- Iteration of decomposition for 2nd moment only:

  result: $(r_n(2), \delta_2, \mathcal{C}_2), \quad \mathcal{E}_K = \{\|W\| \le K_n\}$

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$$\mathbb{P}(\inf_{\mathbf{x}\in\mathcal{C}_2} \|\mathbf{Wx}\|_2 \le \eta \left(\frac{r_n^{(2)}}{\sqrt{n}}\right)^2, \mathcal{E}_K) \le \frac{1}{n\delta_2} \sum_{k=1}^{n} \mathbb{P}(\mathrm{dist}(\mathbf{W}_k, \mathcal{H}_k) < \frac{\eta r_n^{(2)}}{\sqrt{n}}, \mathcal{E}_K).$$

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- For an arbitrary matrix **A**

$$\operatorname{dist}(\mathbf{A}_1, \mathcal{H}_1) \geq \frac{|(\mathbf{B}^{-T}\mathbf{v}, \mathbf{u}) - a_{11}|}{\sqrt{1 + ||\mathbf{B}^{-T}\mathbf{v}||_2^2}},$$

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$$\begin{pmatrix} a_{11} & \mathbf{v}^T \\ \mathbf{u} & \mathbf{B} \end{pmatrix} \tag{1.2}$$

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$$\sup_{a \in \mathbb{R}} \mathbb{P}\left\{ \frac{|(\mathbf{B}^{-T}\mathbf{v}, \mathbf{u}) - a|}{\sqrt{1 + ||\mathbf{B}^{-T}\mathbf{v}||^2}} \leq \varepsilon, \text{ and } \|\mathbf{B}\|_2 \leq K_n \right\} \leq Cn^{-A}$$

$0 < \varepsilon \leq n^{-B}$, constants $A > 0$ and $B > 0$.

- $$\mathbf{Q} = \begin{pmatrix} \mathbf{O}_{n-1} & \mathbf{B}^{-T} \\ \mathbf{B}^{-1} & \mathbf{O}_{n-1} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix},$$

  with $\mathbf{O}_{n-1}$: $(n-1) \times (n-1)$ zero matrix. By definition:

  $$(\mathbf{B}^{-T}\mathbf{v}, \mathbf{u}) = \frac{1}{2}(\mathbf{Q}\mathbf{w}, \mathbf{w}).$$

Decoupling of Quadratic Form

- independent copy

$$\mathbf{w}' = \begin{pmatrix} \mathbf{u}' \\ \mathbf{v}' \end{pmatrix},$$

  with $\mathbf{u}', \mathbf{v}'$ independent copies of $\mathbf{u}, \mathbf{v}$.

- Show

$$\sup_{v \in \mathbb{R}} \mathbb{P}_{\mathbf{w}} \left(|(\mathbf{Qw}, \mathbf{w}) - v| \le 2\varepsilon \right) \le \mathbb{P}_{\mathbf{w},\mathbf{w}'} \left(|(\mathbf{QP}_{J^c}(\mathbf{w} - \mathbf{w}'), \mathbf{P}_J \mathbf{w}) - u| \le 2\varepsilon \right),$$

  with u not depending on $\mathbf{P}_J \mathbf{w} = (\mathbf{P}_J \mathbf{u}, \mathbf{P}_J \mathbf{v})^T$.

- Show

$$\varepsilon_0^{1/2} \sqrt{1 + ||\mathbf{B}^{-T} \mathbf{v}||_2^2} \le ||\mathbf{B}^{-1}||_2 \le \varepsilon_0^{-1} ||\mathbf{QP}_{\mathcal{J}^c}(\mathbf{w} - \mathbf{w}')||_2$$

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with $\mathbf{u}', \mathbf{v}'$ independent copies of $\mathbf{u}, \mathbf{v}$.

- Show

$$\sup_{v \in \mathbb{R}} \mathbb{P}_{\mathbf{w}} \left(|(\mathbf{Q}\mathbf{w}, \mathbf{w}) - v| \le 2\varepsilon \right) \le \mathbb{P}_{\mathbf{w},\mathbf{w}'} \left(|(\mathbf{Q}\mathbf{P}_{J^c}(\mathbf{w} - \mathbf{w}'), \mathbf{P}_J \mathbf{w}) - u| \le 2\varepsilon \right),$$

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- Show

$$\varepsilon_0^{1/2}\sqrt{1 + \|\mathbf{B}^{-T}\mathbf{v}\|_2^2} \le \|\mathbf{B}^{-1}\|_2 \le \varepsilon_0^{-1}\|\mathbf{Q}\mathbf{P}_{J^c}(\mathbf{w} - \mathbf{w}')\|_2$$

Decoupling of Quadratic Form

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- Show

$$\varepsilon_0^{1/2}\sqrt{1 + ||\mathbf{B}^{-T}\mathbf{v}||_2^2} \le ||\mathbf{B}^{-1}||_2 \le \varepsilon_0^{-1}||\mathbf{Q}\mathbf{P}_{\mathcal{J}^c}(\mathbf{w} - \mathbf{w}')||_2$$

- Estimate

$$\sup_{\substack{\mathbf{w}_0=(\mathbf{a},\mathbf{b})^T\in\mathcal{IC}(\delta,r_n^{(2)})\\ w\in\mathbb{R}}} \mathbb{P}_{\mathbf{P}_{\mathcal{J}}\mathbf{w}}\left(|(\mathbf{w}_0,\mathbf{P}_{\mathcal{J}}\mathbf{w})-w|\le\varepsilon_0^{-3/2}\varepsilon\right),$$

where

$$\mathbf{w}_0=\frac{1}{||\mathbf{QP}_{\mathcal{J}^c}(\mathbf{w}-\mathbf{w}')||_2}\begin{pmatrix}\mathbf{B}^{-T}\mathbf{P}_{\mathcal{J}^c}(\mathbf{v}-\mathbf{v}')\\ \mathbf{B}^{-1}\mathbf{P}_{\mathcal{J}^c}(\mathbf{u}-\mathbf{u}')\end{pmatrix}=\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}.$$

- Fix $\mathbf{w}_0$ and w. Rewrite

$$(\mathbf{w}_0, P_{\mathcal{J}}\mathbf{w}) = \sum_{i \in \mathcal{J}} (a_i X_i + b_i Y_i),$$

where $||\mathbf{a}||_2^2 + ||\mathbf{b}||_2^2 = 1$.

- Exist $\mathcal{J}_0 \in \mathcal{J}$ of cardinality $|\mathcal{J}_0| \backsim n$ such that

$$\frac{r_n^{(2)}}{\sqrt{2n}} \le |a_i| \le \frac{1}{\sqrt{\delta n}} \text{ for any } i \in \mathcal{J}_0,$$

- Use Levy concentration inequalities (e.g. Petrov):

Small Ball Probabilities for Sums II

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- Use Levy concentration inequalities (e.g. Petrov):

- Let $X_1, X_2, \dots$ be independent random variables with

$$\mathrm{E}\,X_k = 0 \text{ and } \mathrm{E}\,X_k^2 = \sigma_k^2 > 0.$$

We denote $\sigma^2 = \sum_{k=1}^n \sigma_k^2$ and

$$Q(X, \lambda) = \sup_{a \in \mathbb{R}} \mathbb{P}(|X - a| \le \lambda).$$

- Assume that (1.3) holds. With $S_n = \sum_{k=1}^n X_k$:

$$Q(S_n, \lambda) \le \frac{\sqrt{\lambda}}{(2\sigma^2 - 8\sum_{i=1}^n \mathrm{E}\,X_j^2 \mathbb{I}(|X_j| \ge \lambda/2))^{1/2}}.$$

Thank you for your attention!