# Phases, zeros and fluctuations in complex random energy models

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July 21-25, 2014

- **Complex Random Energy Model: Zeros and Fluctuations.** With Zakhar Kabluchko. Probab. Theory Relat. Fields (2014) 158:159–196.
- **Generalized Random Energy Model at Complex Temperatures.** With Zakhar Kabluchko. arXiv:1402.2142 [math.PR].

# Challenge

- Understand the large scale behaviour of **complex systems** with **strongly inhomogeneous conflicting interactions**.

# This talk

- Spin glasses.
- Random energy models.
  1. **Complex-valued Derrida's random energy model (REM).**
     $\rightsquigarrow$ **phase diagram on the complex plane, fluctuations, zeros of the partition function.**
  2. **Complex-valued Derrida's generalized random energy model (GREM).**
     $\rightsquigarrow$ **phase diagram on the complex plane, fluctuations, zeros of the partition function.**
- ~~**replica method**~~, **probabilistic analysis**.

# Usual formailsm

**Gibbs-Boltzmann distribution**

$$G(\beta)\{\sigma \in \mathrm{d}\sigma\} = \frac{1}{Z(\beta)} \exp(-\beta H(\sigma))\, \mu(\mathrm{d}\sigma),$$

where

- $\sigma = (\sigma_i)_i \rightsquigarrow$ the **configuration** of the system.
- $i \rightsquigarrow$ **sites**.
- $H(\sigma) \rightsquigarrow$ the **energy level** of $\sigma$.
- $\beta \rightsquigarrow$ **inverse temperature**.
- $Z(\beta) = \int \mu(\mathrm{d}\sigma) \exp(-\beta H(\sigma)) \rightsquigarrow$ **partition function**.
- $\mu \rightsquigarrow$ **a priori distribution**.

# Range-free model a la Ising-Lenz, Curie-Weiss

**Range-free** Hamiltonian

$$H(\sigma) = -\frac{1}{?} \sum_{i,j=1}^{N} J_{i,j} \sigma_i \sigma_j,$$

where

- $\sigma_i = \pm 1$ **Ising spins**.
- $i = 1, 2, \ldots, N$ **sites**.
- $J_{i,j} \rightsquigarrow$ pair-wise **interaction strength** (= couplings).
- e.g., $J = \text{const} \rightsquigarrow$ **Curie-Weiss** model (= "homogeneous ferromagnetic interactions"). Here $? = N$.

# Phase transitions

**Free energy:** $F(\beta) = -\frac{1}{\beta}\log Z(\beta)$.

**Log-partition function per site:** $p_N(\beta) = \frac{1}{N}\log Z(\beta)$.

- (Encode much of the behaviour of the Gibbs distribution.)
- **Phase transitions:**

$$p(\beta) := \lim_{N\to\infty} p_N(\beta)$$

**fails** to be **analytic function** of $\beta$.

# Range-free vs. finite-range

Kinds of Models:

- **Range-free** (= infinite-range, mean-field) models on graphs
  - e.g., Erdős–Rényi, configuration model, small-world, . . .
- (as oposed to) **finite-range**
  - e.g., lattice models on $\mathbb{Z}^d$, . . .

# Random range free models: Spin glasses

$$H_N[\omega](\sigma) = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{i,j}[\omega]\sigma_i\sigma_j,$$

where $J_{i,j}$ are real-valued **random variables** (e.g., i.i.d.)

- **N.B. Conflicting interactions** $J_{i,j} > 0$ and $J_{i,j} < 0$ are allowed $\rightsquigarrow$ **frustration**.
- Inhomogeneities.
- **Ex. Sherrington-Kirkpatrick** model $J_{i,j}$ are i.i.d. Gaussian $\mathcal{N}(0,1)$.
- **Q:** Why $N^{-1/2}$ normalization?
- **A:** $\min_{\sigma\in\{\pm1\}^N} H(\sigma) = O(N)$ (**extensive** Statistical Mechanics).

# Phenomena

**Physics predictions (Parisi, Mezard, et. al.):**

- **(Countably) infinitely many pure / metastable states** (in the thermodynamic limit):

$$\text{“}G = \sum_{\alpha=1}^{\infty} w(\alpha) G_\alpha, \quad \sum_{\alpha=1}^{\infty} w(\alpha) = 1\text{”}.$$

- **Ultrametric organisation of pure states:** valley inside valley picture.
- “Continuum of phase transitions”.
- Dynamical phenomena: **aging** (slow relaxation to equilibrium).
- **Chaos**: extreme sensitivity of the pure states to the slightest changes in temperature, external field or disorder.

Mathematics:

- Panchenko, D. (2013). The Sherrington-Kirkpatrick model.

# Motivation

- **Combinatorial optimisation:** finding **ground state**

$$\mathrm{argmin}_{\sigma=\{\pm 1\}^N} H_N(\sigma) = ?$$

  is **NP-complete**.
- **Complex networks:** small world phenomenon, etc.
- **Statistical inference**.
- …

**Ref:** e.g., **Mézard, Parisi, Virasoro (1987); Mézard, Montanari (2007)**.

# Energy as random field

**Fruitfull approach:**

- Treat $H(\cdot)$ as a **random field**.
- **Derrida (1980):**
  - **Focus** on the **field of energy levels** $H(\cdot)$.
  - (**Forget** about **microscopic interactions** $J_{i,j}$.)
- This way we might hope to start exploring the **universality class**.
- **Q:** What is the **"simplest" random field**?
- **A: White noise** (e.g., i.i.d. Gaussian field).

# Derrida's random energy model

**Partition function:**

$$\mathscr{Z}_N(\beta) = \sum_{k=1}^{N} \mathrm{e}^{\beta\sqrt{n}X_k}.$$

- $\{X_k\}_{k=1}^{\infty}$ are **i.i.d.** $\mathscr{N}(0,1)$ **random energies**.
- No microscopic interactions, no spins, just completely random energy levels, ...
- Why bother?

**Log partition function** ($n = \log N$):

$$p(\beta) := \lim_{N\to\infty} \frac{1}{n} \log \mathscr{Z}_N(\beta) = \begin{cases} 1 + \frac{1}{2}\beta^2, & 0 \le \beta \le \sqrt{2}, \\ \sqrt{2}\beta, & \beta \ge \sqrt{2}. \end{cases}$$

$\Rightarrow$ **phase transition.**

**What happens in probabilistic terms?** Well understood: breakdown of the LLN $\rightsquigarrow$ Extreme Value Theory (Bovier, Kurkova, Löwe (2002)).

# Lee-Yang Program (1952)

$$p_N(\beta) := \frac{1}{n} \log |\mathscr{Z}_N(\beta)|$$

**Phase Transitions** $p(\beta)$

$\Updownarrow$

**Analyticity Breaking** $p_N(\beta)$**, as** $N \to \infty$

$\Updownarrow$

($\log$ is non-analytic only at zero, $\mathscr{Z}_N(\cdot)$ is an entire function)

$\Updownarrow$

**Zeros of** $\mathscr{Z}_N(\beta)$**, as** $N \to \infty$**,**

$\beta \in \mathbb{C}$**.**

# Quantum physics

- Interference in inhomogeneous media.
- Schrödinger equations with random potential.
- Quantum Monte Carlo.

**Our caricature:**

$$\mathscr{Z}_N(\beta) = \sum_{k=1}^{N} \mathrm{e}^{\sqrt{n}(\sigma X_k + i\tau Y_k)}, \qquad \beta = (\sigma, \tau) \in \mathbb{R}^2.$$

- $\{(X_k, Y_k)\}_{k=1}^{\infty}$ i.i.d. zero-mean bivariate Gaussian random vectors with

$$\operatorname{Var} X_k = \operatorname{Var} Y_k = 1, \quad \operatorname{corr}(X_k, Y_k) = \rho,$$
$$-1 \le \rho \le 1$$

⇒ **Complex REM.**

# Where are the zeroes of the partition function in the REM?


**Figure:** Complex zeros of $Z_n(\cdot)$. **Simulation** by C. Moukarzel and N. Parga: Physica A 177 (1991). B. Derrida (1991).

# Log-partition function

**Theorem (Derrida (1991), rigorous proof Kabluchko, K. (2012))**

*For every* $\beta \in \mathbb{R}^2$, *the limit*

$$p(\beta) := \lim_{N \to \infty} p_N(\beta)$$

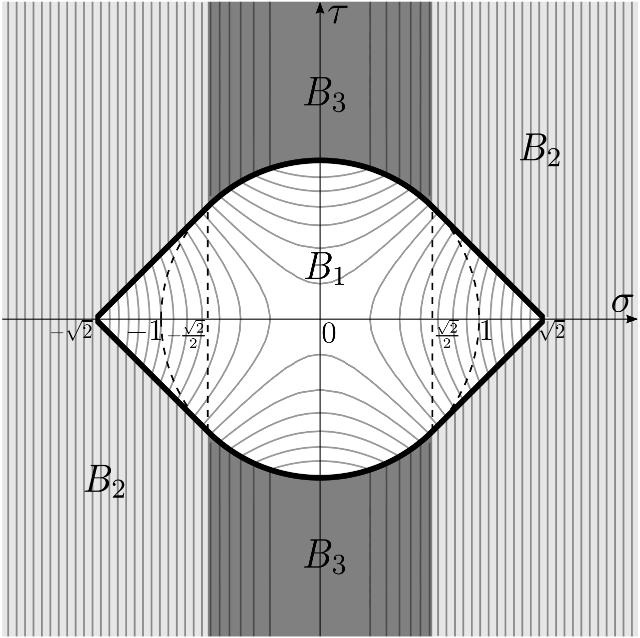
*exists in probability and in* $L^q$, $q \geq 1$, *and is explicitly given as*

$$p(\beta) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3. \end{cases}$$

**N.B.:**

- No dependence on $\rho$ in the limit.

# Complex REM phase diagram in the complex $\beta$-plane



# What happens probabilistically?

Three possible regimes:

- $\mathscr{Z}_n(\beta) \approx \mathbb{E}[\mathscr{Z}_n(\beta)] \rightsquigarrow$ **expectation dominated regime $B_1$.**
- $\mathscr{Z}_n(\beta) \approx \max_k |\exp(\beta\sqrt{n}X_k)| \rightsquigarrow$ **extremes dominated regime $B_2$** (**"glassy phase"**).
- $\mathscr{Z}_n(\beta) \approx \sqrt{\mathrm{Var}[\mathscr{Z}_n(\beta)]} \rightsquigarrow$ **fluctuations/oscillations dominated regime $B_3$** (**"new"**).

# Zeros of the partition function

Empirical distribution of zeros of $\mathscr{Z}_N(\cdot)$ (weighted by multiplicity) $\rightsquigarrow \Xi_N$.

**Theorem (known)**

$$\Xi_N = \frac{1}{2\pi}\Delta \log |\mathscr{Z}_N|.$$

**N.B.** Should be understood in the sense of distributions/generalized functions.

- **Derrida (1991)**.
- Book of **Hough, Krishnapur, Peres, Virág (2009)**.

# Limiting density of zeros

Define $\Xi = 2\Xi_3 + \Xi_{12} + \Xi_{13}$, where

- $\Xi_3 = \text{Lebesgue}|_{B_3}$.
- $\Xi_{13} \rightsquigarrow$ the 1D length measure on $\partial(\bar{B}_1 \cap \bar{B}_3)$ (two circular arcs).
- $\Xi_{12} \rightsquigarrow$ measure with density $\sqrt{2}|\tau|$ w.r.t. the 1D length measure on $\partial(\bar{B}_1 \cap \bar{B}_2)$ (four line segments).

**Theorem (counting extensively many zeros)**

*For every continuous function $f : \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\frac{1}{n} \sum_{\beta \in \mathbb{C}:\ \mathscr{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{P} \frac{1}{2\pi} \int_{\mathbb{C}} f(\beta)\Xi(\mathrm{d}\beta).$$

**N.B.** Exchange $N \to \infty$ and $\Delta$.

# Local structure of zeros of $\mathscr{Z}_N$ in a neighborhood of area $1/n$ of a point in $B_3$

Define $\{\mathbb{G}(t) : t \in \mathbb{C}\}$

$$\mathbb{G}(t) = \sum_{k=0}^{\infty} \xi_k \frac{t^k}{\sqrt{k!}},$$

where $\xi_0, \xi_1, \dots$ are i.i.d. standard complex Gaussian random variables.

**N.B.** $Z \sim N_{\mathbb{C}}(0, s^2)$ iff $Z = X + iY$, where $X, Y \sim N_{\mathbb{R}}(0, s^2/2)$).

**Theorem (local structure of zeros in the oscillatory phase)**

*Let $\beta_0 \in B_3$. For every continuous function $f : \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\sum_{\beta \in \mathbb{C}:\ \mathscr{Z}_N(\beta)=0} f(\sqrt{n}(\beta - \beta_0)) \xrightarrow[N\to\infty]{w} \sum_{\beta \in \mathbb{C}:\ \mathbb{G}(\beta)=0} f(\beta).$$

# Zeros in the high-temperature phase $B_1$

**Theorem**

*Let $K$ be a compact subset of $B_1$. Then, there exists $\varepsilon = \varepsilon(K) > 0$ s.th.*

$$\mathbb{P}[\mathscr{Z}_N(\beta) = 0, \text{ for some } \beta \in K] = O(N^{-\varepsilon}), \quad N \to \infty.$$

**N.B.** Almost sure statement?

# Zeros in the low-temperature phase: Poisson-$\zeta$-function

Define Poisson-$\zeta$-function as

$$\zeta_P(\beta) = \sum_{k=1}^{\infty} \frac{1}{{P_k}^{\beta}}, \quad \operatorname{Re}\beta > \frac{1}{2}, \quad \beta \neq 1,$$

where $P_1 < P_2 < \dots$ are the arrival times of a unit intensity homogeneous **Poisson process** on $\mathbb{R}_+$. N.B. $\zeta_P(\beta) - 1/(1-\beta)$ is a.s. analytic in $\beta > \frac{1}{2}$.

**Theorem (zeros in the low-temperature phase)**

*Let $f : B_2 \to \mathbb{R}$ be a continuous function with compact support. Let $\zeta_P^{(1)}$ and $\zeta_P^{(2)}$ be two independent copies of $\zeta_P$. Then,*

$$\sum_{\beta \in B_2 :\ \mathscr{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{w} \sum_{\substack{\beta \in B_2 : \\ \zeta_P^{(1)}(\beta/\sqrt{2})=0}} f(\beta) + \sum_{\substack{\beta \in B_2 : \\ \zeta_P^{(2)}(\beta/\sqrt{2})=0}} f(-\beta).$$

# Zeros on the circular arc boundaries

**Theorem (zeros on the arcs)**

*Let $\beta_0 = \sigma_0 + i\tau_0$ be such that $\sigma_0^2 + \tau_0^2 = 1$ and $\sigma_0^2 < 1/2$. There exist a complex-valued random variable $\xi$ and a bounded real sequence $\delta_N$ such that the zeros near $\beta_0$ look like*

$$\beta = \beta_0 \left(1 + \frac{2\pi i k + \xi + i\delta_N}{n}\right) + o\left(\frac{1}{n}\right), \quad k \in \mathbb{Z}.$$

# Zeros on the straight linear segment boundaries

**Theorem (zeros on the segments)**

*Let $\beta_0 = \sigma_0 + i\tau_0$ be such that $\sigma_0 > 1/\sqrt{2}$, $\tau_0 > 0$ and $\sigma_0 + \tau_0 = \sqrt{2}$. There exist a complex-valued random variable $\eta$ and a complex sequence $d_N = O(\log n)$ such that the zeros of $\mathscr{Z}_N$ near $\beta_0$ are given by the formula*

$$\beta = \beta_0 + \mathrm{e}^{-\frac{2\pi i}{3}} \frac{1}{n} \left( \frac{2\pi i k + \eta}{\sqrt{2}\tau_0} + d_N \right) + o\left(\frac{1}{n}\right), \quad k \in \mathbb{Z}.$$

# Fluctuations: CLT

**Theorem (CLT)**

*If* $\sigma^2 < 1/2$ *and* $\tau \neq 0$*, then*

$$\frac{\mathscr{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau\rho}}{N^{\frac{1}{2}+\sigma^2}} \xrightarrow[N\to\infty]{w} N_{\mathbb{C}}(0,1).$$

**N.B.** If $\sigma^2 < 1/2$ and $\tau = 0$, then the limiting distribution is real normal (known).

# Fluctuations: Critical CLT

Boundary case $\sigma^2 = 1/2$.

**Theorem (non-standard CLT)**

*If $\sigma^2 = 1/2$ and $\tau \neq 0$, then*

$$\frac{\mathscr{Z}_N(\beta) - N^{1+\frac{1}{2}(\frac{1}{2}-\tau^2)+i\sigma\tau\rho}}{N} \xrightarrow[N\to\infty]{w} N_{\mathbb{C}}(0, 1/2).$$

**N.B.** Truncated variance.

# Fluctuations: Beyond CLT

Fluctuations of $\mathscr{Z}_N(\beta)$ in the domain $\sigma^2 > 1/2$. Let $b_N$ be a sequence such that $\sqrt{2\pi} b_N e^{{b_N}^2/2} \sim N$ as $N \to \infty$. We can take

$$b_N = \sqrt{2n} - \frac{\log(4\pi n)}{2\sqrt{2n}}.$$

**Theorem (extreme value regime)**

*Let $\sigma > 1/\sqrt{2}$, $\tau \neq 0$, and $|\rho| < 1$. Then,*

$$\frac{\mathscr{Z}_N(\beta) - N\mathbb{E}[e^{\sqrt{n}(\sigma X + i\tau Y)} \mathbb{1}_{X<b_N}]}{e^{\sigma\sqrt{n} b_N}} \xrightarrow[N\to\infty]{w} S_{\sqrt{2}/\sigma},$$

*where $S_\alpha$ is a complex isotropic $\alpha$**-stable** random variable with a characteristic function of the form $\mathbb{E}[e^{i\mathrm{Re}(S_\alpha \bar{z})}] = e^{-\mathrm{const}\cdot|z|^\alpha}$, $z \in \mathbb{C}$.*

# Fluctuations: Beyond CLT, details

Under the assumptions of the previous theorem:

$$\frac{\mathscr{Z}_N(\beta)}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} S_{\sqrt{2}/\sigma}, \quad \text{if } \sigma+|\tau|>\sqrt{2},$$

$$\frac{\mathscr{Z}_N(\beta)-N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau\rho}}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} S_{\sqrt{2}/\sigma}, \quad \text{if } \sigma+|\tau|\le\sqrt{2}.$$

Similarly, if $\sigma > 1/\sqrt{2}$, but $\rho = 1$, then we have

$$\frac{\mathscr{Z}_N(\beta)}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} \zeta_P\left(\frac{\beta}{\sqrt{2}}\right), \quad \text{if } \sigma+|\tau|>\sqrt{2},$$

$$\frac{\mathscr{Z}_N(\beta)-N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau}}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} \zeta_P\left(\frac{\beta}{\sqrt{2}}\right), \quad \text{if } \sigma+|\tau|\le\sqrt{2},\ \sigma\neq\sqrt{2}.$$

For $\rho = -1$, replace $\beta$ by $\bar{\beta}$.

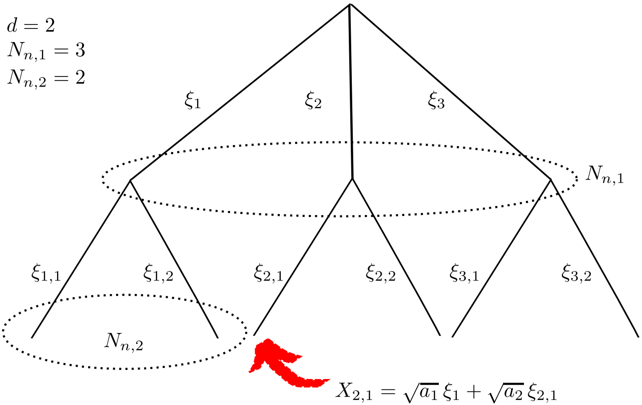
# Generalised Random Energy Model

Define a zero-mean Gaussian random field $\{X_\varepsilon \colon \varepsilon \in \mathbb{S}_n\}$ by

$$X_\varepsilon = \sqrt{a_1}\,\xi_{\varepsilon_1} + \sqrt{a_2}\,\xi_{\varepsilon_1\varepsilon_2} + \ldots + \sqrt{a_d}\,\xi_{\varepsilon_1\ldots\varepsilon_d}.$$

1. **Number of levels** $d \in \mathbb{N}$.
2. **The variances** of the levels $a_1, \ldots, a_d > 0$ (**energetic parameters**).
3. **The branching exponents** $\alpha_1, \ldots, \alpha_d > 1$ (**entropic parameters**).
4. **Branching numbers** $N_{n,1} = [\alpha_1^n], \ldots, N_{n,d} = [\alpha_d^n]$.
5. **Leaves of the tree**
   $\mathbb{S}_n = \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}^d \colon 1 \le \varepsilon_1 \le N_{n,1}, \ldots, 1 \le \varepsilon_d \le N_{n,d}\}$.
6. **Random energies**
   $\{\xi_{\varepsilon_1\ldots\varepsilon_m} \colon 1 \le m \le d, 1 \le \varepsilon_1 \le N_{n,1}, \ldots, 1 \le \varepsilon_m \le N_{n,m}\}$ i.i.d. $\mathcal{N}(0,1)$ r.v.'s.

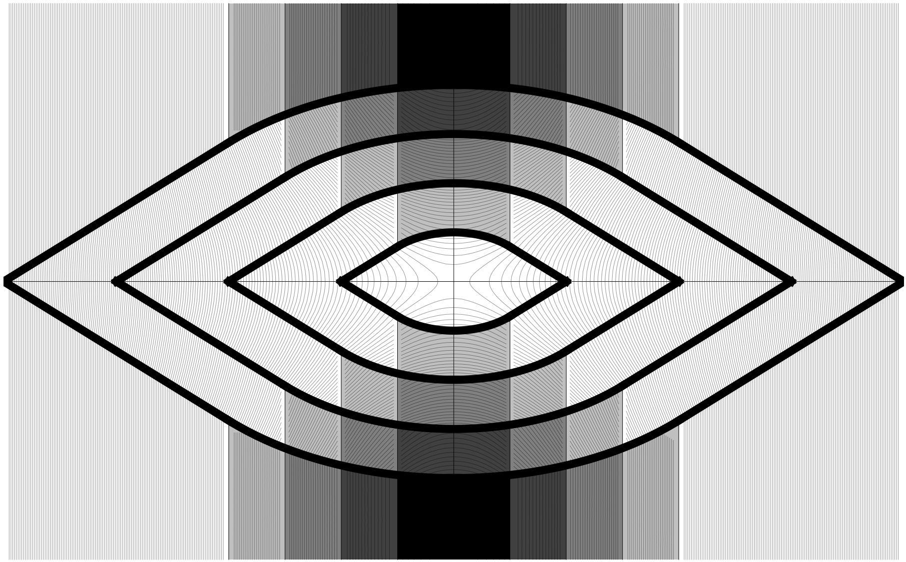
# Tree structure in the GREM




**Cumulative displacement = energy:**

$$X_{\varepsilon_1\varepsilon_2} = \sqrt{a_1}\,\xi_{\varepsilon_1} + \sqrt{a_2}\,\xi_{\varepsilon_1\varepsilon_2}.$$

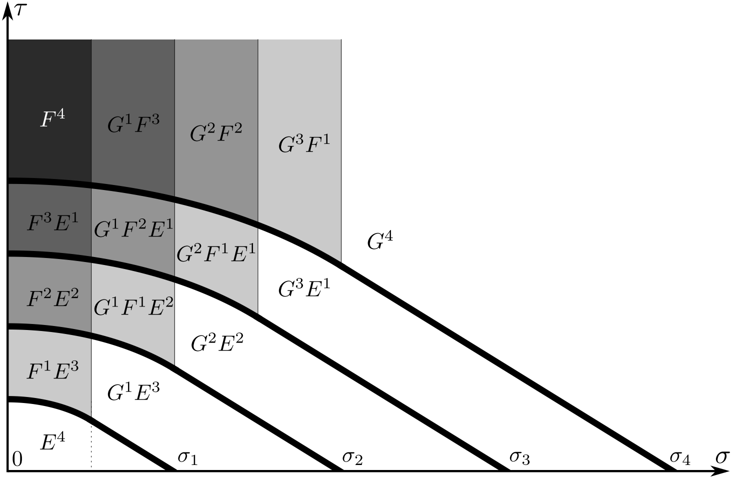
# Phase diagram of the GREM in the complex $\beta$ plane



# Critical inverse temperatures

$$\sigma_k = \sqrt{\frac{2\log \alpha_k}{a_k}}, \quad 1 \le k \le d.$$

$E \rightsquigarrow$ **expectation**, $F \rightsquigarrow$ **fluctuations**, $G \rightsquigarrow$ **glass (EVT).** Let $\sigma_1 < \ldots < \sigma_d$.



# Log-partition function

**Theorem**

*For every $\beta \in \mathbb{C}$, the following limit exists in probability and in $L^q$, for all $q \geq 1$:*

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |\mathscr{Z}_n(\beta)| = \sum_{k=1}^{d} p_k(\beta),$$

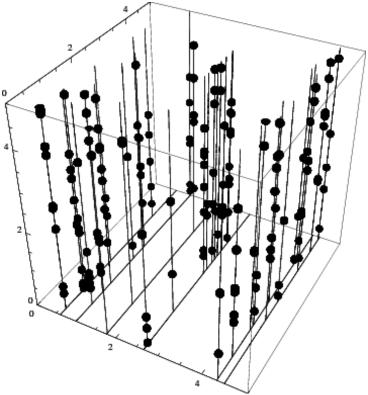
*where*

$$p_k(\beta) = \begin{cases} |\sigma|\sqrt{2a_k \log \alpha_k}, & \text{if } \beta \in \bar{G}_k, \\ \frac{1}{2}\log \alpha_k + a_k\sigma^2, & \text{if } \beta \in \bar{F}_k, \\ \log \alpha_k + \frac{1}{2}a_k(\sigma^2 - \tau^2), & \text{if } \beta \in \bar{E}_k. \end{cases}$$

Confirms and extends **Takahashi (2011)**.

# Fluctuations

**Poisson cascade** $\Pi = \sum_{\varepsilon=(\varepsilon_1,\dots,\varepsilon_d)\in\mathbb{N}^d} \delta(P_{\varepsilon_1}, P_{\varepsilon_1\varepsilon_2}, \dots, P_{\varepsilon_1\dots\varepsilon_d})$, where $\sum_{i=1}^{\infty} \delta(P_{\varepsilon_1\dots\varepsilon_m i})$ a unit intensity Poisson point process on $(0,\infty)$.




**Poisson cascade $\zeta$-function**: $\zeta_P(z_1,\dots,z_d) = \sum_{\varepsilon\in\mathbb{N}^d} P_{\varepsilon_1}^{-z_1} P_{\varepsilon_1\varepsilon_2}^{-z_2} \dots P_{\varepsilon_1\dots\varepsilon_d}^{-z_d}$.

# Fluctuations

**Theorem**

*Let $\beta \in G^{d_1}F^{d_2}E^{d_3}$. Then,*

$$\frac{\mathscr{Z}_n(\beta)}{\mathrm{e}^{c_n(\beta)}} \underset{n\to\infty}{\Longrightarrow} \begin{cases} 1, & \text{if } d_1 = 0 \text{ and } d_2 = 0, \\ N_{\mathbb{C}}(0,1), & \text{if } d_1 = 0 \text{ and } d_2 > 0, \\ \zeta_P\left(\frac{\beta}{\sigma_1}, \ldots, \frac{\beta}{\sigma_{d_1}}\right), & \text{if } d_1 > 0 \text{ and } d_2 = 0, \\ cS_{\sigma/\sigma_1}, & \text{if } d_1 > 0 \text{ and } d_2 > 0. \end{cases}$$

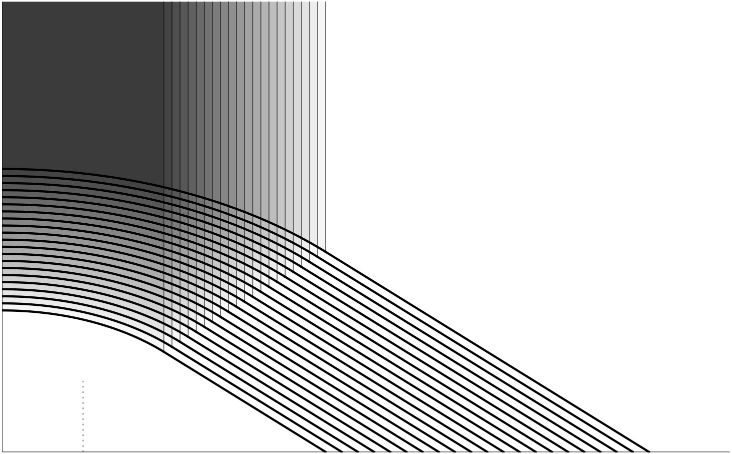
*Here, $\zeta_P$ is the Poisson cascade zeta function and $S_\alpha$ is the isotrpic, complex standard $\alpha$-stable random variable with characteristic function $\mathbb{E}\mathrm{e}^{i\mathrm{Re}(S_\alpha \bar{z})} = \mathrm{e}^{-|z|^\alpha}$, $z \in \mathbb{C}$, where $\alpha \in (0,2)$.*

$$c_{n,k}(\beta) = \begin{cases} \beta\sqrt{na_k}\,u_{n,k}, & \text{if } \beta \in G^k, \\ \frac{1}{2}\log N_{n,k} + a_k\sigma^2 n, & \text{if } \beta \in F^k, \\ \log N_{n,k} + \frac{1}{2}a_k\beta^2 n, & \text{if } \beta \in E^k. \end{cases}$$

$c_n(\beta) = c_{n,1}(\beta) + \ldots + c_{n,d}(\beta)$, $u_{n,k} = \sigma_k\sqrt{na_k}$.

# Continuous GREM = CREM

“Continuum of phase transitions”:



# Complex CREM

- $A\colon [0,1] \to \mathbb{R}$ an increasing concave function $A(1) = 1$.
- $a_1 + \ldots + a_k = A\left(\frac{k}{d}\right), \quad \log \alpha_k = \frac{1}{d} \log \alpha, \quad 1 \le k \le d.$
- Continous GREM: $d \to \infty$.
- $\sigma_t^\infty = \sqrt{\frac{2\log\alpha}{A'(t)}}$.

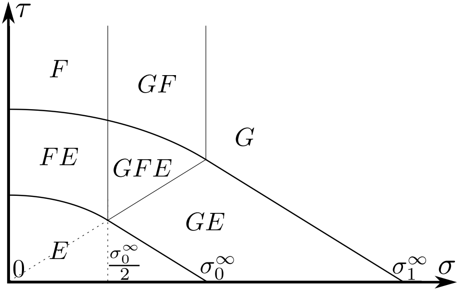



**Figure:** Seven phases $GFE$, $GF$, $FE$, $GE$, $G$, $F$, $E$.

# More of complex random energies: BBM, Gaussian Multiplicative Chaos, ...

- **BBM**, **Gaussian Multiplicative Chaos** $\approx$ CREM with $A(t) = t$
- $\rightsquigarrow \sigma_t^\infty \equiv 1$.
- $\rightsquigarrow$ only three phases: $E, F, G$ (as in the REM).
- Cf. partial results by Derrida, Evans, Speer; Lacoin, Rhodes Vargas; Madaule, Rhodes, Vargas; Madaule, Rhodes, Vargas.