Maximum independent sets in random $d$-regular graphs

Jian Ding, Allan Sly, and Nike Sun

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Outline

1. Maximum independent sets

2. Second moment and condensation

3. Sharp thresholds by counting clusters
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3. Sharp thresholds by counting clusters
Many sparse random CSPs are in by the 1-step replica symmetry breaking universality class of Parisi-Mezard such as $k$-SAT, independent set, coloring.
Many sparse random CSPs are in the 1-step replica symmetry breaking universality class of Parisi-Mezard such as $k$-SAT, independent set, coloring. There is a detailed but non-rigorous theory for their structure and thresholds. A great deal of progress in studying these models rigorously. However, much of the conjectured picture remains unproved, in particular the exact threshold values.
Rigorous Bounds

For sparse CSPs with RSB, threshold behavior long in question. Rigorous bounds on the SAT–UNSAT transition include:

- Random graph coloring: Bollobás '88, Achlioptas–Naor '04, Coja-Oghlan–Vilenchik '13
- Random k-NAE-SAT: Achlioptas–Moore '02, Coja-Oghlan–Zdeborová '12, Coja-Oghlan–Panagiotou '12
- Random k-SAT: Kirousis–Kranakis–Krizanc–Stamatiou '97, Achlioptas–Peres '03, Coja-Oghlan–Panagiotou '13, Coja-Oghlan '14
- Random regular graph independent set: Bollobás '81, McKay '87, Frieze–Luczak '92, Frieze–Suen '94, Wormald '95
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Maximal independent sets in random graphs

Let $A \equiv \mathcal{A}_n \equiv$ maximum size of an independent set in a random graph $G_n$ on $n$ vertices.
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$A_n$ asymptotics?

$A_n/n \rightarrow \alpha_\star$?
Previous work

Solved much earlier on dense Erdős–Rényi graphs $G_{n,p}$: $A_n \sim 2 \log n \log \left[ \frac{1}{1 - p} \right]$ [Grimmett–McDiarmid '75].

Sparse case much harder — numerous partial results on $G_{n,d}$: Bollobás '81, McKay '87, Frieze–Luczak '92, Frieze–Suen '94, Wormald '95 give $A_n/n \approx 2 (\log d)/d$ but not sharp.

For many years, existence of $\alpha^\star$ with $A_n = n\alpha^\star + o(n)$ unknown, even though well known that $A_n$ has only $O(n^{1/2})$ fluctuations.

Existence on $G_{n,d}$, $G_{n,d}/n$ proved by Bayati–Gamarnik–Tetali '10 — super-additivity argument; no information about value of $\alpha^\star$ or fluctuations of $A_n$. 
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Main result

Theorem (DSS). The maximum independent set size $A_n$ in the random $d$-regular graph $G_{n,d}$ has $O(1)$ fluctuations around $n\alpha^* - c^* \log n$ for explicit $\alpha^* = \alpha^*(d)$ and $c^* = c^*(d)$, provided $d \geq d_0$. 
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Solve $\varphi(\lambda) = 0$

\[ \alpha(\lambda) = \log \frac{\lambda}{\lambda - (\lambda - 1)} q^{2} + \frac{\lambda (1-q)}{\lambda - (\lambda - 1)} q - \frac{\lambda d q}{2} + \lambda \frac{1}{(1-q)} \]

$A_n$ has $O(1)$ fluctuations about $n$
Explicit constants

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\[ \varphi(\lambda) = 0 \]

\[ \varphi(\lambda) \equiv \frac{d-2}{2} \log \frac{\lambda}{\lambda - (\lambda - 1)q^2} + \log \frac{\lambda}{\lambda - (\lambda - 1)q} - q \frac{dq/2 + \lambda(1-q)}{\lambda - (\lambda - 1)q^2} \log \lambda \]

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\( \alpha(\lambda) \)

\( A_n \) has \( O(1) \) fluctuations about \( n\alpha(\lambda_*) - \frac{\log n}{2 \log \lambda_*} \)
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First moment upper bound

Let $Z_{n\alpha}$ be the number of independent sets of size $n\alpha$ in $G_n$. 

$$P(Z_{n\alpha} > 0) \leq E[Z_{n\alpha}] = \left(\frac{n}{n\alpha}\right) \left(1 - \frac{d}{n}\right)^{n\alpha(\alpha - 1)/2} = \exp\left\{ n\left[H(\alpha) - \frac{d\alpha^2}{2}\right] + O(\log n) \right\}$$

$H(\alpha) \approx \alpha \log(1/\alpha)$. 

Exponent crosses zero at $\alpha_1 \approx \frac{2\log d}{d}$.  

$A_n \leq n\alpha_1[1 + o(1)]$ on $G_{n,d/n}$. 

Similarly $\alpha_1 \approx \frac{2\log d}{d}$ on $G_{n,d}$. 

Not sharp: $\alpha_1 > \alpha^\star$. 
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H(\alpha) - d\alpha^2/2 \quad (d = 100)
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Second moment fails to locate sharp threshold

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\frac{1}{\mathbb{E}[Z^2]/(\mathbb{E}Z)^2} \leq \mathbb{P}(Z > 0) \leq \mathbb{E}Z
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For MAX-IS on sparse graph ensembles, basic second moment approach fails to locate sharp threshold:
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w.p.p.

For \textbf{MAX-IS} on \textit{sparse} graph ensembles, basic second moment approach \textbf{fails to locate sharp threshold}:

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For MAX-IS on sparse graph ensembles, basic second moment approach fails to locate sharp threshold:

\[ \mathbb{E}Z_{n\alpha} \ll 1 \]
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For **MAX-IS** on **sparse** graph ensembles, basic second moment approach **fails to locate sharp threshold**:

\[
\mathbb{E}[Z^2_{n\alpha}] \asymp (\mathbb{E}Z_{n\alpha})^2
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Second moment fails to locate sharp threshold

\[ \frac{1}{\mathbb{E}[Z^2]/(\mathbb{E}Z)^2} \leq P(Z > 0) \leq \mathbb{E}Z \]

For **MAX-IS** on **sparse** graph ensembles, basic second moment approach **fails to locate sharp threshold**:

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\[
\text{gap}
\]

\text{no conclusions about } \mathbb{P}(Z_{n\alpha} > 0)

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Independent set $\leftrightarrow$ 0/1 configuration ($1 \equiv$ occupied)
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Probability for a 0 to have a **single neighboring 1**? 0 $\leftrightarrow$ 1

$$\mathbb{P} \left( \text{Bin} \left( d, \frac{2 \log d}{d} \right) = 1 \right)$$
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Number of 1's neighboring a 0? On average $2 \log d$

Probability for a 0 to have a single neighboring 1? $0 \leftrightarrow 1$

$$P\left(\text{Bin}\left(d, \frac{2 \log d}{d}\right) = 1\right) = d \frac{2 \log d}{d} \left(1 - \frac{2 \log d}{d}\right)^{d-1}$$
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Typical independent set has at least $\frac{2^n}{d^2}$ nearby solutions
Second moment fails in independent set

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Second moment fails

Non-rigidity does not occur on dense graphs.
Outline

1. Maximum independent sets
2. Second moment and condensation
3. Sharp thresholds by counting clusters
Outline

1 Maximum independent sets

2 Second moment and condensation

3 Sharp thresholds by counting clusters
Breaking the condensation barrier
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Until rather recently, best satisfiability lower bounds remained below condensation threshold $\alpha_c$
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Coja-Oghlan–Panagiotou (2012) for random $k$-NAE-SAT
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**CP ’12** approach: apply second moment method to count clusters rather than assignments.
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CP ’12 approach: apply second moment method to count clusters rather than assignments

Idea: reweight by $2^{-\#f}$ where $\#f$ is the number of vertices which are free.
Whitening algorithm

Approach: Set variables which can be changed to free. Parisi ’02,
Maneva–Mossel–Wainwright ’07
Maneva–Sinclair ’08, Achlioptas–Ricci-Tersenghi ’09

Idea behind survey propagation.
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Idea behind survey propagation.

We determine the exact threshold $\alpha_\star$ by finding a projection which takes entire clusters to single points
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**Whitening algorithm:**

**WHILE** there exists any 0 with a single neighboring 1
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Idea behind survey propagation.

We determine the exact threshold $\alpha^*$ by finding a projection which takes entire clusters to single points

$$f \iff f$$

Whitening algorithm:

WHILE there exists any 0 with a single neighboring 1
DO set both to $f$ and declare them matched

$$0 \iff 1$$

$$f \iff f$$
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Idea behind survey propagation.

We determine the **exact threshold** $\alpha_\star$ by finding a projection which takes **entire clusters** to single points

\[ f \iff f \]

**Whitening algorithm:**

**WHILE** there exists any 0 with a single neighboring 1  
  **DO** set both to f and declare them **matched**  
Set any 0 with no neighboring 1’s to f.
Whitening algorithm

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Observation:
Whitening algorithm

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Whitening algorithm:

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**DO** set both to $f$ and declare them *matched*.

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**Observation:**

Configurations resulting from this procedure can be described by a graphical model.
A chain of swaps
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Graphical model for clusters

After discarding non-maximal configurations, mostly left with $0/1$ configurations where

a. $1$'s can only neighbor $0$'s;

b. $f$'s occur in matched pairs;

c. $0$'s must have at least two neighboring $1$'s.

$Z_{n\alpha} \equiv \text{partition function of graphical model}$

with $n\alpha \leq \#1 + \#f$-pairs

Sharp threshold for $Z_{n\alpha}$ gives $\text{max-is}$ sharp threshold
Graphical model for clusters

After discarding non-maximal configurations, mostly left with 0/1/f configurations where
Graphical model for clusters

After discarding non-maximal configurations, mostly left with $0/1/f$ configurations where

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Graphical model for clusters

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\[
Z_{n\alpha} \equiv \text{partition function of graphical model with } n\alpha \leq \#1\text{'s} + \#f\text{-pairs}
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Sharp threshold for \(Z_{n\alpha}\) gives MAX-IS sharp threshold
Outline of our approach

1. Correspondence between max-is threshold and threshold of simplified \( \frac{1}{f} \) graphical model for clusters

2. Sharp threshold in graphical model for clusters

\[ E[Z_{2n}] \approx (E[Z_{n\alpha}])^2 \| E[Z_{n\alpha}] \ll 1 \rightarrow G_{n,d} \text{ converges locally to } d \text{-regular tree:} \]

Bethe variational principle — relation between (i) local neighborhood profiles optimizing first moment and (ii) fixed points of tree recursions — gives formula for \( \alpha^\star, c^\star \)

Second moment handled by analysis of paired spin model

Most of work, 30+ pages here

3. Variance decomposition to prove \( O(1) \) fluctuations of \( A_n \)
Outline of our approach

1. Correspondence between MAX-IS threshold
Outline of our approach

1. Correspondence between MAX-IS threshold and threshold of simplified $0/1/f$ graphical model for clusters
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Outline of our approach

1. Correspondence between MAX-IS threshold and threshold of simplified 0/1/\$f\$ graphical model for clusters

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\[ \mathbb{E}[Z_{n\alpha}^2] \asymp (\mathbb{E}Z_{n\alpha})^2 \quad \mathbb{E}Z_{n\alpha} \ll 1 \]
Outline of our approach

1. Correspondence between MAX-IS threshold and threshold of simplified 0/1/\(f\) graphical model for clusters

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\[ \mathbb{E}[Z^2_{n\alpha}] \asymp (\mathbb{E}Z_{n\alpha})^2 \quad \mathbb{E}Z_{n\alpha} \ll 1 \quad \rightarrow \]

\(G_{n,d}\) converges locally to \(d\)-regular tree:
Outline of our approach

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*Most of work, 30+ pages here*
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Thank you!