Poisson-Dirichlet statistics for the extremes of the 2D discrete Gaussian free field

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joint work with Louis-Pierre Arguin

July 24, 2014

Outline

1. 2D discrete Gaussian free field
   - Definition, properties
   - Maximum, log-number of high points
   - REM free energy
2. Main results
   - 1-step Replica Symmetry Breaking
   - Poisson-Dirichlet weights
3. Sketch of the proof
   - Ghirlanda-Guerra Identities
   - A larger box
   - A perturbed model and the Bovier-Kurkova technique
   - Computation of its 2-GREM free energy

Gaussian free field on $A \subset \mathbb{Z}^2$

The Gaussian free field (GFF) on a box $A \subset \mathbb{Z}^2$ with Dirichlet boundary condition is the centered Gaussian field $(\phi_v, v \in A)$ with the covariance matrix given by the Green function:

$$G_A(v, v') := E_v \left[\sum_{k=0}^{\tau_A} \mathbf{1}_{\{S_k = v'\}} \right],$$

where $(S_k)_k$ is a simple random walk with $S_0 = v$ of law P_v killed at the first exit time of A, denoted by τ_A.

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log-correlated Gaussian field!

Maximum and high-points

Theorem (Bolthausen, Deuschel, and Giacomin '01)

$$\lim_{n\to\infty} \frac{\max_{v\in V_n^\delta} \phi_v}{\log n^2} = \sqrt{\frac{2}{\pi}}, \qquad \textit{in probability.}$$

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Theorem (Daviaud '06)

For all $0 < \lambda < 1$,

$$\lim_{n\to\infty} \frac{1}{\log n^2} \log \#\{v \in V_n^\delta : \phi_v \geq \lambda\sqrt{\frac{2}{\pi}} \log n^2\} = 1 - \lambda^2, \qquad \textit{in probability.}$$

REM free energy

$$Z_n(\beta) := \sum_{v \in V_n} e^{\beta \phi_v}, \qquad f_n(\beta) := \frac{1}{\log n^2} \log Z_n(\beta).$$

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Proposition

The free energy is the same as the REM. With $\beta_c := \sqrt{2\pi}$, *we have*

$$f(\beta) := \lim_{n\to\infty} f_n(\beta) = \begin{cases} 1 + \frac{\beta^2}{2\pi}, & \text{if } \beta \le \beta_c, \\ \sqrt{\frac{2}{\pi}}\beta, & \text{if } \beta \ge \beta_c, \end{cases} \qquad \text{a.s. and in } L^1.$$

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- weak freezing
- The free energy $f(\beta)$ is differentiable at all β
- Proof: using Laplace's method, direct consequence of Bolthausen-Deuschel-Giacomin '01 and Daviaud '06 results in V_n^δ + comparison argument with i.i.d. random variables for the upper bound

Main results: 1-RSB

The Gaussian field $(\phi_v, v \in V_n)$ is 1-RSB: overlap distribution

$$G_{\beta,n}(v) := \frac{e^{\beta\phi_v}}{Z_n(\beta)}, \qquad q(v,v') := \frac{\mathbb{E}[\phi_v\phi_{v'}]}{\frac{1}{\pi}\log n^2} \left(\approx 1 - \frac{\log \|v-v'\|^2}{\log n^2}\right).$$

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Theorem (Arguin-Z. '13)

The joint distribution of overlaps is the same as the REM: for $\beta > \beta_c$

$$x_\beta(q) := \lim_{n\to\infty} \mathbb{E}\left[G_{\beta,n}^{\times 2}\{q_{12} \le q\}\right] = \begin{cases} \frac{\beta_c}{\beta}, & \text{for } 0 \le q < 1, \\ 1, & \text{for } q = 1. \end{cases}$$

The limit law is $\frac{\beta_c}{\beta}\delta_0 + (1 - \frac{\beta_c}{\beta})\delta_1$.

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- The result for non-hierarchical fields was conjectured by Carpentier & Le Doussal '00.

Notations

- For $0 < \alpha < 1$, let $\eta = (\eta_i, i \in \mathbb{N})$: atoms of a Poisson point process on $(0, \infty)$ of intensity measure $s^{-\alpha-1}\, ds$.

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- A Poisson-Dirichlet variable ξ of parameter α is a probability measure on the space of decreasing weights $\boldsymbol{s} = (s_1, s_2, \dots)$ with $1 \geq s_1 \geq s_2 \geq \cdots \geq 0$ and $\sum_i s_i \leq 1$ which has the same law as

$$\xi \sim \left(\frac{\eta_i}{\sum_j \eta_j}, i \in \mathbb{N} \right)_{\downarrow},$$

where $\downarrow$ stands for the decreasing rearrangement.

Main results: PD weights

Theorem (Arguin-Z. '13)

Let $\beta > \beta_c$ *and* $\xi = (\xi_k, k \in \mathbb{N})$ *be a Poisson-Dirichlet variable of parameter* β_c/β. *For any continuous function* $F : [0,1]^{\frac{s(s-1)}{2}} \to \mathbb{R}$ *of the overlaps of* s *replicas:*

$$\lim_{n\to\infty} \mathbb{E}\left[G_{\beta,n}^{\otimes s}\left(F(q(v_l, v_{l'}))\right)\right] = E\left[\sum_{k_1\in\mathbb{N},\dots,k_s\in\mathbb{N}} \xi_{k_1}\dots\xi_{k_s}\ F(\delta_{k_l k_{l'}})\right].$$

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- Poisson-Dirichlet statistics for the extremes of log-correlated fields conjectured by Fyodorov & Bouchaud '08.

PD weights from GG Identities

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- Using a concentration result under the Gibbs measure at every β where the free energy is differentiable (Panchenko '10) and Gaussian integration by parts, we can prove that $(G_{\beta,n})$ satisfy Ghirlanda-Guerra identities, i.e. (writing $q_{l,l'}$ for $q(v_l, v_{l'})$)

$$\mathbb{E}G_{\beta,n}^{\times s+1}\Big[q_{1,s+1}\ F(q_{ll'})\Big]$$
$$= \frac{1}{s}\mathbb{E}G_{\beta,n}^{\times 2}\Big[q_{12}\Big]\mathbb{E}G_{\beta,n}^{\times s}\Big[F(q_{ll'})\Big] + \frac{1}{s}\sum_{k=2}^{s}\mathbb{E}G_{\beta,n}^{\times s}\Big[q_{1k}\ F(q_{ll'})\Big] + o_n(1).$$

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- for the trivial overlaps 0-1, GG identities imply PD weights (cf PD characterization in Talagrand '03, using joint moments of the PD-weights).

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Lemma

For any $\rho \in (0,1)$,

$$\lim_{n\to\infty} G_{\beta,n}(A^c_{n,\rho}) = 0, \qquad \textit{in } \mathbb{P}\textit{-probability.}$$

A perturbed model: the $(\alpha, \boldsymbol{\sigma})$-GFF

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- For $0 < \alpha < 1$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{R}_+^2$, introduce

$$\psi_v := \sigma_1 \phi_{[v]_\alpha} + \sigma_2 \left(\phi_v - \phi_{[v]_\alpha}\right)$$

The Bovier-Kurkova technique

- For $0 < \alpha < 1$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) = (1, 1+u)$ with $u \in (-1, 1)$

Recall

$$x_\beta(r) := \lim_{n\to\infty} \mathbb{E}\big[G_{\beta,n}^{\otimes 2}(q(v,v') \leq r)\big], \qquad f_{n,\rho}^{(\sigma,\alpha)}(\beta) := \frac{1}{\log n^2} \log \sum_{v \in A_{n,\rho}} e^{\beta \psi_v}.$$

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Assuming that we can compute the free energy of the $(\alpha, \boldsymbol{\sigma})$-GFF restricted to $A_{n,\rho}$, we have:

Lemma

$$\frac{\beta^2}{\pi} \int_\alpha^1 x_\beta(r) dr = \lim_{\rho\to 0} \frac{\partial}{\partial u} \Big(\lim_{n\to\infty} \mathbb{E} f_{n,\rho}^{(\sigma,\alpha)}(\beta) \Big)\Big|_{u=0}$$

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Proof: Gaussian integration by parts + convexity arguments + careful attention to the overlaps between points close to the boundary.

Free energy of the perturbed model

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Proposition

Let $V_{12} := \sigma_1^2\alpha + \sigma_2^2(1-\alpha)$. *Then, for any* $\rho < \alpha$, *and for all* $\beta > 0$,

- **Case 1:** *if* $\sigma_1 \leq \sigma_2$

$$\lim_{n\to\infty} f_{n,\rho}^{(\sigma,\alpha)}(\beta) = f(\beta;V_{12}),$$

- **Case 2:** *if* $\sigma_1 \geq \sigma_2$,

$$\lim_{n\to\infty} f_{n,\rho}^{(\sigma,\alpha)}(\beta) = \alpha f(\beta;\sigma_1^2) + (1-\alpha) f(\beta;\sigma_2^2),$$

where the convergence holds almost surely and in L^1.

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Proof:

- Lower bound: log-number of high points in V_n^δ for the $(\alpha,\boldsymbol{\sigma})$-GFF (Daviaud's argument used recursively) + Laplace's method
- Upper bound: comparison with a non-homogeneous 2-level GREM.

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Free energy of the REM

The limiting free energy of the REM model consisting of n^2 i.i.d. Gaussian variables of variance $\frac{\sigma^2}{\pi} \log n^2$ is given by:

$$f(\beta;\sigma^2) := \begin{cases} 1 + \frac{\beta^2\sigma^2}{2\pi}, & \text{if } \beta \leq \beta_c(\sigma^2) := \frac{\sqrt{2\pi}}{\sigma}, \\ \sqrt{\frac{2}{\pi}}\sigma\beta, & \text{if } \beta \geq \beta_c(\sigma^2). \end{cases}$$

Basic properties of the 2D-GFF

Lemma

Let $B \subset A$ *be subsets of* $\mathbb{Z}^2$*. Let* $(\phi_v, v \in A)$ *be a GFF on* A*. Then*

$$\mathbb{E}[\phi_v | \mathcal{F}_{B^c}] = \mathbb{E}[\phi_v | \mathcal{F}_{\partial B}], \qquad \forall v \in B,$$

and

$$(\phi_v - \mathbb{E}[\phi_v | \mathcal{F}_{\partial B}], v \in B)$$

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Lemma

There exists a function $a : \mathbb{Z}^2 \times \mathbb{Z}^2 \mapsto [0, \infty)$ of the form

$$a(v, v') = \frac{1}{\pi} \log \|v - v'\|^2 + \frac{2\gamma_0 \log 8}{\pi} + O(\|v - v'\|^{-2})$$

(where γ_0 denotes the Euler's constant) such that $a(v, v) = 0$ and

$$G_A(v, v') = E_v \left[a(v', S_{\tau_A}) \right] - a(v, v') \ .$$